

*ANALYSIS AND CONTROL OF LINEAR  
SWITCHED SYSTEMS*

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<sup>1</sup>These notes consist of a reasoned collection of papers up to 2009

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# Chapter 1

## Introduction

These notes aim at reviewing some results on stability analysis and stabilizing control synthesis for continuous time switched linear systems. The notes are articulated into 5 chapters. In the first four chapters we consider autonomous switched systems described by

$$\dot{x}(t) = A_{\sigma(t)}x(t), \quad x(0) = x_0 \quad (1.1)$$

defined for all  $t \geq 0$  where  $x(t) \in \mathbb{R}^n$  is the state,  $\sigma(\cdot) : \mathbb{R} \rightarrow \{1, 2, \dots, N\}$  is the switching rule,  $x_0$  is the initial condition and

$$A_{\sigma(t)} \in \{A_1, \dots, A_N\} \quad (1.2)$$

It is clear that this model naturally imposes a discontinuity on  $A_{\sigma(t)}$  since this matrix must jump instantaneously from  $A_i$  to  $A_j$  for some  $i \neq j = 1, \dots, N$  once switching occurs. In other words,  $A_{\sigma(t)}$  is constrained to jump among the  $N$  vertices of the matrix polytope  $\{A_1, \dots, A_N\}$ .

In Chapter 2 we first consider the problem stability of (1.1), (1.2) under an arbitrary switching signal  $\sigma(\cdot)$ . Then, we move to the problem of determining time-dependent strategies  $\sigma(t)$  that ensure the stability of the resulting time-varying linear system. This problem calls for the concept of *dwell time* and *average dwell time*. In Chapter 3, we pass to the problem of determining stabilizing switching rules  $\sigma(t) = \xi_s(x(t))$  that depend on the measure of the system's state. Then, in Chapter 4 the performance index

$$J = \int_0^{\infty} x(t)Q_{\sigma(t)}x(t)dt \quad (1.3)$$

is introduced and we revise some possible solutions to the optimal control problem for switched systems, i.e. the determination of a state-feedback switching rule  $\sigma(t) = \xi_s(x(t))$  that minimizes the performance  $J$  in (1.3). In this same chapter a thorough analysis of the optimal switching rule for second-order oscillating systems is also developed. In Chapter 5 some recent results on the stabilization of switched systems with incomplete measurements are collected. In this framework, we assume that the system's state is not available for measurements and the designer only has to rely on the output equation

$$y(t) = C_{\sigma(t)}x(t) \quad (1.4)$$

The stabilization problem consists in the determination of a switching rule  $\sigma(t) = \xi_o(y_{\tau \leq t}(\tau))$ , depending on the past values of the output variable (1.4), capable to stabilize the closed-loop system.

Stability of continuous time switched linear systems have been addressed by several authors, [4], [6], [11], [12], [27], [15], [16], [17], [20] and [22], among others. While the survey papers [6] and [16] give a complete and detailed description on the problems arising in this area, the recent paper [11], dealing with extensions of LaSalle's Invariance Principle provides an interesting discussion on a collection of results on uniform stability of switched systems.

Generally speaking, when  $\sigma(\cdot)$  is state independent, that is, when it is a *a priori* piecewise constant signal, the reported stability conditions are obtained using a family of symmetric and positive definite matrices  $\{P_1, \dots, P_N\}$  each one associated to the correspondent matrix of the set  $\{A_1, \dots, A_N\}$  such that a Lyapunov function  $v(x(t))$  is non increasing with respect to  $\sigma(t)$  at every switching time. In Chapter 2, for minimum dwell time design preserving global stability it is assumed that each matrix of the set  $\{A_1, \dots, A_N\}$  is asymptotically stable but the non increasing condition on the Lyapunov function is relaxed. It is replaced by the weaker condition that at every switching time  $t_k$  the sequence  $v(x(t_k))$ , for  $k = 0, \dots, \infty$ , converges uniformly to zero. In some instances, our design procedure for the determination of the minimum dwell time, based on a quadratic guaranteed cost, is related to the results of [21] assuming further that the switching rule is not *a priori* given but can be taken arbitrarily, among the feasible ones, see [9]. For comparison purpose a simple second order example is solved and it is shown that the estimation of the minimum dwell time provided in this paper is sensibly better than the one obtained from the classical result of [17]. The results obtained in this context has some resemblance with those achieved in [24], where the characterization of the exponential growth rate of switched system is provided. However, much work is needed to establish the possible links between these two papers. The average dwell time results are those provided in [10], for Hurwitz matrices and [41] when there are both stable and unstable matrices. Notice that the dwell time calculation provided in the first part of Chapter 2 also suggests a way to solve the state-feedback stabilization problem for a input driven switched system characterized by the pairs  $(A_i, B_i)$ . Indeed, under mild assumptions it is possible to design matrices  $K_i$  such that to stabilize the closed-loop systems  $A_i + B_i K_i$ . Hence one can compute the upper bound of the dwell time to establish the maximum time duration of the control law. The general problem of minimization of the dwell time as a function of the design local control laws  $K_i$  is still open.

In Chapter 3, for switched systems with  $\sigma(\cdot)$  being state dependent, the stability condition is expressed in terms of a set of inequalities that we call *Lyapunov-Metzler inequalities* because the variables involved are a set of symmetric and positive definite matrices  $\{P_1, \dots, P_N\}$  and a Metzler matrix  $\Pi$ . The point to be noticed is that our asymptotical stability condition does not require any stability property associated to each individual matrix of the set  $\{A_1, \dots, A_N\}$  and it contains as special cases the quadratic stability condition and the well known *average* stability condition provided in [15], [10] and the references therein. An important point of our main result is that it includes the stability of possible sliding modes, a fact that in the particular case  $N = 2$  was observed in [15]. It is also important to stress that in [20] we can find some stability results related to the same problem (without the analysis of sliding modes) but restricted to the special case  $N = 2$  which does not require the formalism based on the Lyapunov-Metzler inequalities introduced here. In our general case, the price to be paid, however, is the non-convex nature of the the Lyapunov-Metzler inequalities being thus difficult to solve numerically. From this previous result, a more conservative but easier to solve asymptotical stability condition is proposed. It is important to express that these stability conditions do not suffer of a common drawback appearing, for example, in [13] where sliding modes are excluded and whose eventual occurrence has to be a posteriori verified. Adopting the more stringent condition that  $A_\sigma$  belongs to the convex combination of matrices  $A_1, \dots, A_N$  the control design falls precisely into the well known class of LPV control systems already

analyzed and solved for state and output feedback, [25], [30].

In Chapter 4 the theory of optimal control of switched systems is recalled pursuing the approach that hinges on the Hamilton-Jacobi equation. In particular the finite horizon problem is dealt with and an algorithm is provided based on gridding of the unitary sphere. Moreover, the particular class of second order oscillating systems is considered and the infinite horizon optimal control problem is addressed. To this regard, an algorithm providing the optimal conic switching surfaces is discussed.

The stability conditions expressed in terms of the Lyapunov-Metzler inequalities is developed further in Chapter 5 to cope with the determination of lower and upper bounds on the optimal switching control and output feedback switching control design. It is important to stress that a simple generalization of the Lyapunov-Metzler inequalities provides a solution to the Hamilton-Jacobi-Bellman inequality, an useful property for optimal cost lower bound calculation, see [28]. These problems are addressed in a general framework where the quadratic cost is defined from a set of external impulse-type perturbations. Throughout some simple numerical examples of third order are included for illustration purposes. A more realist practical application of a switched linear system of fourth order is included. The problem consists on the design of a switching control strategy for semi-active suspensions in road vehicles, and is motivated by the paper [29], where an optimal control algorithm has been devised. Finally, a complete analysis of second order oscillating switched system is carried out and a algorithm to find the optimal control law is provided, see [42].

Very little attention has been devoted to the design of stabilizing output feedback control laws. The reader is requested to see [6], [16] and [15] for a rather complete review on stability of continuous time switched linear systems, where special attention is given to the case of switching between two linear systems. The same reference also provides a discussion on hybrid feedback control based on output measurements which can not be directly generalized to cope with the problem addressed in Chapter 5.

The notation used throughout is standard. Capital letters denote matrices and small letters denote vectors. For scalars, small Greek letters are used. For real matrices or vectors ( $'$ ) indicates transpose. For square matrices  $\text{Tr}(X)$  denotes the trace function of  $X$  being equal to the sum of its eigenvalues and, for the sake of easing the notation of partitioned symmetric matrices, the symbol  $(\bullet)$  denotes generically each of its symmetric blocks. The set  $\mathcal{M}$  denotes the set of all Metzler matrices, composed by square matrices  $\Pi \in \mathbb{R}^{N \times N}$  of fixed dimensions with nonnegative off diagonal elements. The subset denoted as  $\mathcal{M}_c$  is composed by Metzler matrices satisfying the normalization constraints  $\sum_{i=1}^N \pi_{ij} = 0$  for all  $j = 1, \dots, N$ . Hence, each matrix in  $\mathcal{M}_c$  has a null (unitary) Perron-Frobenius eigenvalue associated to a nonnegative eigenvector  $\nu \geq 0 \in \mathbb{R}^N$ . The unitary simplex defined for all vectors  $\lambda \in \mathbb{R}^N$  such that  $\lambda_i \geq 0$ , for all  $i = 1, \dots, N$  and  $\sum_{i=1}^N \lambda_i = 1$  is denoted by  $\Lambda$ . Given matrices  $U_1, \dots, U_N$  of compatible dimensions and  $\lambda \in \Lambda$ , the matrix  $U_\lambda := \sum_{i=1}^N \lambda_i U_i$  denotes a matrix obtained by a convex combination. The  $n \times n$  identity matrix is denoted as  $I_n$ . Finally,  $\delta(t)$  denotes the unitary impulse and the square norm of a trajectory  $s(t)$  defined for all  $t \geq 0$ , denoted  $\|s\|_2^2$  equals  $\|s\|_2^2 := \int_0^\infty s(t)'s(t)dt$ , see [5].



## Chapter 2

# Time Switching Control

This chapter considers switched linear system defined by the model (1.1) and (1.2). First, it discusses the ideas underlying the verification of stability under arbitrary switching laws. Then, the attention will be focused on the design of time switching control laws.

### 2.1 Stability under arbitrary switching

Let us consider the switched system

$$\dot{x}(t) = A_{\sigma(t)}x(t), \quad x(0) = x_0 \quad (2.1)$$

We want to address the following problem: under which conditions the system is asymptotically stable for *any* admissible<sup>1</sup>  $\sigma(\cdot)$ ?

Notice first that the signal  $\sigma(t) = i, \forall t$ , is admissible. This means that a necessary condition for stability under arbitrary switching<sup>2</sup> is that all matrices  $A_i, i = 1, 2, \dots, N$  are Hurwitz. Unfortunately, this condition is not sufficient. A simple counterexample is provided by the two triangular matrices

$$A_1 = \begin{bmatrix} -1 & -5 \\ 0 & -1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -1 & 0 \\ 3 & -1 \end{bmatrix}$$

Indeed, consider the  $2T$  periodic signal characterized by

$$\sigma(t) = \begin{cases} 2 & t \in [0, T) \\ 1 & t \in [T, 2T) \end{cases}$$

and the transition matrix  $\Phi(t, \tau)$  of the periodic system

$$\dot{x}(t) = A_{\sigma(t)}x(t)$$

It turns out that the monodromy matrix (transition matrix over one period) is

$$\Phi(2T, 0) = e^{A_2 T} e^{A_1 T}$$

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<sup>1</sup>Here admissible means that in finite time only a finite number of switching can occur. For every piecewise constant switching signal the system is linear and time-varying. thus, asymptotic stability and exponential stability do coincide.

<sup>2</sup>We say that the system is GUAS (Global Uniform Asymptotically Stable) is for each admissible switching signal the associated time-varying linear system is asymptotically stable.

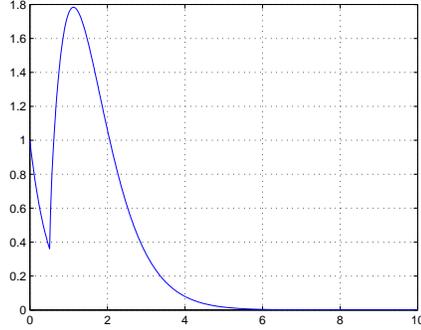


Figure 2.1: The maximum absolute characteristic multiplier as a function of  $T$ .

The periodic system is asymptotically stable if and only if the monodromy matrix has all eigenvalues (characteristic multipliers) inside the open unit disk. In Figure 2.1 is plotted the maximum absolute value of the two characteristic multipliers as a function of  $T$ . It turns out that for  $T = 1$  (for example) the system is unstable, so that the above switching strategy is destabilizing.

On the other hand, a simple sufficient condition for GUAS can be formulated by means of the Lyapunov inequalities

$$A_i'P + PA_i < 0, \quad i = 1, 2, \dots, M \quad (2.2)$$

It is indeed clear that the function

$$V(x) = x'Px(t) \quad (2.3)$$

is a Lyapunov function for any admissible signal  $\sigma(t)$ , since

$$\dot{V}(x(t)) = x'(t)(A_{\sigma(t)}'P + PA_{\sigma(t)})x(t) < 0$$

along the trajectories of the system. The function (2.3) is a Common Lyapunov Function (CLF) for the switched system, in that

$$V(x) > 0, \quad \dot{V}(x) = \frac{\partial V(x)}{\partial x} \dot{x} < 0, \quad x \neq 0$$

for any switching signal  $\sigma(t)$ . Moreover, it is quadratic in the state, being  $V(x) = x'Px(t)$ , and henceforth is referred to as Common Quadratic Lyapunov Function (CQLF).

Unfortunately, there are systems which are asymptotically stable under arbitrary switching and do not admit any CQLF. However, it can be shown that a linear switched system is GUAS if and only if it is possible to find a CLF. A technique to find the CLF refers to the so-called homogeneous Lyapunov functions, see [58], [59]. For instance consider

$$A_1 = \begin{bmatrix} -1 & -1 \\ 1 & -1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -1 & -10 \\ 0.1 & -1 \end{bmatrix}$$

To see that this system does not admit any CQLF, consider, without any loss of generality, the matrix

$$P = \begin{bmatrix} 1 & r \\ r & q \end{bmatrix}$$

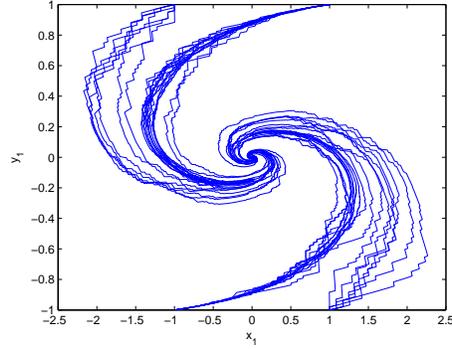


Figure 2.2: system for various switching signals, randomly generated.

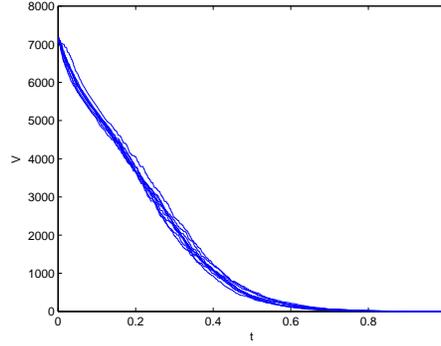


Figure 2.3: The CLF for various switching signals, randomly generated.

which is positive definite if and only if  $q > r^2$ . Then compute  $\Gamma_1 = A_1'P + PA_1$  and  $\Gamma_2 = A_2'P + PA_2$ . It turns out that  $\Gamma_1$  and  $\Gamma_2$  are negative definite if and only if

$$q^2 > 1 - \frac{(r-3)^2}{8}, \quad q^2 > 100 - \frac{(r-300)^2}{800}$$

As can be easily seen, no values of  $q$  satisfy the inequalities, and hence the system does not admit any CQLF. However, there exist the CLF of degree 8

$$V(x) = \xi' P \xi$$

where

$$\xi = \begin{bmatrix} x_1^4 \\ x_1^3 x_2 \\ x_1^2 x_2^2 \\ x_1 x_2^3 \\ x_2^4 \end{bmatrix}, \quad P = \begin{bmatrix} 1 & 3.649 & -14.323 & -5.49 & 6.807 \\ \star & 69.34 & 9.023 & -282.004 & 182.001 \\ \star & \star & 1181.813 & -375.17 & -693.818 \\ \star & \star & \star & 5911.771 & -4520.587 \\ \star & \star & \star & \star & 11393.280 \end{bmatrix}$$

In Figure 2.2 it is plotted the phase portrait of the system's state for some randomly generated switching signals. On the other hand, Figure 2.3 shows the CLF  $V(x)$  for various switching signals, starting from  $x(0) = [1 \ 1]$ .

To end this section, notice that it is always possible to associate with a GUAS system a CLF that is homogeneous of degree 2 and in particular one CLF that takes the form

$$V(x) = \max_{i=1,2,\dots,k} \{(l'_i x)^2\}$$

where  $l'_i$ ,  $i = 1, 2, \dots, k$  are suitable row vectors and  $k$  is a large enough positive integer. Analogously, the following result holds

**Theorem 1** *The system is exponentially stable under arbitrary switching if and only if there exist matrices  $W \in \mathbb{R}^{N \times n}$ ,  $Q_i \in \mathbb{R}^{N \times N}$ ,  $N \geq n$ , such that*

$$WA_i = Q_i W_i, \quad \mu_\infty(Q_i) < 0, \quad \forall i \quad (2.4)$$

□

where  $\mu_\infty(Q_i) < 0 = \max_j [Q_i]_{jj} + \sum_{k \neq j} |[Q_i]_{jk}|$ , see the recent research monograph [2].

## 2.2 RMS under arbitrary switching

The techniques used to determine if a switched system is stable under arbitrary switching can be extended to cope with performance requirements. Herein we briefly consider the *root mean square* property of a switched system. To be precise, let us consider the switched system

$$\dot{x}(t) = A_{\sigma(t)}x(t) + B_{\sigma(t)}u(t) \quad (2.5a)$$

$$y(t) = C_{\sigma(t)}x(t) + D_{\sigma(t)}u(t) \quad (2.5b)$$

where  $A_i$ ,  $i = 1, 2, \dots, N$ , are Hurwitz matrices. It is clear that, under the assumption that the system is asymptotically stable for any switching signal, it makes sense to consider the problem of finding the minimum  $\gamma > 0$  for which

$$\sup_{w \in L_2(0, \infty)} \frac{\|y\|_2}{\|w\|_2} < \gamma \quad (2.6)$$

Notice that such

$$\gamma \geq \max_i \{\gamma_i\}$$

where  $\gamma_i$  is the  $H_\infty$  norm associated with the stationary system  $(A_i, B_i, C_i, D_i)$ .

