Exercises of the course “Constrained and Robust Control”,

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The final test consists in the solution of (at least) one exercise of Part A and (at least) one exercize of part B.

Exercises of the course
“Robust and Constrained Control”: part A

Exercise 1: rotating arm

Given the rotating arm with equation

$$\ddot{\theta}(t) = a\sin(\theta(t)) + \tau(t)$$

with control torque $\tau(t)$ and a uncertain parameter subject to the bounds

$$0 < \alpha \le a \le \beta,$$

let $\bar{\theta}$ be a constant reference angle. Let $x_1(t) = \theta(t) - \bar{\theta}$, $\dot{x}_1(t) = \dot{\theta}(t) = x_2(t)$ and $u(t) = \tau(t) - \bar{\tau}$, where $\bar{\tau}$ is the equilibrium torque. Then,

$$0 = a\sin(\bar{\theta}) + \bar{\tau}.$$

Write the system in the form

$$\begin{aligned}
\dot{x}_1(t) &= x_2(t) \\
\dot{x}_2(t) &= \left[a\,\frac{\sin(\theta) - \sin(\bar{\theta})}{\theta - \bar{\theta}}\right](\theta - \bar{\theta}) + u = [w]x_1 + u
\end{aligned}$$

with uncertain parameter $w(t)$.

1. Find proper bounds for w.
2. Find a Control Lyapunov Function (hint: set $w = 0$ and find the CLF for the nominal system).
3. Write a gradient–based control and find the gain $\gamma > 0$.
4. Assume that the speed has to be bounded as $|x_2| \le 1$. Consider the “auxiliary” control

   $$\hat{x}_2 = -\mathrm{sat}[x_1]$$

   or something similar such as

   $$\hat{x}_2 = -\frac{2}{\pi}\mathrm{atan}[x_1]$$

   and use backstepping to stabilize the system.
5. Write a simulation code for the closed–loop system.

Exercise 2: the LMI

Consider the uncertain system

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & a \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ b \end{bmatrix} u(t),$$

where parameters a and b are uncertain (and time-varying) subject to the bounds

$$0 < \alpha \leq a(t) \leq \beta, \quad 0 < \gamma \leq b(t) \leq \delta.$$

1. Write a set of LMIs (Linear Matrix Inequalities) to find a linear state feedback compensator
$$u = Kx$$
along with a quadratic Lyapunov function
$$\Psi(x) = x^\top P x$$

2. Install the free software CVX (`http://cvxr.com/cvx/`) and write a code which, given the positive values α, β, γ, δ, provides the linear compensator and the quadratic Lyapunov function.

Exercise 3: Bertinoro pendulum

Consider the system

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 + w(t) & -\alpha \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

with

$$|w(t)| \leq \omega < 1$$

and $\alpha > 0$ "small". This is an example of a system that is asymptotically stable for any constant w, but is unstable for time-varying $w(t)$.

1. Find a "destabilizing strategy", namely a function $w(t)$ (which might be not continuous) such that the time-varying system has a diverging trajectory.
2. Write a code that simulates the system and depicts the diverging trajectory.
3. Show that such a $w(t)$ can be chosen "switching", namely, assuming values only on the extrema $w(t) = \pm\omega$.
4. (Optional.) Consider now

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 + w(t) & +\alpha \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix},$$

   with $\alpha > 0$ small. Now the system is unstable for any constant w. Yet, with a time varying $|w(t)| \leq \omega < 1$, considered as a control action, you can "stabilize" it (if α is small enough). Find a suitable control law $w(t)$. Hint: consider the system in reverse time $\tau = -t$, change the sign to $x_2 := -x_2$ and note that the reverse system ...

Exercizes of the course “Constrained and robust control: part B”

Part B1: Robust Stability

Consider the control system in Fig. 1, where G(s) and R(s) are given such that the system with Δ(s)=0 is well posed and asymptotically stable.

Exercise 1: Prove that the system is asymptotically stable for any $\Delta(s) \in H_\infty$ with $\|\Delta(s)\|_\infty \le \alpha$ iff the input sensitivity function has H_∞ norm less than 1/α.

Exercise 2: Let $G(s) = \begin{bmatrix} \dfrac{1}{s+1} \\ \\ \dfrac{2s}{5s+1} \end{bmatrix}$, $R(s) = \begin{bmatrix} \dfrac{1}{s} & \dfrac{1}{s} \end{bmatrix}$. Verify that the closed-loop system with Δ(s)=0 is well posed and asymptotically stable and compute the H_∞ norm of the input sensitivity function and relate it to the complex stability radius. Finally, assume that Δ(s)=Δ real uncertainty. Compute the real stability radius.

Exercise 3: Let the block $\Delta(s)$ describe a polytopic uncertainty, i.e. $\Delta(t) \in \left\{ \begin{bmatrix} -0.5 \\ -0.5 \end{bmatrix}, \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix} \right\}$.

Write a state space description of the closed loop system as

$$\dot{x} = A_\sigma x$$

with σ a switching signal, $\sigma(\tau)$={1,2}. Discuss the quadratic stability under arbitrary switching of the polytopic system.