**Theorem 2** *Assume that there exists a positive definite matrix  $P$  such that*

$$\begin{bmatrix} A'_i P + P A_i & P B_i & C'_i \\ B'_i P & -\gamma^2 I & D'_i \\ C_i & D_i & -I \end{bmatrix} < 0, \quad \forall i \in \mathbb{N} \quad (2.7)$$

*then, for each switching signal  $\sigma$ , the equilibrium solution  $x = 0$  of the switched linear system (2.5) is globally asymptotically stable and*

$$\sup_{w \in L_2, w \neq 0} \int_0^\infty (y' y - \gamma^2 w' w) dt < 0 \quad (2.8)$$

**Proof** First of all notice that (2.7) is equivalent to  $\gamma^2 I - D'_i D_i > 0$  and

$$A'_i P + P A_i + (P B_i + C'_i D_i)(\gamma^2 I - D'_i D_i)^{-1}(P B_i + C'_i D_i)' + C'_i C_i < 0, \quad \forall i \quad (2.9)$$

In particular

$$A'_i P + P A_i < 0$$

so that global asymptotic stability under arbitrary switching is ensured. Also, the state of the system goes to zero for each  $\sigma$  and each input square integrable disturbance  $w$ . This means that, taking  $V(x) = x' P x$ , we have  $V(x(\infty)) = 0$ . Now, compute the derivative of  $V(x)$  along the trajectories of (2.5). Letting

$$w^* = (\gamma^2 I - D'_i D_i)^{-1}(P_i B_i + C'_i D_i)' x$$

from (2.9) it turns out that

$$\begin{aligned} \dot{V}(x) &= x'(A'_\sigma P + P A_\sigma)x + 2x' P B_\sigma w \\ &< -y'y + \gamma^{-2} w' w - (w - w^*)'(\gamma^2 I - D'_i D_i)(w - w^*) \\ &< -y'y + \gamma^{-2} w' w \end{aligned}$$

Integrating from 0 to  $\infty$  and recalling that  $V(x(0)) = V(x(\infty)) = 0$  it follows that

$$\int_0^\infty (y'y - \gamma^2 w' w) dt < 0, \quad \forall \sigma, \quad \forall w \neq 0, \quad w \in L_2$$

■

Consider now inequality (2.7). Taking  $\alpha_i$ ,  $i = 1, 2, \dots, N$  in a simplex, i.e.  $\alpha_i \geq 0$  and  $\sum_i \alpha_i = 1$ , one can multiply (2.7) by  $\alpha_i$ , sum up and use the Schur complement Lemma to obtain

$$A'_\alpha P + P A_\alpha + (P B_\alpha + C'_\alpha D_\alpha)(\gamma^2 I - D'_\alpha D_\alpha)^{-1}(P B_\alpha + C'_\alpha D_\alpha)' + C'_\alpha C_\alpha < 0$$

where

$$\begin{aligned} A_\alpha &= \sum_{i=1}^N \alpha_i A_i, & B_\alpha &= \sum_{i=1}^N \alpha_i B_i \\ C_\alpha &= \sum_{i=1}^N \alpha_i C_i, & D_\alpha &= \sum_{i=1}^N \alpha_i D_i \end{aligned}$$

This means that the polytopic system defined by  $A_\alpha$ ,  $B_\alpha$ ,  $C_\alpha$ ,  $D_\alpha$  has  $H_\infty$  norm less than  $\gamma$  for each choice of  $\alpha$  in the simplex. In conclusion,  $H_\infty$  performances of switched systems under arbitrary switching laws are related to those of polytopic systems. This fact extends a well know result for stability under arbitrary switching, for which quadratic stability is only a conservative sufficient condition. For a thorough discussion on nonconservative solution via polyhedral Lyapunov function, the interested reader is referred to the recent volume [2].

## 2.3 Dwell-time

In this section we assume that each matrix of the set  $\{A_1, \dots, A_N\}$  is asymptotically stable. The problem under consideration can be stated as follows : Determine a minimum dwell time

$T_* > 0$  such that the equilibrium point  $x = 0$  of the system (1.1) is globally asymptotically stable with the time switching control

$$\sigma(t) = i \in \{1, \dots, N\}, \quad t \in [t_k, t_{k+1}) \quad (2.10)$$

where  $t_k$  and  $t_{k+1}$  are successive switching times satisfying  $t_{k+1} - t_k \geq T_*$  for all  $k \in \mathbb{N}$  and the index  $i \in \{1, \dots, N\}$  selected at each instant of time  $t \geq 0$  is arbitrary. Hence, asymptotical stability is preserved whenever  $\sigma(t)$  remains unchanged for a period of time greater or equal to the minimum dwell time  $T_*$ . The complete answer to this problem has been recently worked out by means of polyhedral Lyapunov functions.

**Theorem 3** *The system is exponentially stable for any switching signal in  $D_T$  and only if there exist matrices  $W_i \in \mathbb{R}^{N \times n}$ ,  $Q_i \in \mathbb{R}^{N \times N}$ ,  $N \geq n$ , and  $Z_{ij}$  such that*

$$W_i A_i = Q_i W_i, \quad \mu_\infty(Q_i) < 0, \quad \forall i \quad (2.11)$$

$$W_j e^{A_i T} = Z_{ij} W_i, \quad \|Z_{ij}\|_\infty < 1, \quad \forall i \neq j \quad (2.12)$$

□

It goes without saying that the minimum dwell time is the minimum parameter  $T$  for which the conditions (2.11) and (2.12) are feasible. However, due to computational difficulties, it is worth determining conditions in terms of piecewise quadratic Lyapunov functions. The next theorem provides the theoretical basis towards a possible solution of this problem by characterizing an upper bound for  $T_*$ . It uses the concept of multiple Lyapunov function with the innovation that the classical non increasing assumption at switching times is no longer needed, see [4].

**Theorem 4** *Assume that, for some  $T > 0$ , there exists a collection of positive definite matrices  $\{P_1, \dots, P_N\}$  of compatible dimensions such that*

$$A_i' P_i + P_i A_i < 0, \quad \forall i = 1, \dots, N \quad (2.13a)$$

$$e^{A_i T} P_j e^{A_i T} - P_i < 0, \quad \forall i \neq j = 1, \dots, N \quad (2.13b)$$

*The time switching control (2.10) with  $t_{k+1} - t_k \geq T$  makes the equilibrium solution  $x = 0$  of (1.1) globally asymptotically stable.*

**Proof** Consider, in accordance to (2.10), that  $\sigma(t) = i \in \{1, \dots, N\}$  for all  $t \in [t_k, t_{k+1})$  where  $t_{k+1} = t_k + T_k$  with  $T_k \geq T > 0$  and that at  $t = t_{k+1}$  the time switching control jumps to  $\sigma(t) = j \in \{1, \dots, N\}$ , otherwise the result trivially follows. From (2.13a), it is seen that, for all  $t \in [t_k, t_{k+1})$ , the time derivative of the Lyapunov function  $v(x(t)) = x(t)' P_{\sigma(t)} x(t)$ , along an arbitrary trajectory of (1.1) satisfies

$$\begin{aligned} \dot{v}(x(t)) &= x(t)' (A_i' P_i + P_i A_i) x(t) \\ &< 0 \end{aligned} \quad (2.14)$$

which enables us to conclude that there exist scalars  $\alpha > 0$  and  $\beta > 0$  such that

$$\|x(t)\|^2 \leq \beta e^{-\alpha(t-t_k)} v(x(t_k)), \quad \forall t \in [t_k, t_{k+1}) \quad (2.15)$$

On the other hand, using the inequalities (2.13b) we have

$$\begin{aligned} v(x(t_{k+1})) &= x(t_{k+1})' P_j x(t_{k+1}) \\ &= x(t_k)' e^{A_i T_k} P_j e^{A_i T_k} x(t_k) \\ &< x(t_k)' e^{A_i (T_k - T)} P_i e^{A_i (T_k - T)} x(t_k) \\ &< x(t_k)' P_i x(t_k) \\ &< v(x(t_k)) \end{aligned} \quad (2.16)$$

where the second inequality holds from the fact that for every  $\tau = T_k - T \geq 0$  it is true that  $e^{A_i \tau} P_i e^{A_i \tau} \leq P_i$ . The consequence is that there exists  $\mu \in (0, 1)$  such that

$$v(x(t_k)) \leq \mu^k v(x_0), \quad \forall k \in \mathbb{N} \quad (2.17)$$

which together with (2.15) implies that the equilibrium solution  $x = 0$  of (1.1) is globally asymptotically stable.  $\blacksquare$

This result deserves some comments. First, it is simple to determine the scalars  $\alpha$ ,  $\beta$  and  $\mu$  such that (2.15) and (2.17) hold. Indeed, assuming that  $\{P_1, \dots, P_N\}$  satisfy the conditions of Theorem 4 then, from (2.13a) there exists  $\epsilon > 0$  such that  $A_i' P_i + P_i A_i \leq \epsilon I$  for all  $i = 1, \dots, N$  yielding  $\alpha = \epsilon / \max_i \lambda_{max}(P_i) > 0$  and  $\beta = 1 / \min_i \lambda_{min}(P_i) > 0$ . Furthermore, from (2.13b) there exists  $0 < \mu < 1$  such that  $e^{A_i T} P_j e^{A_i T} \leq \mu P_i$  for all  $i \neq j = 1, \dots, N$  leading to  $v(x(t_{k+1})) \leq \mu v(x(t_k))$  and consequently (2.17). Second, since all matrices of the set  $\{A_1, \dots, A_N\}$  are supposed to be asymptotically stable, the constraints (2.13a) are always feasible and the constraints (2.13b) are satisfied when  $T > 0$  is taken large enough. Third, assuming that matrices  $A_1, \dots, A_N$  are quadratically stable, which is the same to say that they share a positive definite matrix  $P$  such that

$$A_i' P + P A_i < 0, \quad \forall i = 1, \dots, N \quad (2.18)$$

then the inequality (2.13b) is satisfied for  $P_1 = \dots = P_N = P$  for any  $T > 0$  meaning that the switching policy (2.10) may jump from  $i$  to  $j$  arbitrarily fast preserving asymptotical stability. Hence, Theorem 4 contains, as a particular case, the quadratic stability condition. Finally, with  $T > 0$  fixed it is always possible to define a time switching control strategy (2.10) such that  $A_{\sigma(t)}$  is periodic. As a consequence, a necessary condition for the feasibility of constraints (2.13a) and (2.13b) is

$$\theta(T) := \max_{q=1, \dots, n} \left| \lambda_q \left( \prod_{p=1}^N e^{B_p T} \right) \right| < 1 \quad (2.19)$$

where  $\lambda_q(\cdot)$  denotes a generic eigenvalue of  $(\cdot)$  and  $\{B_1, \dots, B_N\}$  are matrices corresponding to any permutation among those of the set  $\{A_1, \dots, A_N\}$ . However, since (2.10) may produce non-periodic policies as well, the necessary condition (2.19) for the existence of a feasible solution to inequalities (2.13), generally does not meet sufficiency. In the sequel, this aspect will be illustrated by means of an example.

In this setting, an upper bound for the minimum dwell time  $T_*$  can be computed by taking the minimum value of  $T$  satisfying the conditions of Theorem 4. Hence, it can be calculated with no big difficulty from the optimal solution of the optimization problem<sup>3</sup>

$$\min_{T > 0, P_1 > 0, \dots, P_N > 0} \{T : (2.13)\} \quad (2.20)$$

which, for each  $T > 0$  fixed, reduces to a convex programming problem with linear matrix inequalities constraints that can be handled by any LMI solver available in the literature to date, see [3] for an important study on systems and LMIs. A line search procedure is then used to deal with the scalar variable  $T > 0$ .

Finally, it is possible to generalize the result of Theorem 4 in order to define a guaranteed cost to go from an arbitrary initial point to the origin, associated to the stabilizing time switching rule (2.10) with  $t_{k+1} - t_k \geq T$  for any fixed  $T > 0$ . To this end we make the assumption that  $\mathcal{T} > 0$  is known such that  $t_{k+1} - t_k \leq \mathcal{T}$  for all  $k \in \mathbb{N}$ . Clearly, these quantities are related through  $\mathcal{T} \geq T \geq T_*$  where the second inequality assures global stability.

<sup>3</sup>This problem should be stated with inf instead of min. All feasible sets of problems expressed in terms of LMIs must be considered closed from the interior within a precision defined by the user.

**Theorem 5** Let  $Q \geq 0 \in \mathbb{R}^{n \times n}$  and  $\mathcal{T} \geq T > 0$  be given. Define the set of symmetric, non-negative definite matrices

$$R_i := \int_0^{\mathcal{T}} e^{A_i^t} Q e^{A_i^t} dt, \quad i = 1, \dots, N \quad (2.21)$$

Assume that there exists a collection of positive definite matrices  $\{P_1, \dots, P_N\}$  of compatible dimensions such that

$$A_i' P_i + P_i A_i + Q < 0, \quad \forall i = 1, \dots, N \quad (2.22a)$$

$$e^{A_i^T} P_j e^{A_i T} - P_i + R_i < 0, \quad \forall i \neq j = 1, \dots, N \quad (2.22b)$$

The time switching control (2.10) with  $\mathcal{T} \geq t_{k+1} - t_k \geq T$  makes the equilibrium solution  $x = 0$  of (1.1) globally asymptotically stable and

$$\int_0^{\infty} x(t)' Q x(t) dt < x_0' P_{\sigma(0)} x_0 \quad (2.23)$$

**Proof** Since for  $Q \geq 0$  and  $\mathcal{T} \geq T > 0$  given, each matrix  $R_i$  defined in (2.21) is nonnegative definite and inequalities (2.22) are satisfied then, inequalities (2.13) are also satisfied. As a consequence, asymptotical stability follows from Theorem 4. On the other hand, using (2.21) together with the inequalities (2.22) we have that  $P_i > R_i$  and

$$\begin{aligned} A_i'(P_i - R_i) + (P_i - R_i)A_i &< -Q - A_i'R_i - R_iA_i \\ &< -Q - \int_0^{\mathcal{T}} \frac{d}{dt} e^{A_i^t} Q e^{A_i^t} dt \\ &< -e^{A_i^T} Q e^{A_i T} \\ &< 0 \end{aligned} \quad (2.24)$$

for all  $i = 1, \dots, N$ . The important consequence of this calculation is that for each  $i = 1, \dots, N$  the inequality  $e^{A_i^T} (P_i - R_i) e^{A_i \tau} \leq (P_i - R_i)$  holds for any  $\tau \geq 0$ . Using this property, taking into account the switching strategy (2.10) with  $t_{k+1} - t_k = T_k \geq T$  and the inequalities (2.22b) one gets

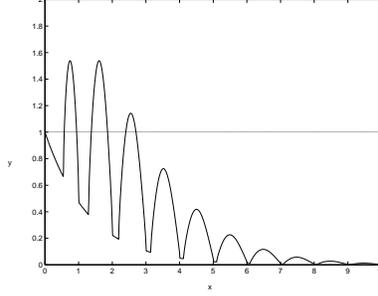
$$\begin{aligned} v(x(t_{k+1})) &= x(t_{k+1})' P_j x(t_{k+1}) \\ &< x(t_k)' e^{A_i(T_k - T)} (P_i - R_i) e^{A_i(T_k - T)} x(t_k) \\ &< x(t_k)' (P_i - R_i) x(t_k) \\ &< v(x(t_k)) - x(t_k)' R_{\sigma(t_k)} x(t_k) \end{aligned} \quad (2.25)$$

which summing up for all  $k \in \mathbb{N}$  and taking into account that  $\mathcal{T} \geq t_{k+1} - t_k$  allows us to write

$$\begin{aligned} \int_0^{\infty} x(t)' Q x(t) dt &= \sum_{k=0}^{\infty} \int_{t_k}^{t_{k+1}} x(t_k)' e^{A_i(t-t_k)} Q e^{A_i(t-t_k)} x(t_k) dt \\ &\leq \sum_{k=0}^{\infty} x(t_k)' R_{\sigma(t_k)} x(t_k) \\ &< v(x_0) \end{aligned} \quad (2.26)$$

which proves the proposed theorem.

It is interesting to observe that the conditions of Theorem 5 are feasible if and only if  $\mathcal{T} \geq T \geq T_*$  and from (2.23) it is seen that a more accurate guaranteed cost is obtained

Figure 2.4: The function  $\theta(T)$ .

whenever the value of  $\mathcal{T}$  is chosen as small as possible. In addition, the choice  $\mathcal{T} = +\infty$  enables us to conclude that the proposed time switching rule (2.10) with  $t_{k+1} - t_k \geq T_*$ , makes the trajectory  $y(t) = Q^{1/2}x(t)$ ,  $t \geq 0$  quadratically integrable. Theorem 5, admits the extreme situation  $\mathcal{T} = T = +\infty$  for which no jump occurs and inequalities (2.22) are verified for

$$P_i = \int_0^\infty e^{A_i t} (Q + \varepsilon I) e^{A_i t} dt > R_i \geq 0, \quad i = 1, \dots, N \quad (2.27)$$

with  $\varepsilon > 0$  arbitrary. When  $\varepsilon > 0$  goes to zero,  $P_i$  goes to  $R_i$  and (2.23) becomes a well known result. On the other hand, for  $T > 0$  arbitrarily small and any  $\mathcal{T} \geq T$ , feasibility holds whenever the set of matrices  $\{A_1, \dots, A_N\}$  admits a common Lyapunov function.

**Example 1** For illustration purpose of the theoretical results obtained so far, let us consider the following example with  $N = 2$  and matrices

$$A_1 = \begin{bmatrix} 0 & 1 \\ -10 & -1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 1 \\ -0.1 & -0.5 \end{bmatrix} \quad (2.28)$$

which are not quadratically stable. First, from problem (2.20), we have calculated an upper bound for the minimum dwell time as being  $T_* \leq 2.76$ . To give an idea of its conservativeness we have calculated from the plot of Figure 2.4 the value  $T_{\text{per}} = 2.71$  corresponding to the necessary condition for stability (2.19), arising from linear periodic systems. Both being very close indicates, for this simple example, a good precision on the determination of  $T_*$ . On the other hand, for comparison purpose we have applied the classical result of [17] for the determination of an alternative upper bound for the minimum dwell time  $T_*$  given by  $T_* \leq \max_{i=1, \dots, N} \{T_i\}$  where

$$T_i = \inf_{\alpha > 0, \beta > 0} \left\{ \frac{\alpha}{\beta} : \|e^{A_i t}\| < e^{(\alpha - \beta t)} \quad \forall t \geq 0 \right\} \quad (2.29)$$

For matrices in (2.28) we have numerically determined  $T_1 = 2.33$  and  $T_2 = 6.66$  yielding an estimation for the minimum dwell time as being  $T_* \leq 6.66$ . Hence, in this particular example, the result provided by the solution of problem (2.20) is much more precise but at expense of a more expressive computational effort. Figure 2.5 has been constructed by simulation of system (1.1) with the time switching rule (2.10),  $t_{k+1} - t_k = 3.0$ , initial conditions  $x_0 = [1 \ 1]'$ ,  $\sigma(0) = 2$  and  $Q = I$ . The family of Lyapunov functions has been calculated from the optimal solution of the following convex programming problem

$$\min_{P_1 > 0, \dots, P_N > 0} \max_{i=1, \dots, N} \{x_0' P_i x_0 : (2.22)\} \quad (2.30)$$

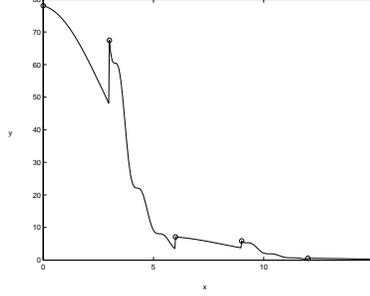


Figure 2.5: The Lyapunov function.

which puts in evidence that a guaranteed cost can be determined for the worst case as far as the initial condition  $\sigma(0)$  appearing in (2.23) is concerned. For  $\mathcal{T} = T = 3.0$ , we have obtained the minimum guaranteed cost equal to  $\delta^* = 100.61$ , valid for both initial conditions. As commented before, the Lyapunov function  $v(x(t)) = x(t)'P_{\sigma(t)}x(t)$  goes to zero as  $t$  goes to infinity however, it is not uniformly decreasing with respect to time. In Figure 5.2, due to the stability conditions of Theorem 5, the discontinuity points, marked with "o", defines a globally convergent sequence  $v(x(t_k))$ , for all  $k \in \mathbb{N}$ . Solving again problem (2.30) but for  $\mathcal{T} = +\infty$  and  $T = 3.0$  the minimum guaranteed cost increases to  $\delta^* = 147.94$  as a consequence of allowing a more flexible switching rule (2.10) with  $t_{k+1} - t_k \geq 3.0$ .

The example above shows that there is a clear improvement on stability conditions, dwell time and guaranteed cost calculations when compared to the results available in the literature to date, see [11], [17]. Notice however, that the conditions in Theorem 4 are still conservative, in that they employ only piecewise quadratic Lyapunov functions. It is possible to diminish the conservativeness by using homogeneous polynomial Lyapunov equations via Kronecker calculus, see [?]. Interestingly, these conditions are strict for second order systems. For instance, the exact minimum dwell time associated with the example above is  $T^* = 2.7078$ .

## 2.4 Average dwell-time

In this section we consider system (1.1), (1.2) and assume first that it is constituted by Hurwitz matrices. For each switching sequence  $\sigma$  and each  $t > \tau \geq 0$  denote by  $N_{\sigma}(\tau, t)$  the number of switchings in the interval  $(\tau, t)$  and let  $\mathcal{S}[\tau_a, N_0]$  the set of switching laws obeying

$$N_{\sigma}(\tau, t) \leq N_0 + \frac{t - \tau}{\tau_a}$$

where  $N_0 \geq 0$  is the so-called chatter bound and  $\tau_a$  is the average dwell time. This means that there may exist some consecutive switchings separated by less than  $\tau_a$ , but the average time interval between consecutive switchings is no less than  $\tau_a$ . We show, see [10], that there exist a sufficiently large  $\tau_a^*$  such that the switching system is stable for any switching rule in  $\mathcal{S}[\tau_a, N_0]$ , with  $\tau_a \geq \tau_a^*$  and any chatter bound  $N_0$ . Indeed, since all matrices  $A_i$ ,  $i = 1, 2, \dots, N$  are Hurwitz, we can write

$$\|e^{A_i t}\| \leq e^{\alpha_i - \lambda_i t}, \quad \forall i$$

Hence, taking  $t \in [t_k, t_{k+1})$  where  $t_k$  is the  $k$ -th switch, we can write

$$\|\Phi(t, 0)\| \leq e^{\alpha(k+1)} e^{-\beta t}$$

where  $\alpha = \max_i \alpha_i$  and  $\beta = \min_i \beta_i$ . Hence, for all switching signals in  $\mathcal{S}[\tau_a, N_0]$  we have

$$\|\Phi(t, 0)\| \leq e^{\alpha(N_0+1)} e^{(\frac{\alpha}{\tau_D} - \beta)t}$$

Letting

$$\tau_a^* = \frac{\alpha}{\beta - \lambda}, \quad \lambda \in (0, \beta)$$

the thesis follows.

Now, we assume that the system is composed by both Hurwitz and non Hurwitz matrices. Following [41], and without loss of generality, we assume that  $A_1, A_2, \dots, A_r$  are non Hurwitz and  $A_{r+1}, A_{r+2}, \dots, A_N$  are Hurwitz. Then it is possible to write

$$\begin{aligned} \|e^{A_i t}\| &\leq e^{\alpha_i + \beta_i t}, & i = 1, 2, \dots, r \\ \|e^{A_i t}\| &\leq e^{\alpha_i - \beta_i t}, & i = r+1, 2, \dots, N \end{aligned}$$

with  $\alpha_i \geq 0$  and  $\beta_i > 0, \forall i$ . Now let

$$\beta^+ = \max_{i=1, \dots, r} \beta_i, \quad \beta^- = \min_{i=r+1, \dots, N} \beta_i$$

and  $T^+(t)$  [ $T^-(t)$ ] the total activation time of the unstable [stable] subsystems in the interval  $[0, t)$ . Finally let  $\mathcal{SW}_{\tau_a}$  the class of switching laws satisfying the following two conditions

$$\inf_{t \geq 0} \frac{T^-(t)}{T^+(t)} \geq \frac{\beta^+ + \beta^*}{\beta^- - \beta^*}, \quad \beta^* \in (\lambda, \beta^-), \quad \lambda \in (0, \beta^-) \quad (2.31)$$

$$\text{The average dwell time is not smaller than } \tau_a \quad (2.32)$$

It is possible to prove that there exists  $\tau_a^*$  sufficiently large such that the switched system is stable for any switching rule in  $\mathcal{SW}_{\tau_a}$  for any  $\tau_a \geq \tau_a^*$  and any chatter bound  $N_0$ . Indeed, for any  $t \in [t_k, t_{k+1})$  it follows

$$\|\Phi(t, 0)\| \leq e^{\alpha(k+1)} e^{\beta^+ T^+(t) - \beta^- T^-(t)}$$

where  $\alpha = \max_i \alpha_i$  and  $\beta^+ = \max_{i=1, \dots, r} \beta_i, \beta^- = \min_{i=r+1, \dots, N} \beta_i$ . Since  $T^+(t) + T^-(t) = t$  a simple computation shows that  $\beta^+ T^+(t) - \beta^- T^-(t) \leq \beta^* t$  so that

$$\|\Phi(t, 0)\| \leq e^{\alpha(N_0+1)} e^{(\frac{\alpha}{\tau_D} - \beta^*)t}$$

The result follows by taking

$$\tau_a^* = \frac{\alpha}{\beta^* - \lambda}$$

Notice that if all matrices are Hurwitz, condition (2.31) is satisfied so that the last formula corresponds to the average dwell time in this case.

## 2.5 RMS with dwell time constraint

Consider again system (2.5) and assume that  $A_i, i = 1, 2, \dots, N$  are Hurwitz matrices. The RMS problem with dwell constraint consists in finding the minimum  $T^* \geq 0$  for which (2.6) holds for any switching signal with commutation instants satisfying  $t_{k+1} - t_k \geq T^*$ . To this end, denote by  $\mathcal{D}_T$  the set of all switching signals satisfying  $t_{k+1} - t_k \geq T, \forall k$ .

Notice first that, being  $\gamma \geq \gamma_i$  (the  $H_\infty$  norm of system  $(A_i, B_i, C_i, D_i)$ ), there exist positive semidefinite matrices  $P_i$  satisfying the Riccati equations

$$A_i'P_i + P_iA_i + (P_iB_i + C_i'D_i)(\gamma^2I - D_i'D_i)^{-1}(P_iB_i + C_i'D_i)' + C_i'C_i = 0 \quad (2.33)$$

with  $A_i + B_i(\gamma^2I - D_i'D_i)^{-1}(P_iB_i + C_i'D_i)'$  Hurwitz. To this end, we need to introduce the following matrices

$$H_i = A_i + B_iL_i \quad (2.34)$$

$$Q_i = (C_i + D_iL_i)'(C_i + D_iL_i) - \gamma^2L_i'L_i \quad (2.35)$$

$$L_i = (\gamma^2I - D_i'D_i)^{-1}(P_iB_i + C_i'D_i)' \quad (2.36)$$

$$S_i = \int_0^\infty e^{H_it}B_i(I - \gamma^{-2}D_i'D_i)^{-1}B_i'e^{H_it}dt \quad (2.37)$$

$$U_i(\tau) = \int_0^\tau e^{H_it}B_i(I - \gamma^{-2}D_i'D_i)^{-1}B_i'e^{H_it}dt \quad (2.38)$$

$$R_i(\tau) = \int_0^\tau e^{H_it}Q_ie^{H_it}dt \quad (2.39)$$

$$(2.40)$$

(2.33) can be factorized as

$$H_i'P_i + P_iH_i + Q_i = 0 \quad (2.41)$$

for all  $i \in \mathbb{N}$ . As indicated before, noticing that the optimal gain  $L_i$  is determined from the unique stabilizing solution to the algebraic Riccati equation (2.33), matrix  $H_i$  is Hurwitz for each  $i \in \mathbb{N}$ . However, since matrix  $Q_i$  for each  $i \in \mathbb{N}$  is not positive definite, the stabilizing solution of the Riccati equation is not a Lypunov matrix associated to the closed loop system, a well known fact in  $\mathcal{H}_\infty$  theory. The next lemma is of key importance since it gives an upper bound to the  $\mathcal{H}_\infty$  cost. It regards the differential Riccati equation

$$-\dot{\Pi} = A_\sigma'\Pi + \Pi A_\sigma + (\Pi B_\sigma + C_\sigma'D_\sigma)(\gamma^2I - D_\sigma'D_\sigma)^{-1}(\Pi B_\sigma + C_\sigma'D_\sigma)' + C_\sigma'C_\sigma = 0 \quad (2.42)$$

**Lemma 1** *Assume that  $\sigma(t) = i$  for  $t \in [t_k, t_{k+1})$  and  $\sigma(t_{k+1}) = j$ . Assume that a bunch of  $N$  positive definite matrices  $Z_i$  are given. Finally, assume that the solution  $\Pi(t)$  of (2.42) with final condition  $\Pi(t_{k+1}) = Z_j$  exists in the interval  $t \in [t_k, t_{k+1})$ . Then, for the switched linear system (2.5), the following upper bound holds*

$$\sup_w \int_{t_k}^{t_{k+1}} (y'y - \gamma^2w'w)dt \leq x(t_k)'\Pi(t_k)x(t_k) - x(t_{k+1})'Z_jx(t_{k+1}) \quad (2.43)$$

where

$$\Pi(t_k) = P_i + e^{H_i'(t_{k+1}-t_k)} \left( (Z_j - P_i)^{-1} - \gamma^{-2}U_i((t_{k+1} - t_k)) \right)^{-1} e^{H_i(t_{k+1}-t_k)}$$

**Proof** The proof follows by computing the differential equation for  $(\Pi(t) - P_i)^{-1}$ , the derivative of  $V(x) = x'\Pi(t)x(t)$  and using classical square completing arguments. ■

From Lemma 1 it is clear that, if  $\Pi(t_{k+1} - t_k) < Z_i$ , for any  $t_{k+1} - t_k \geq T$ , then

$$\sup_{w \in L_2} \int_0^\infty (y'y - \gamma^2w'w)dt \leq \sum_{k=0}^\infty x(t_k)'\Pi(t_k)x(t_k) - x(t_{k+1})'Z_jx(t_{k+1}) \leq x(0)'Z_{\sigma(0)}x(0) \quad (2.44)$$

so that the guaranteed bound is obtained as  $x(0) \rightarrow 0$ . The next theorem states a sufficient condition in terms of LMIs.

**Theorem 6** Assume that, for given  $T > 0$ , and for all  $i, j$ , there exists matrices  $Z_1, Z_2, \dots, Z_M$  such that

$$\begin{bmatrix} A'_i Z_i + Z_i A_i & Z_i B_i & C'_i \\ B'_i Z_i & -\gamma^2 I & D'_i \\ C_i & D_i & -I \end{bmatrix} < 0 \quad (2.45)$$

and

$$\begin{bmatrix} e^{H'_i T} Z_j e^{H_i T} - Z_i + W_i & e^{H'_i T} (Z_j - P_i) \\ * & Z_j - P_i - \gamma^2 S_i^{-1} \end{bmatrix} < 0 \quad (2.46)$$

The following hold:

- a) The equilibrium solution  $x = 0$  of the switched linear system (2.5) is globally asymptotically stable.
- b) Any trajectory of the switched linear system (2.5) with zero initial condition satisfies

$$\sup_w \int_0^\infty (y'y - \gamma^2 w'w) dt < 0, \quad \forall \sigma \in \mathcal{D}_T \quad (2.47)$$

**Proof** We have to ensure that  $\Pi(t_{k+1} - t_k) < Z_i$ , for any  $t_{k+1} - t_k \geq T$ , when  $\Pi(t_{k+1}) = Z_j$ . Letting  $\tau = t_{k+1} - t_k$ , this is tantamount to saying that  $\Pi(0) < Z_i$  when  $\Pi(\tau) = Z_j$ , i.e.

$$Z_i > P_i + e^{H'_i \tau} ((Z_j - P_i)^{-1} - \gamma^{-2} U_i(\tau))^{-1} e^{H_i \tau}, \quad \forall \tau \geq T$$

It is left to the reader to prove that this inequality is ensured by (2.46) when the matrices  $Z_i$  satisfy (2.45). ■

Notice that for  $\gamma \rightarrow \infty$ , the inequalities become

$$\begin{aligned} A'_i Z_i + Z_i A_i + C'_i C_i &< 0 \\ e^{A'_i T} Z_j e^{A_i T} - Z_i + R_i(T) &< 0, \quad P_i \rightarrow \int_0^\infty e^{A'_i t} C'_i C_i e^{A_i t} dt \end{aligned}$$

so that conditions the conditions of Theorem 5 for the  $H_2$  cost are recovered. Moreover, if feasibility occurs for as  $T \rightarrow 0$ , then  $Z_i = Z_j = Z$  so that

$$\begin{bmatrix} A'_i Z_i + Z_i A_i & Z_i B_i & C'_i \\ B'_i Z_i & -\gamma^2 I & D'_i \\ C_i & D_i & -I \end{bmatrix} < 0$$

which ensures that the attenuation  $\gamma$  is guaranteed for  $\sigma \in \mathcal{D}_0$ , see (2.7).

For illustration purpose of the theoretical results obtained so far, let us consider the following example with  $N = 2$  already analyzed in Section 2.3 for dwell time calculations. The matrices of the switching system (2.5) are given by

$$\left[ \begin{array}{c|c} A_1 & B_1 \\ \hline C_1 & D_1 \end{array} \right] = \left[ \begin{array}{cc|c} 0 & 1 & 0 \\ -10 & -1 & 1 \\ \hline 0.8727 & 0 & -0.8727 \end{array} \right] \quad (2.48)$$

$$\left[ \begin{array}{c|c} A_2 & B_2 \\ \hline C_2 & D_2 \end{array} \right] = \left[ \begin{array}{cc|c} 0 & 1 & 0 \\ -0.1 & -0.5 & 1 \\ \hline 0 & 0.3333 & 0.3333 \end{array} \right] \quad (2.49)$$

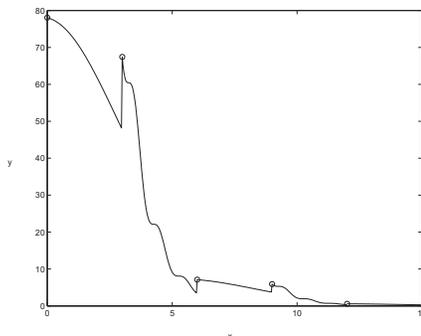


Figure 2.6: The functions  $T(\gamma)$  and  $T_p(\gamma)$ .

and it is important to mention that they are not open loop quadratically stable, in which case the value of  $\gamma$  for which (2.7) holds can not be calculated. The output matrices have been determined in such a way that each transfer function has an unitary  $\mathcal{H}_\infty$  norm, yielding  $\gamma_c = \max_i \{\gamma_i\} = 1$ .

Moreover, with  $T > 0$  fixed it is always possible to define a time-switching control strategy  $\sigma \in \mathcal{D}_T$  such that  $H_{\sigma(t)}$  is periodic. As a consequence, a necessary condition for the feasibility of constraints (2.45) and (2.46) is

$$\theta(T) = \max_{q=1, \dots, n} \left| \lambda_q \left( \prod_{p=1}^N e^{E_p T} \right) \right| < 1 \quad (2.50)$$

where  $\lambda_q(\cdot)$  denotes a generic eigenvalue of  $(\cdot)$  and  $\{E_1, \dots, E_N\}$  are matrices corresponding to any permutation among those of the set  $\{H_1, \dots, H_N\}$ . However, since the conditions of Theorem 6 take into account non-periodic policies as well, the necessary condition (2.50) for the existence of a feasible solution to inequalities (2.45)-(2.46), generally does not meet sufficiency. Hence a relevant function to be determined, based on this necessary condition is

$$T_p(\gamma) = \max_{T>0} \{T : \theta(T) = 1\} \quad (2.51)$$

Figure 2.6 shows in solid line the function  $T(\gamma)$ , in dashdot line the function  $T_p(\gamma)$  against  $\gamma \in (2.3, 7]$  and in dashed line the value of  $T(\infty)$  which is in accordance to the fact that, for this particular example, the minimum dwell time preserving asymptotical stability is  $T_* 2.7078$ . From this figure it is also confirmed that  $T_p(\gamma) \leq T(\gamma)$  for all  $\gamma > \gamma_c$  and that both are decreasing functions. The consequence is that the minimum dwell time is associated  $\gamma = +\infty$ . This is an expected behavior of the function  $T(\gamma)$  since for smaller values of  $\gamma$ , bounded bellow by  $\gamma_c$ , the switched linear system must support richer switching rules without losing stability. This is compensated by the increasing of the corresponding dwell time  $T(\gamma)$ . Figure 2.6 also puts in evidence the good concordance between the functions  $T(\gamma)$  obtained from a sufficient condition assuring inequality (2.47) and  $T_p(\gamma)$  obtained from a necessary condition assuring the same inequality. Although mentioned before, this aspect could be improved but, in our opinion, the results reported in this simple example are precise enough to classify the proposed method as a valid procedure for  $\mathcal{H}_\infty$  and dwell time specification.

## Chapter 3

# State Switching Control

In this chapter we consider once again the system (1.1) where the switching rule satisfies (1.2). The main difference from the previous chapter is that, presently, it is assumed that the switches that occur are based on the value of the state vector. Two main problems can be defined: in the first, tackled in Section 3.1 it is assumed that the state-dependent switching law is given and one has to establish the possible stability of the system only. In the second, tackled in Section 3.2, the state vector  $x(t)$  is available for feedback for all  $t \geq 0$ , and the goal is to determine the function  $u(\cdot) : \mathbb{R}^n \rightarrow \{1, \dots, N\}$ , such that

$$\sigma(t) = u(x(t)) \tag{3.1}$$

makes the equilibrium point  $x = 0$  of (1.1) asymptotically stable.

### 3.1 Stability of a given switched system

In this section we briefly consider a given switched system and we aim at analyzing its stability properties. For instance consider the pair of matrices

$$A_1 = \begin{bmatrix} \gamma & -1 \\ 2 & \gamma \end{bmatrix}, \quad A_2 = \begin{bmatrix} \gamma & -2 \\ 1 & \gamma \end{bmatrix},$$

where  $\gamma$  is a negative number close to zero and consider the switched system

$$\dot{x} = \begin{cases} A_1 x & \text{if } x_1 x_2 \leq 0 \\ A_2 x & \text{if } x_1 x_2 > 0 \end{cases}$$

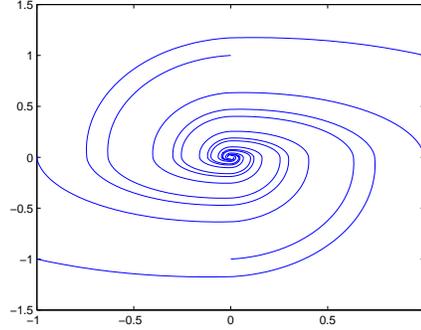
In Figure 3.1 it is shown the phase portrait of this switched system with  $\gamma = -0.1$ . It is seen that the system is asymptotically stable. Indeed we can find a continuous and differentiable function

$$V(x) = x'x$$

which is positive definite and whose derivative along the trajectories of the switched system is negative, since

$$\dot{V}(x) = \begin{cases} x'(A_1 + A_1')x & \text{if } x_1 x_2 \leq 0 \\ x'(A_2 + A_2')x & \text{if } x_1 x_2 > 0 \end{cases} = \begin{cases} 2\gamma x_1^2 + 2\gamma x_2^2 + 2x_1 x_2 & \text{if } x_1 x_2 \leq 0 \\ 2\gamma x_1^2 + 2\gamma x_2^2 - 2x_1 x_2 & \text{if } x_1 x_2 > 0 \end{cases}$$

If the stability analysis with a single Lyapunov function is impossible, then it is possible to

Figure 3.1: Phase portrait with  $\gamma = -0.1$ .

consider multiple Lyapunov functions. For instance consider again the two matrices  $A_1$  and  $A_2$  as before and the switched system

$$\dot{x} = \begin{cases} A_1 x & \text{if } x_1 \geq 0 \\ A_2 x & \text{if } x_1 < 0 \end{cases}$$

and the function

$$V(x) = \begin{cases} x' P_1 x & \text{if } x_1 \geq 0 \\ x' P_2 x & \text{if } x_1 < 0 \end{cases}$$

where

$$P_1 = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 0.5 & 0 \\ 0 & 1 \end{bmatrix}$$

Notice that function  $V(x)$  is continuous in the switching surface  $x_1 = 0$ , and

$$\dot{V}(x) = \begin{cases} x'(P_1 A_1 + A_1' P_1)x & \text{if } x_1 x_2 \leq 0 \\ x'(P_2 A_2 + A_2' P_2)x & \text{if } x_1 x_2 > 0 \end{cases} = \begin{cases} 4\gamma x_1^2 + 2\gamma x_2^2 & \text{if } x_1 \geq 0 \\ \gamma x_1^2 + 2\gamma x_2^2 & \text{if } x_1 < 0 \end{cases}$$

Hence the system is asymptotically stable.