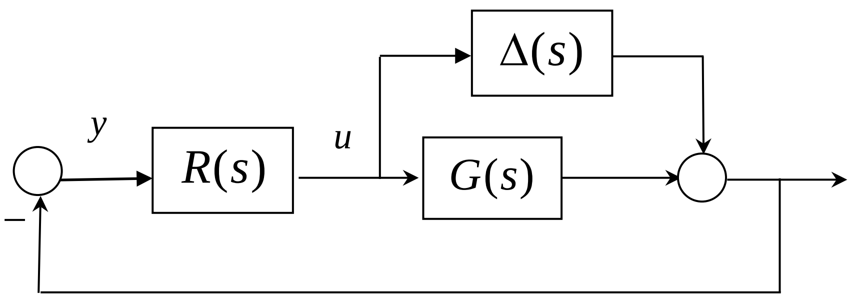

Fig.1

Part B2: State-feedback

Consider the system

$$\dot{x} = Ax + B_1 w + B_2 u$$
$$z = Cx + Du$$

Exercise 4:. Prove that the family of all state feedback laws $u = Kx$ such that the closed-loop system is asymptotically stable is given by

$$K = WS^{-1}$$
$$S > 0$$
$$AS + B_2 W + SA' + W' B_2' < 0$$

Prove that the family of all state-feedback laws $u = Kx$ such that the H_∞ norm of the closed-loop system from w to z is less than $\gamma > 0$ is given by

$$K = WS^{-1}$$
$$S > 0$$
$$\begin{bmatrix} AS + B_2 W + SA' + W' B_2' & SC' + W' D' \\ CS + DW & -\gamma^2 I \end{bmatrix} < 0$$

Exercise 5: Assume that $G(s) = C(sI - A)^{-1} B_2 + D$ has full column normal rank in the extended imaginary axis. Prove that the H_2 norm of the closed-loop system from w to z is minimized by

$$K = -B_2' P - C' D$$

where $P \geq 0$ is the stabilizing solution of the ARE

$$A' P + PA - (PB_2 + C' D)(D' D)^{-1} (B_2' P + D' C) + C' C = 0$$

Exercise 6: Assume that $G(s) = C(sI - A)^{-1} B_2 + D$ has full column normal rank in the extended imaginary axis. Prove that the H_∞ norm of the closed-loop system from w to z is less that g iff
there exists $P \geq 0$, stabilizing solution of the ARE

$$A' P + PA - (PB_2 + C' D)(D' D)^{-1} (B_2' P + D' C) + \frac{1}{\gamma^2} PB_1 B_1' P + C' C = 0$$

In such a case $K = -B_2' P - C' D$ is the state-feedback law minimizing the γ-entropy of the closed-loop system.

Part B3: Filtering

Consider the block scheme in Fig.2, where $G(s) = C(sI - A)^{-1}B + D \in L_\infty$,
$H(s) = M(sI - A)^{-1}B \in L_2$. Assume also that $G(s)$ has full row normal rank in the extended imaginary axis.

Exercise 7: Prove that there exists the canonical spectral factorization of $G(j\omega)G(-j\omega)'$, i.e. a square $G_o(s) \in H_\infty$ with $G_o(s)^{-1} \in H_\infty$ such that $G(j\omega)G(-j\omega)' = G_o(j\omega)G_o(-j\omega)'$, for all ω.
and write a state-space description of system $G_o(s)$ via the stabilizing solution $S \geq 0$ of the Riccati equation

$$AS + SA' - (SC' + BD')(DD')^{-1}(DB' + CS) + BB' = 0$$

Exercise 8: Prove that the filter $F^{opt}(s) = \left[H(s)G(s)(G_0(-s)')^{-1}\right]_{st} G_0(s)^{-1}$ is the optimal H_2 filter, i.e. $F^{opt}(s) \in H_\infty$, $F^{opt}(s)G(s) - H(s) \in H_2$ and $\left\| F^{opt}(s)G(s) - H(s) \right\|_2$ is minimized.

Exercise 9: Prove that a state space description of $F^{opt}(s) = -M(sI - A + LC)^{-1}L$ where we have set $L = -(SC' + BD')(DD')^{-1}$ (optimal observer gain).

Exercize 10: Letting $w = [w_1' \; w_2']'$, $G = [\overline{G} \quad I]$, $H = [\overline{G} \quad 0]'$ with a given $\overline{G}(s) \in L_2$, the problem reduces to the Wiener filtering problem and the above solution can be rewritten as
$F^{opt}(s) = I - G_0(s)^{-1}$
and prove that the solution of point above satisfies the robustness property
$\left\| F^{opt}(s)G(s) - H(s) \right\|_\infty < 2$

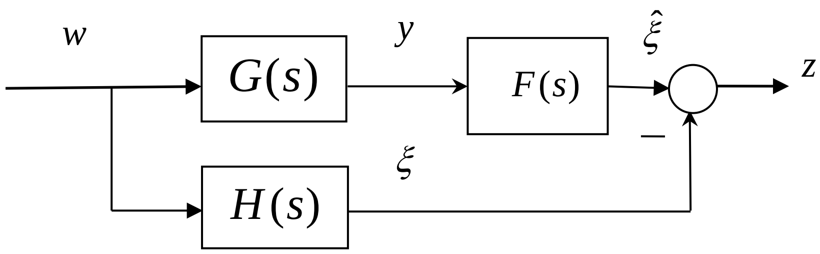

Fig.2