The idea underlying the construction of the above Lyapunov function is to determine two functions, each for each region, with decreasing derivative in the region where the corresponding dynamics is active. For quadratic functions, it is useful in this regard, to resort to a well known result of convex programming, called *S*-procedure.

Let us assume to have two quadratic functions

$$x' Q_i x, \quad i = 1, 2$$

We want to check the following conditions

$$x' Q_0 x > 0 \quad \forall x \quad \text{such that} \quad x' Q_1 x \geq 0 \quad (3.2)$$

It turns out that (S-procedure):

- (i) If condition (3.2) is satisfied than there exists a nonnegative scalar  $\alpha$  such that

$$Q_0 - \alpha Q_1 > 0 \quad (3.3)$$

(ii) If condition (3.3) and there exist  $x_0 \neq 0$  such that  $x_0'Q_1x_0 > 0$ , then condition (3.2) is satisfied.

The condition  $x_0 \neq 0$  such that  $x_0'Q_1x_0 > 0$  is called *constrain qualification*. The proof that (i)  $\rightarrow$  (ii) is very easy and can be extended easily to a finite number of functions,  $x'Q_ix$ ,  $i = 0, 2, \dots, M$ . To be precise, if there exists nonnegative scalars  $\alpha_i$ ,  $i = 1, 2, \dots, M$  such that  $Q_0 - \sum_{i=1}^M \alpha_i Q_i \geq 0$  then  $x'Q_0x \geq 0$  whenever  $x'Q_ix \geq 0$ ,  $i = 1, 2, \dots, M$ . The converse result (ii)  $\rightarrow$  (i) is more difficult to be proven and left to the reader.

Thanks to the  $S$ -procedure, given a switched system constituted by matrices  $A_i$ ,  $i = 1, 2, \dots, M$  and activation regions of the type  $x'S_ix \geq 0$ , the problem is to find positive definite matrices  $P_i$  (yielding functions  $V_i(x) = x'P_ix$ ) such that

$$x'(A_i'P_i + P_iA_i)x < 0, \quad \forall x \text{ such that } x'S_ix \geq 0, \quad i = 1, 2, \dots, M$$

To this aim it is sufficient to find positive definite matrices  $P_i$  and nonnegative scalars such that

$$A_i'P_i + P_iA_i + \alpha_i S_i < 0, \quad i = 1, 2, \dots, M$$

Of course we are interested in functions  $V_i(x)$  which are continuous in the switching surfaces, and hence an additional constraint has to be added. To be precise, if the boundary between  $x'S_ix$  and  $x'S_jx$  is described by  $\{x : f'_{ij}x = 0\}$ , where  $f_{ij}$  is a  $n$ -dimensional vector, then  $P_i - P_j$  must satisfy

$$P_i - P_j = f_{ij}t'_{ij} + t_{ij}f'_{ij}, \quad \forall (i, j) = 1, 2, \dots, M$$

for some  $n$ -dimensional vector  $t_{ij}$ .

However, notice that the fact that the derivative is negative is not sufficient to have asymptotic stability if sliding modes occur. Indeed, consider the matrices

$$A_1 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

and the surfaces

$$S_1 = \begin{bmatrix} -0.0666 & 0.1227 \\ 0.1227 & 0.9487 \end{bmatrix}, \quad S_2 = -S_1$$

It is possible to find  $P_1$  and  $P_2$  satisfying

$$A_i'P_i + P_iA_i + \alpha_i S_i < 0$$

with

$$P_1 = \begin{bmatrix} 0.0645 & -0.3615 \\ -0.3615 & 3.2651 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 0.1311 & -0.4840 \\ -0.4840 & 2.3165 \end{bmatrix}$$

and  $\alpha_1 = 3$ ,  $\alpha_2 = 9$ . It is clear that the function

$$V(x) = \max_{i=1,2} x'P_ix$$

is such that  $\dot{V}(x) < 0$  whenever the derivative exists, i.e.  $x$  such that  $x'(P_1 - P_2)x \neq 0$ . However, the trajectories of the switched system, as shown in Figure 3.2, tend to the unstable sliding surface obtained by letting  $x'(P_1 - P_2)x = 0$ , i.e.  $x_2 = 0.1656x_1$ . Along this surface, the chattering system behaves as the linear combination

$$\dot{x} = (A_1\alpha + A_2(1 - \alpha))x = \begin{bmatrix} 1 & -0.4574 \\ 0.4574 & -0.08523 \end{bmatrix} x$$

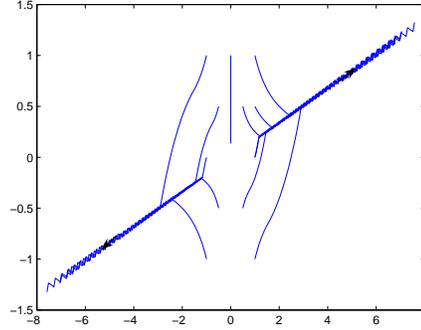


Figure 3.2: Phase portrait with unstable sliding mode

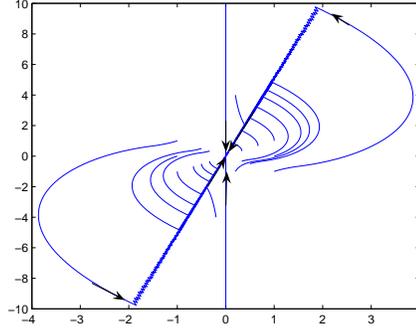


Figure 3.3: Phase portrait with stable sliding mode

obtained with  $\alpha = 0.7562$ . To understand the reason of instability of the Filippov solutions, take a vector  $y$  belonging to the switching surface and check that

$$y'(A_1'P_2 + P_2A_1)y > 0, \quad y'(A_2'P_1 + P_1A_2)y > 0$$

This means that, for each  $i = 1, 2$  it results

$$\begin{aligned} D^+v(y) &= \limsup_{h \rightarrow 0^+} \frac{V(y + hA_i y) - V(y)}{h} \\ &= \max_{i=1,2} y'(A_i'P_i + P_iA_i)y > 0 \end{aligned}$$

Consider now the same switched system and the switching surfaces:

$$s_1(x) = 0.3827x_1 + 0.9239x_2 = 0, \quad s_2(x) = 0.9808x_1 - 0.1951x_2 = 0$$

This means that

$$\sigma(x(t)) = \begin{cases} 1 & s_1(x)s_2(x) < 0 \\ 2 & s_1(x)s_2(x) > 0 \end{cases}$$

The phase portrait of the system is depicted in Figure 3.3. As a result, the switched system is asymptotically stable. However, finding a Lyapunov function is rather complex.

The switched system with the given pair  $A_1, A_2$  was introduced in [60], where it is shown that it does not admit a convex Lyapunov function. However, choosing the switching above surfaces we can conclude that it is indeed stabilizable. The next section is devoted to the state-feedback stabilization problem.

## 3.2 Stabilization

First we discuss a classical stability condition provided in [15] and more recently in [23] as a particular case of switched nonlinear systems. Let us first define the simplex

$$\Lambda := \left\{ \lambda \in \mathbb{R}^N : \sum_{i=1}^N \lambda_i = 1, \lambda_i \geq 0 \right\} \quad (3.4)$$

and assume that there exists  $\lambda_\infty \in \Lambda$  such that  $A_{\lambda_\infty}$  is asymptotically stable. Hence it is possible the determination of  $P > 0$  satisfying the Lyapunov inequality

$$A'_{\lambda_\infty} P + P A_{\lambda_\infty} < 0$$

It turns out that the switching rule with

$$\sigma(t) = u(x(t)) = \arg \min_{i=1, \dots, N} x(t)' (A'_i P + P A_i) x(t) \quad (3.5)$$

makes the equilibrium point  $x = 0$  of the switched system (1.1) globally asymptotically stable. Indeed, considering the Lyapunov function  $v(x(t)) = x(t)' P x(t)$  we have

$$\begin{aligned} \dot{v}(x(t)) &= x(t)' \left( A'_{\sigma(t)} P + P A_{\sigma(t)} \right) x(t) \\ &= \min_{i=1, \dots, N} x(t)' (A'_i P + P A_i) x(t) \\ &= \min_{\lambda \in \Lambda} x(t)' (A'_\lambda P + P A_\lambda) x(t) \\ &\leq x(t)' (A'_{\lambda_\infty} P + P A_{\lambda_\infty}) x(t) \\ &< 0 \end{aligned} \quad (3.6)$$

In conclusion, if a set of matrices admits a Hurwitz convex combination, then there exists a stabilizing state-feedback switching rule such that the closed-loop system is quadratically stable. Also the converse result is true for  $N = 2$ . Precisely, if there exists a state-feedback switching rule such that the closed-loop system is quadratically stable, then  $A_1$  and  $A_2$  admit a convex Hurwitz combination. Indeed, let  $v(x) = x' P x$  be the quadratic Lyapunov function. This means that

$$x' (A'_1 P + P A_1) x < 0$$

for all  $x$  such that  $x' (A'_2 P + P A_2) x \geq 0$  and viceversa. In view of the S-procedure we have

$$A'_1 P + P A_1 + \beta (A'_2 P + P A_2) < 0$$

and hence  $A_\lambda = \alpha A_1 + (1 - \alpha) A_2$  is Hurwitz with  $\alpha = (\beta + 1)^{-1}$ .

To end this point, it is important to keep in mind that, even if it is known that there exists  $\lambda \in \Lambda$  such that  $A_\lambda$  is asymptotically stable, the numerical determination of  $\lambda \in \Lambda$  and  $P > 0$  such that  $A'_\lambda P + P A_\lambda < 0$  is not a simple task due to the nonlinear nature of this equation.

Now, let associate with the simplex  $\Lambda$  a set of positive definite matrices  $\{P_1, \dots, P_N\}$ . This fact enables us to introduce the following piecewise quadratic Lyapunov function

$$v(x) := \min_{i=1, \dots, N} x' P_i x = \min_{\lambda \in \Lambda} \left( \sum_{i=1}^N \lambda_i x' P_i x \right) \quad (3.7)$$

As it will be clear in the sequel, this Lyapunov function is crucial to our purposes, see [1] and the references therein. However, it presents some difficulties to be handled including the fact that it is not differentiable everywhere. To analyze this aspect the set  $I(x) = \{i : v(x) = x' P_i x\}$  plays a central role since  $v(x)$  fails to be differentiable on  $x \in \mathbb{R}^n$  such that  $I(x)$  is composed by more than one element or, in other words, when the result of the minimization indicated in (3.7) is not unique, [19]. A main role is played by the the class of Metzler matrices denoted by  $\mathcal{M}$  and constituted by all matrices  $\Pi \in \mathbb{R}^{N \times N}$  with elements  $\pi_{ij}$ , such that

$$\pi_{ij} \geq 0 \quad \forall i \neq j, \quad \sum_{i=1}^N \pi_{ij} = 0 \quad \forall j \quad (3.8)$$

It is clear that any  $\Pi \in \mathcal{M}$  presents an eigenvalue at the origin of the complex plane since  $c' \Pi = 0$  where  $c' = [1 \ \dots \ 1]$ . In addition, it is well known from the Frobenius-Perron's theorem that the eigenvector associated to the null eigenvalue of  $\Pi$  is non-negative yielding the conclusion that there always exists  $\lambda_\infty \in \Lambda$  such that  $\Pi \lambda_\infty = 0$ . The next theorem summarizes the main result of this section.

**Theorem 7** *Assume that there exist a set  $\{P_1, \dots, P_N\}$  of positive definite matrices and  $\Pi \in \mathcal{M}$  satisfying the Lyapunov-Metzler inequalities*

$$A_i' P_i + P_i A_i + \sum_{j=1}^N \pi_{ji} P_j < 0, \quad i = 1, \dots, N \quad (3.9)$$

*The state switching control (3.1) with*

$$u(x(t)) = \arg \min_{i=1, \dots, N} x(t)' P_i x(t) \quad (3.10)$$

*makes the equilibrium solution  $x = 0$  of (1.1) globally asymptotically stable.*

**Proof** It follows from the Lyapunov function (3.7) which, as we have said before, is not differentiable for all  $t \geq 0$ . For this reason we need to deal with the Dini derivative (see [8])

$$D^+ v(x(t)) = \limsup_{h \rightarrow 0^+} \frac{v(x(t+h)) - v(x(t))}{h} \quad (3.11)$$

Assume, in accordance to (3.10), that at an arbitrary  $t \geq 0$ , the state switching control is given by  $\sigma(t) = u(x(t)) = i$  for some  $i \in I(x(t))$ . Hence, from (5.19) and the system dynamic equation (1.1), applying the result of Theorem 1, pp. 420 of [14] we have

$$\begin{aligned} D^+ v(x(t)) &= \limsup_{h \rightarrow 0^+} \frac{v(x(t) + h A_i x(t)) - v(x(t))}{h} \\ &= \min_{l \in I(x(t))} x(t)' (A_i' P_l + P_l A_i) x(t) \\ &\leq x(t)' (A_i' P_i + P_i A_i) x(t) \end{aligned} \quad (3.12)$$

where the inequality holds from the fact that  $i \in I(x(t))$ . Finally, remembering that  $\Pi \in \mathcal{M}$  and that  $x(t)'P_jx(t) \geq x(t)'P_ix(t)$  for all  $j \neq i = 1, \dots, N$  once again due to the fact that  $i \in I(x(t))$ , using the Lyapunov-Metzler inequalities (3.9) one gets

$$\begin{aligned} D^+v(x(t)) &< -x(t)' \left( \sum_{j=1}^N \pi_{ji} P_j \right) x(t) \\ &< - \left( \sum_{j=1}^N \pi_{ji} \right) x(t)' P_i x(t) \\ &< 0 \end{aligned} \tag{3.13}$$

which proves the proposed theorem since the Lyapunov function  $v(x(t))$  defined in (3.7) is radially unbounded.

It is important to observe that Theorem 7 does not require that the set  $\{A_1, \dots, A_N\}$  be composed exclusively by asymptotically stable matrices. Indeed, with  $\Pi \in \mathcal{M}$ , a necessary condition for the Lyapunov-Metzler inequalities to be feasible with respect to  $\{P_1, \dots, P_N\}$  is matrices  $A_i + (\pi_{ii}/2)I$  for all  $i = 1, \dots, N$  be asymptotically stable. Since  $\pi_{ii} \leq 0$  this condition does not imply on the asymptotical stability of  $A_i$ . However, an interesting case occurs when all matrices  $\{A_1, \dots, A_N\}$  are asymptotically stable for which the choice  $\Pi = 0$  is possible and the state switching strategy proposed preserves stability. Furthermore, if the set  $\{A_1, \dots, A_N\}$  is quadratically stable then the Lyapunov-Metzler inequalities admit a solution  $P_1 = \dots = P_N = P$  and  $I(x(t)) = \{1, \dots, N\}$  for all  $t \geq 0$ . In this classical but particular case, at any  $t \geq 0$ , the control law  $u(x(t)) = i \in \{1, \dots, N\}$  can be chosen arbitrarily and asymptotical stability is guaranteed. Hence, Theorem 7, contains as a particular case (since the Lyapunov-Metzler inequalities do not depend on  $\Pi$  anymore) the quadratic stability condition.

**Remark 1** (Chattering)

*Another important feature of Theorem 7 is that chattering in the switching when occurs is always stable. Indeed, assume that  $x \in \mathbb{R}^n$  belongs to a certain region  $\mathcal{C}$  of the state space where the cardinality of  $I(x)$  is greater than one. From the Lyapunov function (3.7), a switching from  $i \in I(x)$  to  $j \in I(x)$  is possible only if  $x'(A'_i P_j + P_j A_i)x \leq x'(A'_i P_i + P_i A_i)x < 0$  where the last inequality follows directly from (3.9). Hence, we conclude that whenever  $x \in \mathcal{C}$  the time derivative of the positive definite function  $v(x) = x' P_j x$  is strictly negative along all trajectories such that  $\dot{x} \in \text{co}\{A_i x : i \in I(x)\}$  which implies that they are asymptotically stable. In the particular case characterized by  $N = 2$ , this aspect has already been treated in [15]. In and [20] it is commented the fact that a Lyapunov function like (3.7) but with  $\min$  replaced by  $\max$  does not exhibit this property, in which instance the chattering must be ruled out. In this sense, the numerical procedure propose in [13] for the determination of a switching state dependent control has to be further qualified in order to prevent chattering since when it occurs instability may be observed.*

In the literature, the Lyapunov-Metzler inequalities with  $\Pi \in \mathcal{M}$  fixed, have been introduced in order to study the *Mean-Square* (MS) stability of Markov Jump Linear Systems (MJLS). In that context, the Metzler matrix  $\Pi = \Pi_0 \in \mathcal{M}$  is given and  $\Pi'_0$  represents the infinitesimal transition matrix of a Markov chain  $\sigma(t)$  governing the dynamical system (1.1). In this respect, each component of the vector  $\lambda(t) \in \Lambda$  is the probability of the Markov chain to be on the  $i$ -th logical state and obeys the differential equation

$$\dot{\lambda}(t) = \Pi_0 \lambda(t), \quad \lambda(0) = \lambda_0 \in \Lambda \tag{3.14}$$

where the eigenvector  $\lambda_\infty \in \Lambda$  associated to the null eigenvalue of  $\Pi_0$  represents the stationary probability vector. Hence, using the fact that the stochastic system under consideration is said to be MS-stable if

$$\lim_{t \rightarrow +\infty} E(\|x(t)\|^2) = 0 \quad (3.15)$$

for any initial state  $x(0)$  and any initial probability pattern  $\lambda_0 \in \Lambda$ , it has been shown (see e.g. [7]) that the system is MS-stable if and only if there exists a set of positive definite matrices  $\{P_1, \dots, P_N\}$  satisfying the Lyapunov-Metzler inequalities (3.9) for  $\Pi = \Pi_0$ . Numerically speaking, this is a simple case, since (3.9) reduces to a set of linear matrix inequalities.

A relevant point to be discussed now concerns the existence of a solution of the Lyapunov-Metzler inequalities (3.9) with respect to the variables  $\Pi \in \mathcal{M}$  and  $\{P_1, \dots, P_N\}$ . Standard Kronecker calculus shows that for  $\Pi \in \mathcal{M}$  fixed, a solution with respect to the remaining variables exists if and only if the  $Nn^2$ -dimensional square matrix  $\mathcal{J} := \mathcal{A} + \mathcal{B}\mathcal{C}$  is asymptotically stable, where

$$\mathcal{A} = \begin{bmatrix} A'_1 \oplus A'_1 & 0 & \cdots & 0 \\ 0 & A'_2 \oplus A'_2 & \cdots & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & \cdots & A'_N \oplus A'_N \end{bmatrix} \quad (3.16)$$

and

$$\mathcal{B} = \Pi' \begin{bmatrix} 0_{N-1} \\ I_{N-1} \end{bmatrix} \otimes I_{n^2}, \quad \mathcal{C} = \begin{bmatrix} -1_{N-1} & I_{N-1} \end{bmatrix} \otimes I_{n^2} \quad (3.17)$$

with the symbols  $\oplus$  and  $\otimes$  indicating the Kronecker sum and Kronecker product respectively<sup>1</sup>,  $0_{N-1}$  denoting a row vector of  $N-1$  zeros components and  $1_{N-1}$  denoting a column vector of  $N-1$  ones components. Hence, the existence of a solution to (3.9) reduces to the existence of  $\Pi \in \mathcal{M}$  rendering matrix  $\mathcal{J}$  asymptotically stable. A possible approach to verify the existence of such a matrix is based on the observation that any  $\alpha \geq 0$  and  $\Pi \in \mathcal{M}$  implies  $\alpha\Pi \in \mathcal{M}$ , which from the introduction of this new degree of liberty makes possible to verify the existence of  $\alpha \geq 0$  such that  $\mathcal{J}(\alpha) := \mathcal{A} + \alpha\mathcal{B}\mathcal{C}$  is asymptotically stable. Putting aside the situation on which all matrices  $\{A_1, \dots, A_N\}$  are asymptotically stable making possible to set  $\alpha = 0$ , let us consider the other extreme situation corresponding to  $\alpha \rightarrow +\infty$ . Simple determinant manipulations show that a certain number of eigenvalues goes to  $-\infty$  while the other ones that remain finite, coincide with the *invariant zeros* of the triple  $(\mathcal{A}, \mathcal{B}, \mathcal{C})$ .

Fortunately, these invariant zeros can be determined with no big difficulty from the definition

$$\begin{bmatrix} \mu I - \mathcal{A} & \mathcal{B} \\ \mathcal{C} & 0 \end{bmatrix} \begin{bmatrix} \xi \\ \eta \end{bmatrix} = 0$$

with the key observation that matrix  $\mathcal{C}$  being constant, that is independent of  $\alpha$  and  $\Pi$ , imposes to the solution of  $\mathcal{C}\xi = 0$  a vector of compatible dimension with the particular structure  $\xi' = [x' \ \cdots \ x']$ ,  $x \in \mathbb{R}^{n^2}$ . In addition, taking  $\lambda_\infty \in \Lambda$  such that  $\Pi\lambda_\infty = 0$ , multiplying each sub-equation above by  $\lambda_{\infty i}$  and summing up, it follows that

$$\left( \mu I - \sum_{i=1}^N \lambda_{\infty i} A'_i \oplus A'_i \right) x = 0$$

which, can be rewritten as

$$(\mu I - A'_{\lambda_\infty} \oplus A'_{\lambda_\infty}) x = 0 \quad (3.18)$$

<sup>1</sup>While the Kronecker product is more or less standard, the sum requires a formal definition. In this respect we define the Kronecker sum of two matrices  $D$  and  $E$  as  $D \oplus E = D \otimes I + I \otimes E$ . It is important to recall that the eigenvalues of the Kronecker sum  $D \oplus E$  are given by all sums of all eigenvalues of  $D$  and  $E$ .

where  $A_{\lambda_\infty} = \sum_{i=1}^N \lambda_{\infty i} A_i$ . Therefore, as  $\alpha$  goes to infinity, the eigenvalues of  $\mathcal{J}(\alpha)$  that remain finite, tend to the eigenvalues of  $A'_{\lambda_\infty} \oplus A_{\lambda_\infty}$  which are in the left hand plane if and only if so are the eigenvalues of  $A_{\lambda_\infty}$ . This means that, if there exists  $\lambda_\infty \in \Lambda$  such that  $A_{\lambda_\infty}$  is asymptotically stable, then any  $\Pi_0 \in \mathcal{M}$  satisfying  $\Pi_0 \lambda_\infty = 0$  and  $\alpha$  a sufficiently large positive number provide  $\Pi = \alpha \Pi_0 \in \mathcal{M}$  such that the Lyapunov-Metzler inequalities are feasible with respect to the remaining variables  $\{P_1, \dots, P_N\}$ .

**Example 2** *To illustrate the above point, let us consider a simple example with  $N = 2$ , the pair of matrices*

$$A_1 = \begin{bmatrix} 0 & 1 \\ 2 & -9 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 1 \\ -2 & 8 \end{bmatrix} \quad (3.19)$$

and

$$\Pi_0 = \begin{bmatrix} -0.51 & 0.49 \\ 0.51 & -0.49 \end{bmatrix} \in \mathcal{M} \quad (3.20)$$

The eigenvector associated to the null eigenvalue of  $\Pi_0$  is given by  $\lambda'_\infty = [0.49 \ 0.51]$ . We have determined numerically that the Lyapunov-Metzler inequalities (3.9) have a solution of the form  $\Pi = \alpha \Pi_0$ , for all  $\alpha \geq 615.7374$ , in accordance to the fact that the invariant zeros of the triple  $(A, B, C)$  are  $-0.33, -0.33, -0.33 \pm j0.226$  which as discussed before, can alternatively be obtained from the eigenvalues of the asymptotically stable matrix  $A_{\lambda_\infty} = 0.49A_1 + 0.51A_2$ , taking all sums.

The Lyapunov-Metzler inequalities introduced in Theorem 7 are difficult to be solved, since one has to search over the parameters of a Metzler matrix. However, a simple (yet more conservative) numerical procedure based on line search can be settled to determine its solution. This aspect will be considered next.

### 3.3 Guaranteed cost

Let us introduce a guaranteed quadratic cost associated to the proposed state switching control law (3.10).

**Lemma 2** *Let  $Q \geq 0$  be given. Assume that there exist a set of positive definite matrices  $\{P_1, \dots, P_N\}$  and  $\Pi \in \mathcal{M}$  satisfying the Lyapunov-Metzler inequalities*

$$A'_i P_i + P_i A_i + \sum_{j=1}^N \pi_{ji} P_j + Q < 0, \quad i = 1, \dots, N \quad (3.21)$$

*The state switching control (3.1) with  $u(x(t))$  given by (3.10) makes the equilibrium solution  $x = 0$  of (1.1) globally asymptotically stable and*

$$\int_0^\infty x(t)' Q x(t) dt < \min_{i=1, \dots, N} x'_0 P_i x_0 \quad (3.22)$$

**Proof** It has the same pattern of the proof of Theorem 7. The Lyapunov function (3.7) and the Lyapunov-Metzler inequalities (3.21) yield

$$D^+ v(x(t)) < -x(t)' Q x(t) \quad (3.23)$$

which after integration gives

$$\begin{aligned} v(x(t)) - v(x(0)) &= \int_0^t D^+ v(x(\tau)) d\tau \\ &< - \int_0^t x(\tau)' Q x(\tau) d\tau, \quad \forall t \geq 0 \end{aligned} \quad (3.24)$$

proving thus the proposed lemma since due to the asymptotical stability,  $v(x(t))$  goes to zero as  $t$  goes to infinity.

The numerical determination, if any, of a solution of the Lyapunov-Metzler inequalities with respect to the variables  $(\Pi, \{P_1, \dots, P_N\})$  is not a simple task and certainly deserves additional attention. The main source of difficulty stems from its non-convex nature due to the products of variables and so LMI solvers do not apply. Perhaps, a point to be further investigated is that its particular structure with  $\pi_{ji}$  being scalars may help on the design of an interactive method based on relaxation.

In this paper we pursue an alternative route. The main idea is to get a simpler, although certainly more conservative stability condition that can be expressed by means of LMIs being thus solvable by the machinery available in the literature to date. The next theorem shows that working with a subclass of Metzler matrices, characterized by having the same diagonal elements, this goal is accomplished.

**Theorem 8** *Let  $Q \geq 0$  be given. Assume that there exist a set of positive definite matrices  $\{P_1, \dots, P_N\}$  and a scalar  $\gamma > 0$  satisfying the modified Lyapunov-Metzler inequalities*

$$A_i' P_i + P_i A_i + \gamma(P_j - P_i) + Q < 0, \quad j \neq i = 1, \dots, N \quad (3.25)$$

*The state switching control (3.1) with  $u(x(t))$  given by (3.10) makes the equilibrium solution  $x = 0$  of (1.1) globally asymptotically stable and*

$$\int_0^\infty x(t)' Q x(t) dt < \sum_{i=1}^N x_0' P_i x_0 \quad (3.26)$$

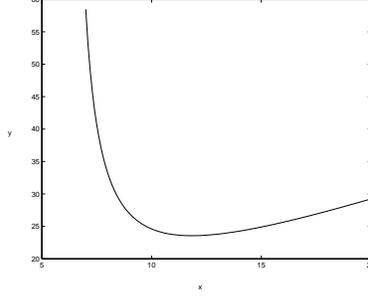
**Proof** The proof follows from the choice of  $\Pi \in \mathcal{M}$  such that  $\pi_{ii} = -\gamma$  and the remaining elements satisfying

$$\gamma^{-1} \sum_{j \neq i=1}^N \pi_{ji} = 1 \quad (3.27)$$

for all  $i = 1, \dots, N$ . Taking into account that  $\pi_{ji} \geq 0$  for all  $j \neq i = 1, \dots, N$  multiplying (5.24c) by  $\pi_{ji}$ , summing up for all  $j \neq i = 1, \dots, N$  and finally multiplying the result by  $\gamma^{-1} > 0$  we get

$$\begin{aligned} A_i' P_i + P_i A_i + Q &< - \sum_{j \neq i=1}^N \pi_{ji} (P_j - P_i) \\ &< - \sum_{j=1}^N \pi_{ji} P_j \end{aligned} \quad (3.28)$$

which being valid for all  $i = 1, \dots, N$  are the Lyapunov-Metzler inequalities (3.21). From Lemma 2, the upper bound (3.22) holds which trivially implies that (3.26) is verified. The proposed theorem is thus proved.  $\blacksquare$

Figure 3.4: Guaranteed cost as a function of  $\gamma$ .

The basic theoretical features of Theorem 7 and Lemma 2 are still present in Theorem 8. The most important is that the asymptotic stability of the set of matrices  $\{A_1, \dots, A_N\}$  still is not required. In addition, notice that the guaranteed cost (3.26) is clearly worse than the one provided by Lemma 2 but the former being convex makes possible to solve the problem

$$\min_{\gamma > 0, P_1 > 0, \dots, P_N > 0} \left\{ \sum_{i=1}^N x_0' P_i x_0 : (3.25) \right\} \quad (3.29)$$

by LMI solvers and line search. The next example illustrates some aspects of the theoretical results obtained so far.

**Example 3** Consider the system (1.1) with  $N = 2$  and matrices  $\{A_1, A_2\}$  given by

$$A_1 = \begin{bmatrix} 0 & 1 \\ 2 & -9 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 1 \\ -2 & 2 \end{bmatrix} \quad (3.30)$$

which, as it can be easily verified by inspection, are both unstable. Considering  $Q = I$  and the initial condition  $x_0 = [1 \ 1]'$ , problem (3.29) has been solved by line search fixing  $\gamma$  and minimizing its objective function, denoted by  $\delta(\gamma)$ , with respect to the remaining variables. Figure 3.4 shows the behavior of the function  $\delta(\gamma)$  which enables us to determine its minimum value  $\delta^* = 23.56$ , corresponding to  $\gamma^* = 11.80$ . It is important to stress that, in this particular example, the function  $\delta(\gamma)$  has a unique minimum. However, we do not have any evidence that this is a generic property valid in all cases. Figure 3.5 shows the trajectories of the state variable  $x(t) \in \mathbb{R}^2$  versus time for the system controlled by the state switching rule  $\sigma(t) = u(x(t))$  given by (3.10) with the positive definite matrices

$$P_1 = \begin{bmatrix} 6.7196 & 1.6293 \\ 1.6293 & 1.0222 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 6.0825 & 2.1293 \\ 2.1293 & 2.2206 \end{bmatrix} \quad (3.31)$$

obtained from the optimal solution of problem (3.29). As it can be seen, the proposed control strategy is very effective to stabilize the system under consideration.

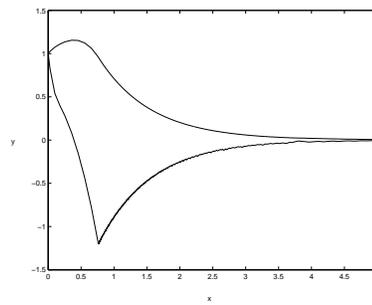


Figure 3.5: Time simulation of the state switching control.

## Chapter 4

# Optimal control

The problem of determining optimal control laws for hybrid and switched systems has been widely investigated in the last years, both from theoretical and from computational point of view [44], [50], [47], [48], [49]. For continuous-time switched systems, most of the literature studied necessary and/or sufficient conditions for a trajectory to be optimal, with the introduction of new versions of the minimum principle [31], [34], [35], [36] [51], [52]. The problem is also investigated in [37] for the case of two subsystems. More in detail, in [53], [35], the switched system is embedded into a larger family of nonlinear systems that can be handled directly by classical control theory. This idea was further exploited in [37], where necessary conditions for optimality of the embedded problem are derived using the maximum principle. When the necessary condition indicate an optimal solution of bang-bang type, a solution for the original switched problem may be derived. In [38], the problem of optimal control of autonomous switched systems was studied for a quadratic cost functional on an infinite horizon and a fixed number of switches. In this setting, the optimal control law can be computed by a discretization of the unitary semi-sphere. In later works, the same procedure was extended to the case where an infinite number of switches are allowed, [39], [40].

A special class of optimal control problems concerns autonomous switched systems, where the continuous control is absent and only the switching signal must be determined [54]. In particular, the sequence of active subsystems may be arbitrary, or it may be subject to constraints given as a pre-specified sequence with arbitrary length or as an arbitrary sequence with pre-specified length.

This chapter is organized as follows. The first section studies the optimal control problem for an autonomous linear switched system on a finite time interval. The switched system is embedded into a larger family of nonlinear systems; Necessary conditions for optimality on a finite horizon are developed using Hamilton-Jacobi-Bellman equation. No constraints are imposed on the switching and the performance index contains no penalty on the switching. Exploiting some properties of the optimal control, a numerical procedure for the solution of the problem based on the discretization of the state space is proposed.

In the second section the simple but important class of second order oscillating systems is considered and an algorithm is provided to find the optimal switching rule over an infinite horizon.

## 4.1 Problem formulation

In this paper we consider the following autonomous linear switched system

$$\begin{cases} \dot{x}(t) = A_{\sigma(t)}x(t) \\ x(t_0) = x_0 \end{cases} \quad (4.1)$$

where  $x \in \mathbb{R}^n$  is the continuous state and

$$\sigma(t) : [0, t_f] \rightarrow \mathcal{S} = \{1, \dots, N\}$$

is a piecewise constant function of time, called switching signal. We say that the subsystem  $\Sigma_s$  is active at time  $t$  when  $\sigma(t) = s$ . The state trajectory evolution of such a system can be controlled by choosing an appropriate switching sequence  $\Sigma = \{(t_0, s_0), (t_1, s_1), \dots, (t_K, s_K)\}$  defined in  $[t_0, t_f]$ , with  $0 \leq K \leq \infty$ ,  $t_0 \leq t_1 \leq \dots \leq t_K \leq t_f$ , and  $s_k \in \mathcal{S}$ . This switching sequence indicates that  $\sigma(t) = s_k$ ,  $\forall t \in [t_k, t_{k+1})$ , so that  $\dot{x}(t) = A_{s_k}x(t)$  in  $[t_k, t_{k+1})$ . No assumptions about the number of switchings nor about the sequence of active subsystems are made. However, for the switched system to be well-behaved, we consider only non Zeno sequences, which switch at most a finite number of times in every finite interval  $[t_i, t_j]$  with  $0 \leq t_i < t_j \leq t_f$ . Finally, the state of system (4.1) does not undergo jump discontinuities at the switching times.

Quadratic optimal control problem for autonomous linear switched system can be defined introducing a quadratic cost functional to be minimized. Assuming that both the subsystems and the cost functional are time invariant, it is possible to set the initial time to  $t_0 = 0$  without loss of generality. The cost functional to be minimized over all admissible switching sequences is given by

$$J(x_0, x, \sigma) = \int_0^{t_f} \frac{1}{2}x(t)^T Qx(t)dt + \frac{1}{2}x(t_f)^T Sx(t_f) \quad (4.2)$$

where  $x(t)$  is a solution of (4.1) with the switching signal  $\sigma(t)$ . The matrices  $Q$  and  $S$  are assumed to be symmetric and positive semidefinite. The optimal switching signal, the corresponding trajectory and the optimal cost functional will be denoted as  $\sigma^\circ(t, x_0)$ ,  $x^\circ(t)$  and  $J(x_0, x^\circ, \sigma^\circ)$  respectively.

In order to obtain a more tractable optimal control problem, the switched system (4.1) is embedded [35] into the larger family

$$\begin{cases} \dot{x}(t) = \sum_{s \in \mathcal{S}} u_s(t)A_sx(t) \\ x(0) = x_0 \end{cases} \quad (4.3)$$

parameterized by  $N$  variables  $u_s(t)$  subject to the constraints

$$\begin{cases} u_s(t) \geq 0, \quad \forall s \\ \sum_{s \in \mathcal{S}} u_s(t) = 1 \end{cases} \quad (4.4)$$

The vector  $u(t) = [u_1(t) \dots u_N(t)]^T$  can be regarded as a piecewise-continuous input of the embedding system. The set of trajectories of the embedding system contains the trajectory of the switched system, obtained constraining  $u(t)$  to be a simplex, i.e. a vector with  $u_i(t) = 1$  and  $u_j(t) = 0$ ,  $j \neq i$  when  $\sigma(t) = i$ .

The constraints regarding the discrete range of  $u(t)$  can be handled following optimal control theory in Pontryagin [56], [55]. Moreover, if the optimal solution of the embedding problem  $u(t)$  is the vertex of a simplex, it is also the optimal solution of the original switched problem, otherwise only a suboptimal solution can be determined [37].

The quadratic optimal control problem for the embedding system (4.3) is thus reformulated as follows. Given a fixed final time  $t_f$ , find the control input  $u^\circ(t)$  and the corresponding state trajectory  $x^\circ(t)$  such that the cost functional

$$J(x_0, x, u) = \int_0^{t_f} \frac{1}{2} x(t)^T Q x(t) dt + \frac{1}{2} x(t_f)^T S x(t_f) \quad (4.5)$$

evaluated for  $x(t) = x^\circ(t)$  is minimum. Of course, the infinite horizon optimal control problem is obtained by letting  $t_f \rightarrow \infty$ .

## 4.2 Finite-time optimal control

In this section we consider the optimal control problem in a finite horizon length.

### 4.2.1 Solution of the embedding optimal control

In the classical control theory, global sufficient conditions for optimality have been developed as a strengthening of the necessary conditions. Sufficient conditions introduce certain assumptions about the regularity of the functions involved and about the behaviour of the cost functional which must satisfy the Hamilton-Jacobi-Bellman equation [55], [56]. It is easy to see that even the simple case of a linear autonomous switched systems with quadratic cost functional does not match all these hypothesis. Introducing the concept of generalized solution and with suitable assumptions, however, such conditions may still be applicable at least for those cases where the optimal trajectories are non Zeno.

First of all, we cannot rely on the differentiability of the solution of (4.3). Nonetheless, for the non Zeno trajectories, the consequent mathematical difficulties can be overcome considering the definition of a solution in the sense of Carathéodory [57], namely a function  $x(t) : \mathbb{R}^+ \rightarrow \mathbb{R}^n$  is said to be a solution of (4.3), if it is absolute continuous on each compact subset of  $\mathbb{R}^+$  and it satisfies (4.3) for *almost all*  $t \geq 0$ .

The hamiltonian function relative to system (4.3) and cost functional (4.5) is given by

$$H(x, u, p) = \frac{1}{2} x^T Q x + p^T \sum_{s \in \mathcal{S}} u_s A_s x \quad (4.6)$$

A Pontragn triple  $(x^\circ(t), u^\circ(t), p^\circ(t))$  is defined as one triple satisfying the conditions

$$\dot{x}^\circ(t) = \sum_{s \in \mathcal{S}} u_s^\circ(t) A_s x^\circ(t), \quad x^\circ(0) = x_0 \quad (4.7)$$

$$H(x^\circ(t), u^\circ(t), p^\circ(t)) < H(x^\circ(t), u, p^\circ(t)), \quad \forall u \in \mathcal{U} \quad (4.8)$$

$$-\dot{p}^\circ(t) = \sum_{s \in \mathcal{S}} u_s^\circ(t) A_s^T p^\circ(t) + Q x^\circ(t) \quad (4.9)$$

$$p^\circ(t_f) = S x^\circ(t_f) \quad (4.10)$$

Letting  $p^\circ(t) = P(t)x^\circ(t)$  the above conditions can be rewritten as follows

$$\dot{x}^\circ(t) = \sum_{s \in \mathcal{S}} u_s^\circ(t) A_s x^\circ(t), \quad x^\circ(0) = x_0 \quad (4.11)$$

$$-\dot{P}(t) = \sum_{s \in \mathcal{S}} u_s^\circ(t) A_s^\top P(t) + P(t) \sum_{s \in \mathcal{S}} u_s^\circ(t) A_s + Q \quad (4.12)$$

$$P(t_f) = S \quad (4.13)$$

$$u^\circ(t) = \arg \min_{u \in \mathcal{U}} (x^\circ(t))^\top P(t) \sum_{s \in \mathcal{S}} u_s^\circ(t) A_s x^\circ(t) \quad (4.14)$$

**Theorem 9** Let  $u^\circ(t)$  defined in  $[0, t_f]$  be an optimal control and  $x^\circ(t)$ , with  $x^\circ(0) = x_0$  the associated state trajectory. Then, if such a control is optimal it satisfies

$$u^\circ(t) = \arg \min_{u \in \mathcal{U}} (x^\circ(t))^\top P(t) \sum_{s \in \mathcal{S}} u_s^\circ(t) A_s x^\circ(t)$$

where

$$-\dot{P}(t) = \sum_{s \in \mathcal{S}} u_s^\circ(t) A_s^\top P(t) + P(t) \sum_{s \in \mathcal{S}} u_s^\circ(t) A_s + Q, \quad P(t_f) = S$$

## 4.2.2 Solution of the switched optimal control

The optimal control  $u^\circ(t)$  for the embedding system (??) can be bang-bang (switching) but also have singular arcs (sliding modes) satisfying Filippov trajectories. Notice indeed the min of a convex combination coincide with the min of the vertices. We now rewrite the equations of the necessary conditions in the unknown  $\sigma(t)$ , but being prepared that the optimal solution (or candidate ones) may have sliding modes in finite intervals of times from 0 to  $t_f$ .

**Theorem 10** Let

$$\sigma^\circ(t, x_0) = \arg \min_{s \in \mathcal{S}} \{x^\circ(t)^\top P(t) A_s x^\circ(t)\}$$

where  $\sigma^\circ(t, x_0) : [0, t_f] \times \mathbb{R}^n \rightarrow \mathcal{S}$  be an admissible switching signal relative to  $x_0$  and  $x^\circ(t, x_0)$  is optimal then it satisfies

$$-\dot{P}(t) = A_{\sigma^\circ(t, x_0)}^\top P(t) + P(t) A_{\sigma^\circ(t, x_0)} + Q, \quad P(t_f) = S$$

and the associated optimal cost is

$$J(x_0, x^\circ, \sigma^\circ) = \frac{1}{2} x_0^\top P(0) x_0$$

Note that for a linear switched system and quadratic cost functional, the optimal switching signal shows some interesting properties which can be exploited to simplify the numerical determination of the optimal solution. For instance it is invariant upon scaling of the initial state  $x_0$ . Thanks to this fact, an equivalent formulation of the candidate optimal solutions can be obtained referring to a normalized state vector. Such a formulation may help during the numerical integration of (??).

**Corollary 1** Let  $\xi(t) = \frac{x(t)}{\|x(t)\|}$  then the switching signal

$$\sigma^\circ(t, x_0) = \tilde{\sigma}^\circ \left( t, \frac{x_0}{\|x_0\|} \right) = \tilde{\sigma}^\circ(t, \xi_0)$$

where  $\tilde{\sigma}^\circ(t, \xi_0)$  is the solution of the system of differential equations

$$\begin{cases} \dot{\xi}(t) = (A_{\tilde{\sigma}^\circ(t, \xi_0)} - \text{trace}(A_{\tilde{\sigma}^\circ(t, \xi_0)} \xi(t) \xi(t)^T) I) \xi(t) \\ -\dot{P}(t) = A_{\tilde{\sigma}^\circ(t, \xi_0)}^T P(t) + P(t) A_{\tilde{\sigma}^\circ(t, \xi_0)} + Q \\ \tilde{\sigma}^\circ(t, \xi_0) = \arg \min_{s \in \mathcal{S}} \{ \text{trace}(P(t) A_s \xi(t) \xi(t)^T) \} \end{cases} \quad (4.15)$$

with the split boundary conditions

$$\begin{cases} \xi(0) = \xi_0 = \frac{x_0}{\|x_0\|} \\ P(t_f) = S \end{cases} \quad (4.16)$$

satisfies the necessary conditions. The value of the optimal cost functional is

$$J(x_0, x^\circ, \tilde{\sigma}^\circ) = \frac{1}{2} x_0^T P(0) x_0 \quad (4.17)$$

*Proof* Observing that for all  $t$  except the switching instants

$$\begin{aligned} \frac{d}{dt} \|x(t)\| &= \frac{d}{dt} \sqrt{x_1(t)^2 + \dots + x_n(t)^2} = \\ &= \frac{2x_1(t)\dot{x}_1(t) + \dots + 2x_n(t)\dot{x}_n(t)}{2\sqrt{x_1(t)^2 + \dots + x_n(t)^2}} = \frac{x(t)^T A_{\tilde{\sigma}^\circ(t, \xi_0)} x(t)}{\|x(t)\|} \end{aligned}$$

we can write

$$\begin{aligned} \dot{\xi}(t) &= \frac{\dot{x}(t)}{\|x(t)\|} - \frac{x(t) \frac{d}{dt} \|x(t)\|}{\|x(t)\|^2} = \\ &= A_{\tilde{\sigma}^\circ(t, \xi_0)} \frac{x(t)}{\|x(t)\|} - \frac{x(t)}{\|x(t)\|} \frac{\frac{d}{dt} \|x(t)\|}{\|x(t)\|} = \\ &= A_{\tilde{\sigma}^\circ(t, \xi_0)} \xi(t) - \xi(t) \frac{x(t)^T}{\|x(t)\|} A_{\tilde{\sigma}^\circ(t, \xi_0)} \frac{x(t)}{\|x(t)\|} = \\ &= A_{\tilde{\sigma}^\circ(t, \xi_0)} \xi(t) - \text{trace}(\xi(t)^T A_{\tilde{\sigma}^\circ(t, \xi_0)} \xi(t)) \xi(t) = \\ &= (A_{\tilde{\sigma}^\circ(t, \xi_0)} - \text{trace}(A_{\tilde{\sigma}^\circ(t, \xi_0)} \xi(t) \xi(t)^T) I) \xi(t) \end{aligned}$$

Finally, from the properties of the trace operator it follows that

$$\begin{aligned} \arg \min_{s \in \mathcal{S}} \{ x^T(t) P(t) A_s x(t) \} &= \\ &= \arg \min_{s \in \mathcal{S}} \{ \text{trace}(P(t) A_s x(t) x^T(t)) \} = \\ &= \arg \min_{s \in \mathcal{S}} \{ \text{trace}(P(t) A_s \xi(t) \xi^T(t)) \} \end{aligned}$$

### 4.2.3 Numerical determination of the optimal switching signal

The determination of the control signal both in the embedding and in the switching case cannot be performed through a simple integration of a differential matrix equation of Lyapunov type (as in the linear case). The methodology proposed in Corollary 1 requires the solution of a

nonlinear system of differential equations (4.15) with the split boundary conditions (4.16), due to the dependence of the system structure on the switching signal.

This problem goes under the name of ‘two point boundary value problem’, as opposed to usual single point boundary value problems. While in the single point case it is always possible to start an acceptable solution at one edge of the interval and continue it through the interval by numerical integration, in the two point case the boundary conditions at the starting (final) point do not determine a unique solution to start with. Additional troubles come from the discrete nature of the switching signal. The easiest way to solve a two point boundary value is to use the ‘shooting technique’ [46], where a two boundary problem is reduced to an initial (final) value problem with a random choice of the initial (final) conditions to complete the boundary conditions at one end of the time interval. The equations are then integrated with standard techniques and corrections are made for the initial guess; the process is repeated until convergence is reached.

Since the initial condition  $x_0$  is given but arbitrary, a slight modification of the shooting technique may be adopted. The space of the solutions is systematically explored, choosing an arbitrary value for the unspecified terminal condition (the final state  $x(t_f)$ ) and computing the corresponding optimal solution integrating backward in time. Computation continues until the state space is so well covered with optimal solutions that a suboptimal solution can be determined for any arbitrary initial state.

The invariance of the time-dependent switching rule upon scaling of the initial state comes in handy to reduce the region of the state space to explore. We can restrict, for example, to the set of final states with a given norm

$$\mathcal{B}_f = \{\xi_f : \|\xi_f\| = 1\} \quad (4.18)$$

since the same scaling applies to the final states, too. Note that, if  $x \in \mathbb{R}^n$  then  $\mathcal{B}_f$  is an hypersurface of dimension  $n - 1$ . As an example, if  $x \in \mathbb{R}^2$ , a possible choice for the terminal hypersurface is the unit semicircle.

**Algorithm 1** *Procedure for the computation of a suboptimal switching sequence*

1. Consider a suitable discretization of the terminal hypersurface (4.18) by letting

$$\bar{\mathcal{B}}_f = \{\xi_f^{(i)} : \xi_f^{(i)} \in \mathcal{B}_f, i = 1, \dots, N_f\} \quad (4.19)$$

2. For each point belonging to  $\bar{\mathcal{B}}_f$  equations (4.15) are integrated backward in time, with the one point boundary condition

$$\begin{cases} \xi(t_f) = \xi_f^{(i)} \\ P(t_f) = S \end{cases} \quad (4.20)$$

in order to determine the initial point of the trajectory  $x_0^{(i)}$  and the corresponding switching sequence  $\Sigma^{(i)} = \{(t_1^{(i)}, s_1^{(i)}), \dots, (t_K^{(i)}, s_K^{(i)})\}$ .

3. Given a generic  $x_0$  compute  $\xi_0 = \frac{x_0}{\|x_0\|}$

(a) if  $\xi_0 = \xi_0^{(i)}$  for some  $i$ , then the optimal control law  $\Sigma^{(i)}$  is applied forward, remembering that the switching signal is invariant upon scaling of the initial state;

(b) if  $\xi_0 \neq \xi_0^{(i)}$  for all  $i$ , then the control law  $\Sigma^{(j)}$  with

$$j = \arg \min_{i=1, \dots, N_f} \|\xi_0 - \xi_0^{(i)}\|$$

is applied forward, obtaining a suboptimal solution to the switched control problem.

The proposed procedure is quite simple to implement; however its applicability tends to be reduced as the dimension of the state space or the number of points on the terminal hypersurface increase. Numerical problems may also appear during the integration of (4.15).

#### 4.2.4 A Numerical Example

Consider a linear switched system (4.1) with three stable second-order subsystems

$$A_1 = \begin{bmatrix} 0 & 1 \\ -2 & -1 \end{bmatrix} \quad A_2 = \begin{bmatrix} 0 & 2 \\ -1 & -1 \end{bmatrix} \quad A_3 = \begin{bmatrix} 0 & 1.5 \\ -1 & -1.5 \end{bmatrix}$$

and the cost functional (4.2) with  $t_f = 2$  and with

$$Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad S = \begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix}$$

Equations (4.15) are integrated backward in time, considering as a terminal boundary the points on the unit semicircle  $\mathcal{B}_f = \{\xi_f : \xi_f = [\cos(\theta) \ \sin(\theta)]^T, \theta \in [0, \pi)\}$ . Fig. 4.2.4 shows the optimal trajectories obtained for the switched system, when the semicircle is divided into 20 points uniformly distributed. Fig. 4.2 shows the same trajectories scaled so that the initial point of each trajectory (marked with a small circle) lies on the unit semicircle. It is apparent that such points are not uniformly distributed on the semicircle, even if the final points were so. In the general case, it is not possible to foresee how well the state space will be covered starting from a particular discretization of the terminal hypersurface.

Fig. 4.3 shows how the value of the cost functional is affected by the interpolation proposed in Algorithm 1, comparing the optimal value of the cost functional with the suboptimal value obtained with the algorithm previously described. The suboptimal cost functional (crosses on the figure) is obtained computing the optimal control law through backward integration for 20 points on the unit semicircle and then applying Algorithm 1 to 60 points equally distributed on this surface. The optimal cost (solid line) is obtained considering a finer discretization of the terminal hypersurface (120 points). In this particular example the range of worsening due to suboptimality is within 10% and it is concentrated in the areas less covered by initial points.

### 4.3 The switching oscillating system

In this section our analysis focuses on a second order system of the form

$$\ddot{y}(t) = -\alpha_i \dot{y}(t) - \beta_j y(t) + w(t) \quad (4.21)$$

where  $y(t) \in \mathbb{R}$ ,  $i \in \Omega_\alpha = \{1, 2, \dots, n_\alpha\}$ ,  $j \in \Omega_\beta = \{1, 2, \dots, n_\beta\}$ , and the values of  $\alpha_i$  and  $\beta_j$  are known parameters. In mechanical systems  $\alpha_i$  can be interpreted as the damping coefficient and  $\beta_j$  as the stiffness coefficient. The input  $w(t)$  is a scalar disturbance to be specified later.

The above model lends itself to describe a large variety of physical systems, whose coefficients may be switched within a finite set in order to improve some given performance. We say that

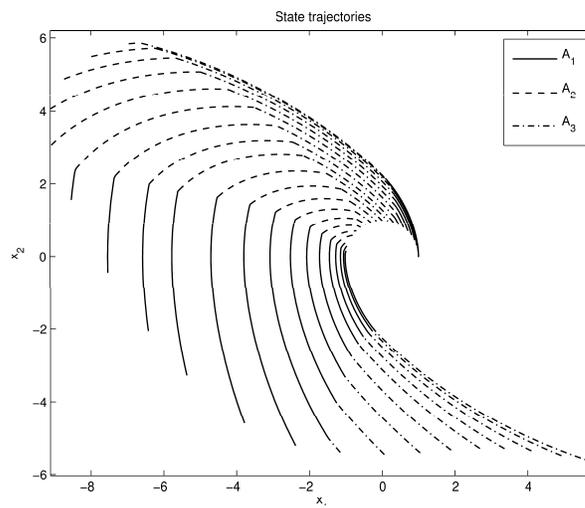


Figure 4.1: Optimal trajectories in the state space with final point on unit semicircle, obtained through backward integration of (4.15).

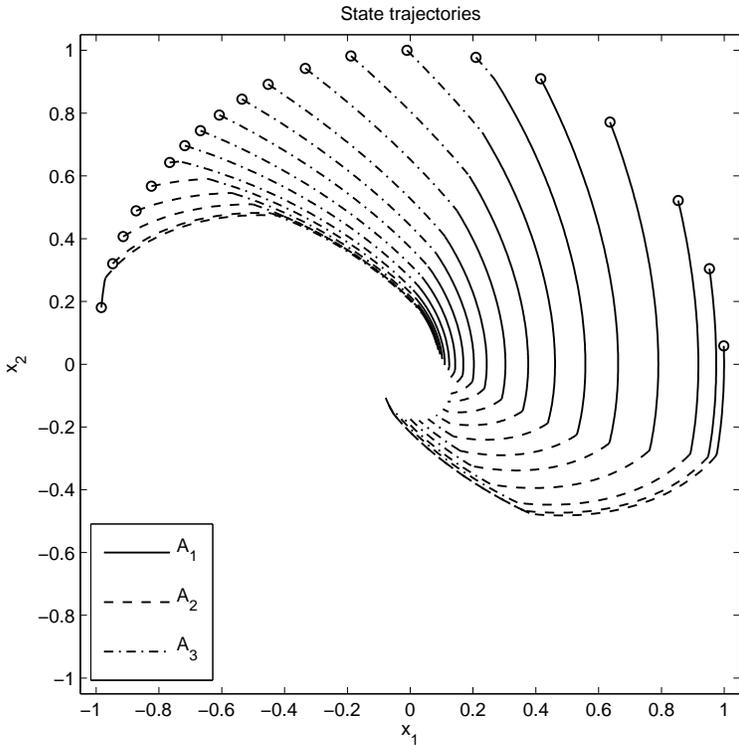


Figure 4.2: Optimal trajectories in the state space obtained through scaling in order to have initial point on the unit semicircle.

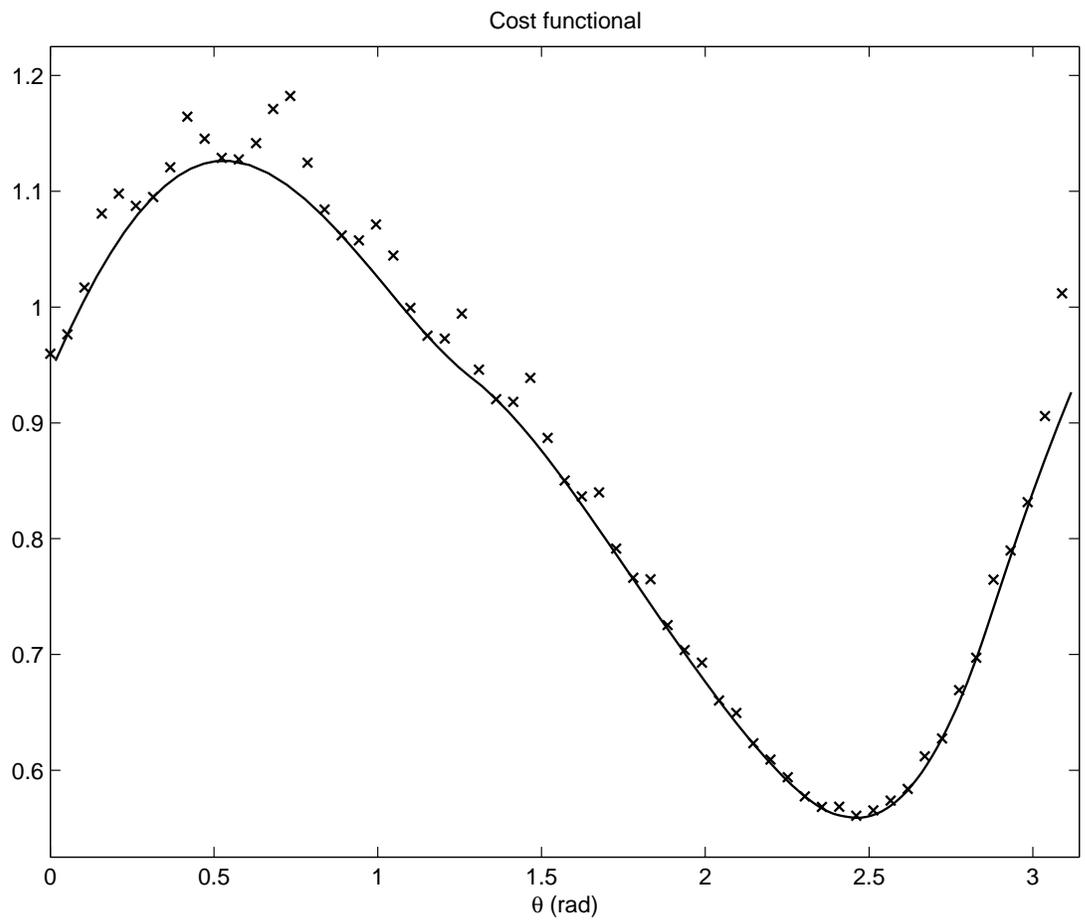


Figure 4.3: Comparison between optimal (solid) and suboptimal (cross) cost functional, with  $x_0 = [\cos(\theta) \sin(\theta)]^T$

the system is operating in the  $(i, j)$  mode when the underlying parameters take the values  $(\alpha_i, \beta_j)$ . Let  $\sigma(t) \in \Omega_\alpha \times \Omega_\beta$  represent the switching signal. As  $\sigma(t)$  changes, the evolution of the system is switched from one mode to another. Notice that the positiveness of  $\alpha_i$  and  $\beta_j$  is a necessary and sufficient condition for the stability of the single  $(i, j)$  mode. However, in general, even if all modes are stable, there might exist a switching signal that makes the resulting time-varying system unstable, [15].

Let us now introduce the performance variable (scalar or 2-dimensional vector)

$$z(t) = \gamma_j y(t) + \delta_i \dot{y}(t)$$

and the performance index

$$J = \int_0^\infty z(t)' z(t) dt \quad (4.22)$$

The (vector) coefficients  $\gamma_j$ ,  $j = 1, 2, \dots, n_\beta$  and  $\delta_i$ ,  $i = 1, 2, \dots, n_\alpha$ , may depend on the switching signal  $\sigma(t)$  in order to weight differently the contribution of the individual modes in the performance index.

Our aim is at finding a state-feedback strategy  $\sigma = u(y, \dot{y})$  that minimizes  $J$  when  $w(\cdot) = 0$  and the initial state  $(y(0), \dot{y}(0))$  is given, albeit arbitrary. Notice that this problem admits a solution whenever the switched system is stabilizable, see [15]. This occurs for instance when a single  $(i, j)$  mode is stable. The problem generalizes to switched system the classical linear quadratic optimal control theory. It is interesting to stress that the solution to this problem also provides the optimal switching strategy in the case when the initial state is zero and  $w(t)$  is an impulsive signal. Indeed, the latter situation reduces to the former by taking an initial state  $y(0) = 0$  and  $\dot{y}(0) = 1$ . In addition the optimal strategy minimizes the variance of  $z(t)$  when  $w(t)$  is a white noise process.

### 4.3.1 Computation of the optimal switching

The optimal control problem for the switched system can be solved by a suitable adaptation of the Hamilton-Jacobi equation, see e.g. [18]. To compact the notation we are well advised to rewrite the system in state-space form

$$\dot{x}(t) = A_{\sigma(t)} x(t) + B w(t) \quad (4.23)$$

$$z(t) = E_{\sigma(t)} x(t) \quad (4.24)$$

where

$$x = \begin{bmatrix} y \\ \dot{y} \end{bmatrix}, \quad A_\sigma = \begin{bmatrix} 0 & 1 \\ -\beta_j & -\alpha_i \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad E_\sigma = [ \gamma_j \quad \delta_i ]$$

The solution to the optimal control problem exists if it is possible to compute a continuous, piecewise differentiable and positive definite function  $V(y, \dot{y}) = V(x)$  satisfying

$$0 = \min_\sigma \left( \frac{\partial V}{\partial x} A_\sigma x + x' E'_\sigma E_\sigma x \right) \quad (4.25)$$

The optimal switching rule is then given by

$$\sigma = u(y, \dot{y}) = u(x) = \arg \min_\sigma \left( \frac{\partial V}{\partial x} A_\sigma x + x' E'_\sigma E_\sigma x \right) \quad (4.26)$$

and  $V(x(0))$  represents the optimal value of the performance index when  $x(0)$  is the initial state. It is obvious that a sufficient condition for the existence of the optimal solution is the

existence of a stabilizing switching rule. For instance, this condition is guaranteed when one of the modes is already stable or when there exists a stable convex combination of the  $M = n_\alpha n_\beta$  modes, see e.g. [15].

The solution to equation (4.25) can be found through an iterative numerical procedure. It is expedient to perform a change of coordinates from the phase plane  $(y, \dot{y})$  to the polar coordinates  $(\rho, \theta)$ . To this purpose we write

$$x = \begin{bmatrix} \rho \cos(\theta) \\ \rho \sin(\theta) \end{bmatrix}, \quad W(\rho, \theta) = V(x), \quad \frac{\partial V}{\partial x} = \begin{bmatrix} \frac{\partial W}{\partial \rho} & \frac{\partial W}{\partial \theta} \end{bmatrix} \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\rho^{-1} \sin(\theta) & \rho^{-1} \cos(\theta) \end{bmatrix}$$

Notice now that the optimal switching rule is invariant with respect to a scaling of the norm of  $x(0)$  and a change of sign. Consequently, for each real number  $\epsilon$  and each initial state  $x(0) \in \mathbb{R}^2$ , we have  $V(\epsilon x(0)) = \epsilon^2 V(x(0))$ . This reflects in simple constraints for  $W(\rho, \theta)$ , namely  $W(\rho, \theta) = \rho^2 \bar{W}(\theta)$  and  $\bar{W}(\theta - \pi) = \bar{W}(\theta)$ . By using the polar coordinates and recalling the definitions of  $A_\sigma$  and  $E_\sigma$ , equation (4.25) can be equivalently rewritten as

$$0 = \min_{\sigma} H(\theta, \sigma) \quad (4.27)$$

where

$$\begin{aligned} H(\theta, \sigma) &= 2 \sin(\theta) ((1 - \beta_j) \cos(\theta) - \alpha_i \sin(\theta)) \bar{W} \\ &\quad - (\sin(\theta)^2 + \beta_j \cos(\theta)^2 + \alpha_i \sin(\theta) \cos(\theta)) \frac{d\bar{W}}{d\theta} \\ &\quad + (\gamma_j \cos(\theta) + \delta_i \sin(\theta))' (\gamma_j \cos(\theta) + \delta_i \sin(\theta)) \end{aligned} \quad (4.28)$$

As obvious, the role of  $\rho$  becomes immaterial and the only unknown is the function  $\bar{W}(\theta)$ . This means that the switching surfaces are straight line in the phase plane. Moreover, being  $H(\theta + \pi, \sigma) = H(\theta, \sigma)$ , such surfaces turn out to be symmetric with respect to the origin and the modes activation regions are cones, as already known, see e.g. [39].

The problem is then to find a solution  $\bar{W}^\circ(\theta)$ ,  $\theta \in [0, \pi)$ , and the optimal switching strategy  $\sigma$  as a function of  $\theta$ , namely

$$\sigma^\circ = u^\circ(\theta) = \arg \min_{\sigma} H(\theta, \sigma) \quad (4.29)$$

We have devised a simple discretization algorithm to work out the solution. Precisely, consider a discretization of the upper unit semicircle  $\theta = k\Delta\theta$ ,  $\Delta\theta = \frac{\pi}{N}$ ,  $k = 0, 1, \dots, N-1$  and take the symmetric approximation of the derivative, i.e.

$$\frac{d\bar{W}}{d\theta} \simeq \frac{\bar{W}(\theta + \Delta\theta) - \bar{W}(\theta - \Delta\theta)}{2\Delta\theta}, \quad \bar{W}(-\Delta\theta) = \bar{W}((N-1)\Delta\theta), \quad \bar{W}(\pi) = \bar{W}(0)$$

Now letting

$$s = \begin{bmatrix} \sigma(0) \\ \sigma(\Delta\theta) \\ \sigma(2\Delta\theta) \\ \vdots \\ \sigma((N-1)\Delta\theta) \end{bmatrix}, \quad v = \begin{bmatrix} \bar{W}(0) \\ \bar{W}(\Delta\theta) \\ \bar{W}(2\Delta\theta) \\ \vdots \\ \bar{W}((N-1)\Delta\theta) \end{bmatrix}, \quad h(s) = \begin{bmatrix} H(0, \sigma(0)) \\ H(\Delta\theta, \sigma(\Delta\theta)) \\ H(2\Delta\theta, \sigma(2\Delta\theta)) \\ \vdots \\ H((N-1)\Delta\theta, \sigma((N-1)\Delta\theta)) \end{bmatrix}$$

we can rewrite (4.28) as

$$h(s) = L(s)v + m(s) \quad (4.30)$$

where the  $N^2$  square matrix  $L(s)$  and the vector  $m(s)$  can be easily deduced from (4.28). Notice that  $L(s)$  is a tridiagonal matrix except for the first and last rows. The algorithm starts with an initial vector  $v_{(0)}$ , for instance a vector with identical positive entries, or the one obtained from the Lyapunov function of a stable mode. Then, the core of the algorithm is based on equations (4.27), (4.29) and (4.30). The main iteration step is to compute

$$s_{(i)} = \begin{bmatrix} \sigma_{(i)}(0) \\ \sigma_{(i)}(\Delta\theta) \\ \sigma_{(i)}(2\Delta\theta) \\ \vdots \\ \sigma_{(i)}((N-1)\Delta\theta) \end{bmatrix}$$

and  $v_{(i+1)}$  in the following way

$$\begin{aligned} s_{(i)} &= \arg \min_s (L(s)v_{(i)} + m(s)) \\ v_{(i+1)} &= -L(s_{(i)})^{-1}m(s_{(i)})\eta + (1 - \eta)v_{(i)} \end{aligned}$$

where the above minimization of the vector  $L(s)v_{(i)} + m(s)$  is considered elementwise and  $\eta \in (0, 1]$  is a parameter controlling the smoothness of the solution. The algorithm ends when  $\|v_{(i^*+1)} - v_{(i^*)}\|$  is smaller than a given tolerance. The entries of  $s_{(i^*)}$  yield the optimal control strategy in the  $\theta$  grid points. Finally, the optimal value of the performance index is  $J^o = \rho(0)^2 \bar{W}(\theta(0))$ . This last value, in the grid points, can be found by taking the appropriate entry of vector  $v_{(i^*)}$ . The convergence analysis of the algorithm as well as its computational complexity are worth of further investigation. However, the algorithm was tested in many examples and convergence was always observed when at least one mode was stable.

### 4.3.2 A special case

This section is mainly devoted to discuss the special situation of equation (4.21) when the stiffness parameter  $\beta_j$  is fixed, i.e.  $\Omega_\beta = \{1\}$ ,  $\beta_1 = \beta > 0$ , and the damping parameter  $\alpha_i$  may switch between two values, i.e.  $\Omega_\alpha = \{1, 2\}$ ,  $\alpha_1 = \alpha_{min} \geq 0$ ,  $\alpha_2 = \alpha_{max} > \alpha_{min}$ . For simplicity we set  $\alpha_{min} = 0$ . We assume that the performance index is the integral of  $\ddot{y}(t)^2$ , so that  $\delta_i = \alpha_i$  and  $\gamma_j = \beta$ . In mechanical systems this corresponds to minimizing the integral of the squared acceleration. The case when also the parameter  $\beta_j$  can switch is briefly discussed at the end of the section.

The algorithm presented in the previous section has been run for different values of  $\beta$  and  $\alpha_{max}$  and  $N = 500$ . In all outcomes the optimal switching surfaces have the shape drawn in Figure 4.41. As can be noticed, one commutation occurs when the velocity  $\dot{y}$  changes its sign, whereas the second commutation is triggered by the crossing of a straight line with angle  $\theta^*(\alpha_{max}, \beta)$ . Therefore, the optimal strategy suggests that a null damping coefficient is more effective when  $y$  and  $\dot{y}$  have the same sign and the ratio  $\dot{y}/y$  is below a given threshold, namely  $\tan(\theta^*)$ . Figure 4.5 shows the value (in degrees) of  $\theta^*(\alpha_{max}, \beta)$  as a function of  $\alpha_{max}$  for different values of  $\beta$ . In order to illustrate the role of the switching rule, in Figure 4.6 the phase portrait of the optimal switched system is plotted for the particular choice  $\alpha_{max} = 1$ ,  $\beta = 1$ .

Finally, we have computed the performance index corresponding to the particular initial condition  $\theta(0) = \pi/2$  and  $\rho(0) = 1$ . In Figure 4.7 the optimal performance index  $J^o$  is plotted against  $\alpha_{max}$  for different values of  $\beta$ . The dashed curves correspond to the  $L_2$  performance associated with the constant damping coefficient  $\alpha_{max}$ . It is apparent that the switched damping improves significantly on the constant specially for high values of  $\alpha_{max}$ .

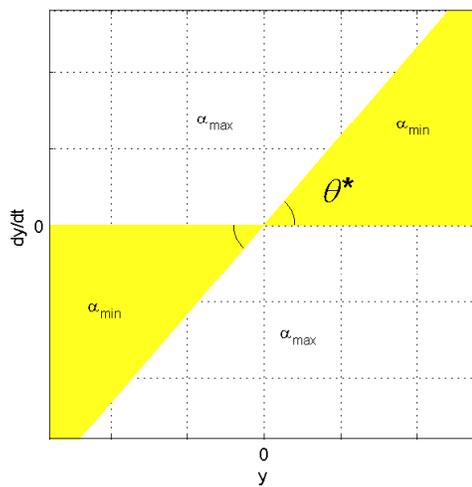
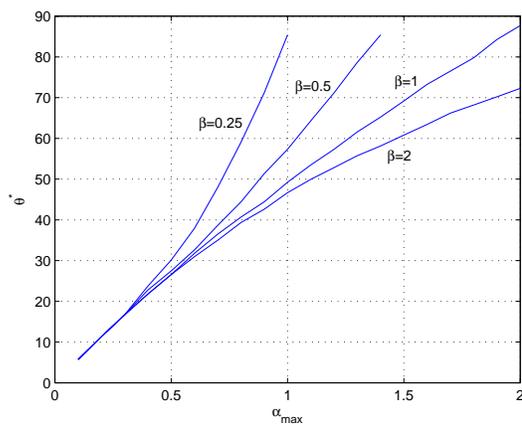


Figure 4.4: Shape of the switching surfaces

Figure 4.5:  $\theta^*(\alpha_{max}, \beta)$  as a function of  $\alpha_{max}$  for different values of  $\beta$

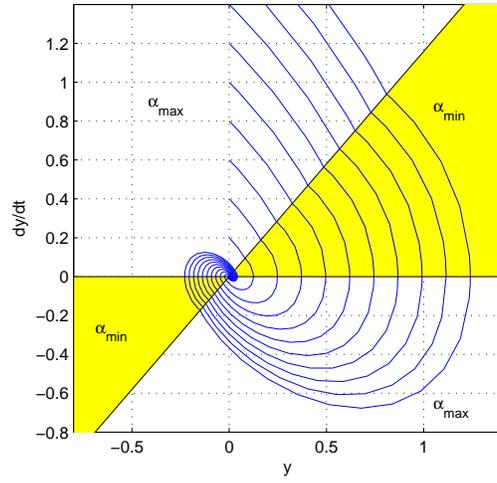


Figure 4.6: Phase portrait of the optimal switched system for  $\alpha_{max} = 1$  and  $\beta = 1$

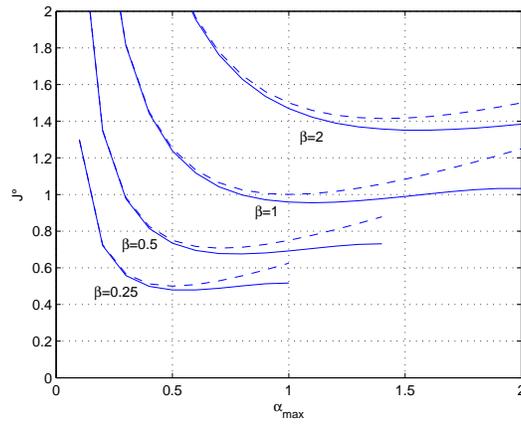


Figure 4.7: Optimal performance index with  $\theta(0) = \pi/2$  and  $\rho(0) = 1$



### 4.3.3 An application

This section discusses a practical application of the optimal switching control design presented before. Precisely, we consider the problem of comfort-oriented control of a semi-active suspension system in road vehicles. Our aim is to compare the achievable performance with the one provided by the classical switching rule based on the so-called two-state Sky-Hook (SH) approach, [43]. The model is as follows:

$$\begin{aligned} M\ddot{\xi}(t) &= -c(t)(\dot{\xi}(t) - \dot{\xi}_t(t)) - k(\xi(t) - \xi_t(t)) + k\Delta_s - Mg \\ m\ddot{\xi}_t(t) &= c(t)(\dot{\xi}(t) - \dot{\xi}_t(t)) + k(\xi(t) - \xi_t(t)) - k_t(\xi_t(t) - \xi_r(t)) - k\Delta_s + k_t\Delta_t - mg \\ \dot{c}(t) &= -\eta c(t) + \eta c_{in}(t) \end{aligned}$$

where  $\xi(t)$ ,  $\xi_t(t)$  and  $\xi_r(t)$  are the vertical position of the body, the unsprung mass and the road profile, respectively. The coefficients  $M$  and  $m$  are the quarter-car body mass and the unsprung mass (tire, wheel, brake, etc...), respectively. The parameters  $\eta$ ,  $k$  and  $k_t$  are the bandwidth of the active shock absorber, the stiffness of the suspension spring and of the tire, respectively. The coefficients  $\Delta_s$  and  $\Delta_t$  are the length of the unloaded suspension spring and of the tire. Finally,  $c(t)$  and  $c_{in}(t)$  are the actual and requested damping coefficients of the passive shock-absorber. In order to simplify the computations we assume that  $\eta$  is large enough so that  $c(t) \sim c_{in}(t)$ . Moreover we consider a genuine switching strategy, so that  $c(t) = c_i$  can assume only two values, namely  $c_1 = c_{min} \geq 0$  and  $c_2 = c_{max} > c_1$ , to be specified later on.

The control objective consists in minimizing the chassis vertical acceleration  $\ddot{\xi}(t)$  by a suitable choice of the control variable  $c(t) \in \{c_{min}, c_{max}\}$ . In the classical two-state SH approach [43], the system is switched according to the sign of  $\dot{\xi}(t)(\dot{\xi}(t) - \dot{\xi}_t(t))$ . In order to fit this example in the framework of the present paper, let us take the variations  $\delta\xi(t)$  and  $\delta\xi_t(t)$  of  $\xi(t)$  and  $\xi_t(t)$  around an equilibrium point associated with zero road profile, arriving to the system

$$M\delta\ddot{\xi}(t) = -c_i(\delta\dot{\xi}(t) - \delta\dot{\xi}_t(t)) - k(\delta\xi(t) - \delta\xi_t(t)) \quad (4.31)$$

$$m\delta\ddot{\xi}_t(t) = c_i(\delta\dot{\xi}(t) - \delta\dot{\xi}_t(t)) + k(\delta\xi(t) - \delta\xi_t(t)) - k_t(\delta\xi_t(t) - \xi_r(t)) \quad (4.32)$$

Notice that this is a 2-DOF system. In order to apply the optimal switching control design previously discussed, we make the (realistic) assumption that  $k_t$  is sufficiently high so that the displacement of the tire can be approximated by the road profile, i.e.  $\delta\xi_t(t) \simeq \xi_r(t)$ . Consequently, letting  $y(t) = \delta\xi(t) - \xi_r(t)$ , the approximated model can be written as

$$\ddot{y}(t) = -\frac{c_i}{M}\dot{y}(t) - \frac{k}{M}y(t) + \ddot{\xi}_r(t)$$

Thus, we have recovered equation (4.21) with  $\alpha_i = c_i/M$ ,  $\beta_j = \beta = k/M$  and  $w(t) = \ddot{\xi}_r(t)$ . Moreover, to improve comfort, it is advisable to minimize the integral of  $\ddot{y}(t)^2$ . The situation is exactly the one discussed in Section 4, and, consequently, the optimal switching surfaces are those qualitatively depicted in Figure 1. The following parameters have been selected, see [29]:  $M = 400kg$ ,  $m = 50kg$ ,  $k = 2.0 \times 10^4 N/m$ ,  $k_t = 2.5 \times 10^5 N/m$ ,  $c_1 = c_{min} = 3.0 \times 10^2 Ns/m$  and  $c_2 = c_{max} = 3.9 \times 10^3 Ns/m$ . The optimal switching angle has been computed on the basis of  $\alpha_{max}$  and  $\beta$  through the numerical algorithm of Section 3 with  $N = 500$  grid points. It turns out  $\theta^* = 86.6^\circ$ .

Two sets of simulations have been carried out, by applying both the Sky-Hook (SH) and the optimal switching (OS) control laws to the 2-DOF system (4.31), (4.32). The first set of simulations refers to the response to a unit impulse on the road acceleration  $w(t)$ , namely a ramp on the road profile. The first row of Table 1 reports the integral of the squared chassis

	OS	SH	PS <sub>1</sub>	PS <sub>2</sub>
$\int_0^\infty \ddot{y}(t)^2 dt$ for $\ddot{\xi}_r = \delta(t)$	7.446	8.288	26.548	8.307
$\frac{\int_0^{20} \ddot{y}(t)^2 dt}{\int_0^{20} \ddot{\xi}_r(t)^2 dt}$ for $\ddot{\xi}_r \sim WN$	0.623	0.787	3.558	0.719

Table 4.1: Performance of the different control strategies under an impulsive or a white noise disturbance

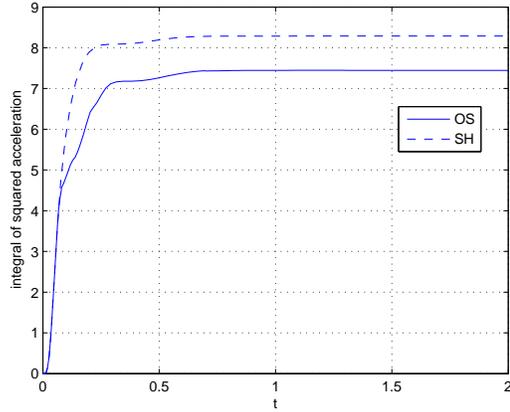


Figure 4.10: Time history of the integral of  $\ddot{y}(t)^2$  due to an impulse of  $\ddot{\xi}_r(t)$

acceleration obtained with different control strategies. The notation PS<sub>1</sub> and PS<sub>2</sub> refers to a passive suspension with fixed damping coefficient equal to  $c_{min}$  and  $c_{max}$ , respectively. As apparent from Table 1, the algorithm OS outperforms all other strategies.

Figure 4.10 shows the integral of the square of the chassis acceleration against time. It can be seen that OS is capable of lowering the acceleration in the transient better than SH, even if its design is based on a simplified 1-DOF model.

In the second set of simulations the road profile  $\xi_r(t)$  has been generated as the double integral of a sample realization of a white noise process with power  $\chi^2 = 0.1$ . The performance of the four algorithms above has been measured as the power attenuation on the chassis acceleration, namely the ratio

$$\Theta_T = \frac{\int_0^T \ddot{y}(t)^2 dt}{\int_0^T \ddot{\xi}_r(t)^2 dt}$$

This value, for  $T = 20$  sec., is reported in the second row of Table 1. Figure 4.11 shows the behavior of the acceleration. The plot has been restricted to an interval of 2 seconds, in order to better represent the effects of the commutations. The OS strategy outperforms SH at the price of faster switching commutation and shorter dwell intervals.

Finally the power attenuation  $\Theta_T$  as a function of  $T$  is plotted in Figure 4.12.

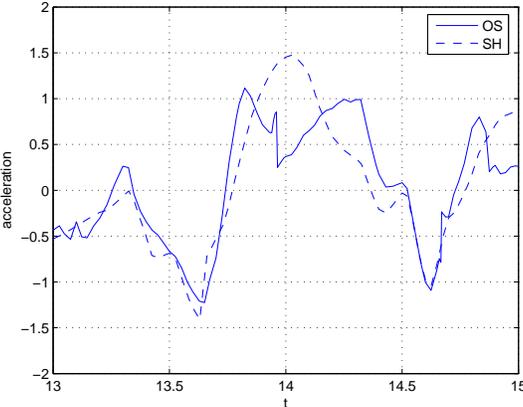


Figure 4.11: Chassis acceleration during a short interval under a random road profile

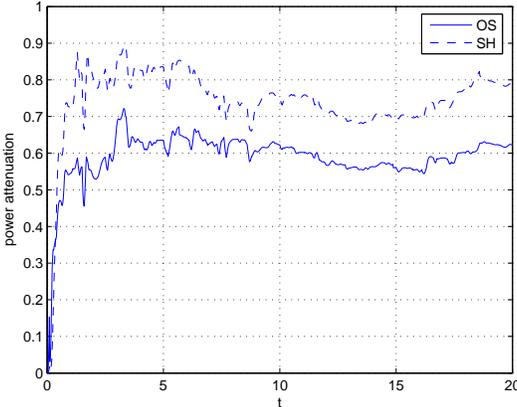


Figure 4.12: Power attenuation under a random road profile



# Chapter 5

## Output feedback control

### 5.1 Preliminaries

Consider linear switched systems of the following form

$$\dot{x}(t) = A_{\sigma(t)}x(t) + Bw(t) \quad (5.1a)$$

$$y(t) = C_{\sigma(t)}x(t) + Dw(t) \quad (5.1b)$$

$$z(t) = E_{\sigma(t)}x(t) \quad (5.1c)$$

which evolves from zero initial condition. The vectors  $x(t) \in \mathbb{R}^n$ ,  $w(t) \in \mathbb{R}^m$ ,  $y(t) \in \mathbb{R}^p$  and  $z(t) \in \mathbb{R}^q$  denote the state, the exogenous disturbance, the measured output and the controlled output variables, respectively. The switching signal is represented by a function  $\sigma(t)$  defined as

$$\sigma(t) : t \geq 0 \rightarrow \mathbb{N} := \{1, 2, \dots, N\} \quad (5.2)$$

making clear that at each instant of time  $t \geq 0$  one and only one among  $N$  known linear systems defined by matrices

$$\mathcal{S}_i := \begin{bmatrix} A_i & B \\ C_i & D \\ E_i & 0 \end{bmatrix}, \quad \forall i \in \mathbb{N} \quad (5.3)$$

are switched on. To ease presentation we have considered that the controlled variable  $z(t)$  does not depend directly on the external disturbance  $w(t)$ . Certainly, based on the results provided here, the reader does not have difficulty to treat more general situations.

Assuming that  $w(t)$  is an impulse disturbance (to be precisely defined afterwards) and that a quadratic cost functional  $J(\sigma)$ , as in equation (1.3), is given, the purpose of this paper is to design an output feedback control law of the form

$$\sigma(t) = u(y(\tau)), \quad \forall \tau \leq t \quad (5.4)$$

in such a way that the origin  $x = 0$  is a globally asymptotically stable equilibrium point. Moreover, a quantitative measure on the quality of the proposed policy (5.4) with respect to the optimal one is provided. This last requirements in given in terms of a lower and an upper bound  $J_{inf}$  and  $J_{sup}$  such that

$$J_{inf} \leq \inf_{\sigma \in \mathcal{S}} J(\sigma) \leq J(u) \leq J_{sup} \quad (5.5)$$

where  $\mathcal{S}$  defines the set of stabilizing switching rules. This last point is of particular importance since as it is largely recognized, the determination of the optimal policy and consequently the

correspondent minimum cost  $\inf_{\sigma \in \mathbb{N}} J(\sigma)$  is extremely hard even for linear switched systems constituted by a small number of linear systems of low order. This problem will be tackled in more details in Chapter 4. Now consider again the switched autonomous system

$$\dot{x}(t) = A_{\sigma(t)}x(t) \quad (5.6)$$

and the switching rule

$$u(x) := \arg \min_{i \in \mathbb{N}} x' P_i x \quad (5.7)$$

where  $P_i$  are suitable positive definite matrices. The next, results, already introduced in Chapter 3, provides an upper bound for the optimal cost.

**Theorem 11** *Let  $Q_i \geq 0, i \in \mathbb{N}$  be given. The following statements are true : If there exist a set of positive definite matrices  $\{P_1, \dots, P_N\}$  and  $\Pi \in \mathcal{M}_c$  satisfying the Lyapunov-Metzler inequalities*

$$A_i' P_i + P_i A_i + \sum_{j=1}^N \pi_{ji} P_j + Q_i < 0 \quad (5.8)$$

for all  $i \in \mathbb{N}$  then the state feedback switching control  $\sigma(t) = u(x(t))$  makes the equilibrium solution  $x = 0$  of (5.6) globally asymptotically stable and

$$\int_0^{\infty} x(t)' Q_{\sigma(t)} x(t) dt \leq v(x_0) \quad (5.9)$$

As for a lower bound, the following result can be stated.

**Theorem 12** *Let  $Q_i \geq 0, i \in \mathbb{N}$  be given and define the function  $V(x) := \max_{i \in \mathbb{N}} x' P_i x$ . The following statements are true : If there exist a set of positive definite matrices  $\{P_1, \dots, P_N\}$  and  $\Pi \in \mathcal{M}_c$  satisfying the inequalities*

$$A_j' P_i + P_i A_j + \sum_{k=1}^N \pi_{ki} P_k + Q_j \geq 0 \quad (5.10)$$

for  $i, j \in \mathbb{N} \times \mathbb{N}$  then the following lower bound holds

$$\inf_{\sigma \in \mathcal{S}} \int_0^{\infty} x(t)' Q_{\sigma(t)} x(t) dt \geq V(x_0) \quad (5.11)$$

**Proof** The proof of part a) follows from the determination of the Dini derivative of function  $V(x(t))$  along any trajectory of (5.6). Considering the set  $I(x) := \{i : V(x) = x' P_i x, i \in \mathbb{N}\}$  and  $\sigma = j \in \mathbb{N}$  arbitrary, making use of (5.10) we obtain

$$\begin{aligned} \dot{V}_+(x) &= \max_{l \in I(x)} x'(A_{\sigma}' P_l + P_l A_{\sigma})x \\ &\geq x'(A_j' P_i + P_i A_j)x \\ &\geq -x' Q_j x - \sum_{k=1}^N \pi_{ki} x' P_k x \\ &\geq -x' Q_j x \end{aligned} \quad (5.12)$$

where we have used the fact that  $x' P_i x \geq x' P_k x$  for all  $k \in \mathbb{N}$  and that  $\Pi \in \mathcal{M}_c$ . Consequently  $\dot{V}_+(x) + x' Q_{\sigma} x \geq 0$  for all  $(x, \sigma) \in \mathbb{R}^n \times \mathbb{N}$ , which by integration from zero to infinity yields the desired result (5.11) since the optimal trajectory satisfies  $x(0) = x_0$  and  $x(\infty) = 0$ .

Theorem 12 allows an useful interpretation on the existence of an optimal control policy. Inequalities (5.10) are always feasible when  $Q_i > 0$  for all  $i \in \mathbb{N}$  as it can be readily verified with  $P_i \rightarrow 0$ . On the other hand, if (5.10) admits an unbounded feasible solution then the lower bound  $V(x_0) \rightarrow +\infty$  and we can conclude that the optimal control problem (5.11) does not admit a stabilizing solution. To prevent this undesirable situation let us consider  $\nu > 0 \in \Lambda$  the eigenvector associated to the null eigenvalue of  $\Pi \in \mathcal{M}_c$  and  $\lambda \in \Lambda$ . Multiplying (5.10) successively by  $\nu_i > 0$  and  $\lambda_j \geq 0$  and summing up for all  $i, j \in \mathbb{N} \times \mathbb{N}$  we obtain

$$A'_\lambda P_\nu + P_\nu A_\lambda + Q_\lambda \geq 0 \quad (5.13)$$

Hence, assuming that there exists  $\lambda \in \Lambda$  such that  $A_\lambda$  is asymptotically stable, the inequality (5.10) implies that

$$\sum_{i=1}^N \nu_i P_i \leq \int_0^\infty e^{A'_\lambda t} Q_\lambda e^{A_\lambda t} dt \quad (5.14)$$

Since the right hand side of (5.14) is bounded, the conclusion is that the lower bound of the optimal cost (5.11) is bounded as well. Moreover, it is important to remember that under the same condition, that is, the existence of an asymptotically stable convex combination, from Theorem 7 the Lyapunov-Metzler inequalities admit a solution providing thus a stabilizing control and an upper bound to the optimal cost. In the next section these results are generalized to cope with the more general models for switched linear systems given in (5.1).

## 5.2 Closed Loop Performance

In this section, the following version of the switched linear system (5.1) is considered where, for the moment, the output variable is not taken into account. Assume that

$$\dot{x}(t) = A_{\sigma(t)}x(t) + Bw(t) \quad (5.15a)$$

$$z(t) = E_{\sigma(t)}x(t) \quad (5.15b)$$

evolves from zero initial condition and that  $\sigma(x)$  is a stabilizing switching state feedback control. For each  $k = 1, \dots, m$  an exogenous input of the form  $w(t) = e_k \delta(t)$  where  $e_k \in \mathbb{R}^m$  is the  $k$ th column of the identity matrix  $I_m$  is applied and the corresponding controlled output is obtained. Based on this, we define the following cost functional associated to the stabilizing control policy  $\sigma(x)$  as being

$$J(\sigma) := \sum_{k=1}^m \|z_k\|_2^2 \quad (5.16)$$

The interpretation of this cost steams from the fact that for a fixed stabilizing control policy  $\sigma(x)$ , any trajectory of (5.15) with zero initial condition and  $w(t) = e_k \delta(t)$  is alternatively provided by the same equations subject to the initial condition  $x(0) = B e_k$  and input  $w(t) = 0$ . This fact is also important to make clear that matrix  $B$  in (5.15a) can be considered, with no loss of generality, independent of  $\sigma \in \mathbb{N}$ . Indeed, if the input matrix were dependent on the switching policy then the initial condition would be  $x(0) = B_{\sigma(0)} e_k$ , with  $B_{\sigma(0)}$  being a fixed matrix for all  $k = 1, \dots, m$ . Hence, the results obtained so far can be applied to get lower and upper bounds to the optimal cost (5.16) for both continuous and discrete time cases.

**Theorem 13** *Consider the switched linear system (5.15) with zero initial condition and define  $Q_i := E'_i E_i$  for all  $i \in \mathbb{N}$ . If there exist a set of positive definite matrices  $\{P_1, \dots, P_N\}$  and  $\Pi \in$*

$\mathcal{M}_c$  ( $\mathcal{M}_d$ ) satisfying the inequalities (5.8) then the switching control strategy  $\sigma(t) = u(x(t))$  given in (5.7) is such that

$$J(\sigma) \leq \min_{i \in \mathbb{N}} \text{Tr}(B' P_i B) \quad (5.17)$$

**Proof** It follows from Theorem 11. Indeed, considering successively the initial condition  $x(0) = B e_k$  and  $w(t) = 0$  we have

$$\begin{aligned} J(\sigma) &\leq \sum_{k=1}^m \min_{i \in \mathbb{N}} (B e_k)' P_i (B e_k) \\ &\leq \min_{i \in \mathbb{N}} \sum_{k=1}^m (B e_k)' P_i (B e_k) \\ &\leq \min_{i \in \mathbb{N}} \text{Tr}(B' P_i B) \end{aligned} \quad (5.18)$$

which proves the proposed theorem.

In contrast to the result provided by Theorem 11 where the performance of each control policy was dependent on the initial state  $x_0 \in \mathbb{R}^n$ , Theorem 13 shows how to associate an unique stabilizing policy to a series of impulse-type perturbations applied to each external input channel. The consequence is somewhat similar to that observed in the classical  $H_2$  Theory of LTI systems where the control policy is effective to deal with perturbations of a wide frequency range acting on each input channel. In the next theorem the same reasoning is applied to lower bound calculations.

**Theorem 14** Consider the switched linear system (5.15) with zero initial condition and define  $Q_i := E_i' E_i$  for all  $i \in \mathbb{N}$ . If there exist a set of positive definite matrices  $\{P_1, \dots, P_N\}$  and  $\Pi \in \mathcal{M}_c$  ( $\mathcal{M}_d$ ) satisfying the inequalities (5.10), then the following lower bound holds

$$\inf_{\sigma \in \mathbb{N}} J(\sigma) \geq \max_{i \in \mathbb{N}} \text{Tr}(B' P_i B) \quad (5.19)$$

**Proof** Considering successively the initial condition  $x(0) = B e_k$  and  $w(t) = 0$ , Theorem 12 yields

$$\begin{aligned} \inf_{\sigma \in \mathbb{N}} J(\sigma) &\geq \sum_{k=1}^m \max_{i \in \mathbb{N}} (B e_k)' P_i (B e_k) \\ &\geq \max_{i \in \mathbb{N}} \sum_{k=1}^m (B e_k)' P_i (B e_k) \\ &\geq \max_{i \in \mathbb{N}} \text{Tr}(B' P_i B) \end{aligned} \quad (5.20)$$

which proves the proposed theorem.

The numerical determination of the upper and lower bounds of the optimal switching policy is involved and costly. The main difficulty is concentrated on the determination of the Metzler matrix  $\Pi \in \mathbb{R}^{N \times N}$  which certainly requires further research efforts. For the moment this difficulty is circumvented by replacing the search for a Metzler matrix by the determination of a scalar  $\gamma$ , as indicated in Corollary 8. This approach certainly introduces some conservativeness on the calculation of the final bounds but is numerically efficient. However, for  $\Pi \in \mathbb{R}^{N \times N}$  fixed, the associated lower and upper bounds follow from the solution of convex programming problems. Indeed, the minimization of the right hand side of (5.17) written as

$$\min_{i \in \mathbb{N}} \left\{ \min_{P_1, \dots, P_N \in \Phi(\Pi)} \text{Tr}(B' P_i B) \right\} \quad (5.21)$$

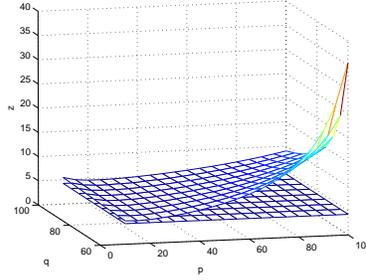


Figure 5.1: Upper and Lower bounds.

where  $\Phi(\Pi)$  is the convex set of all positive definite matrices  $P_i, i \in \mathbb{N}$  satisfying the LMIs (5.8) for some fixed Metzler matrix  $\Pi \in \mathbb{R}^{N \times N}$ , shows that the matrices  $P_i, i \in \mathbb{N}$  can be calculated from the internal minimization for each  $i \in \mathbb{N}$  and afterwards those correspondent to the minimum cost are selected. Similar reasoning can be applied to get the maximum lower bound. The next example illustrates the results obtained so far.

**Example 4** Consider a continuous time switched linear system (5.15a)-(5.15b) defined by the following matrices

$$A_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & -2 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -2 & -2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ -1 & -1 \\ 0 & 1 \end{bmatrix} \quad (5.22)$$

which are not stable but admit a stable convex combination. Matrices  $Q_1 = \text{diag}\{1, 1, 2\}$  and  $Q_2 = \text{diag}\{2, 1, 1\}$  define the associated cost  $J(\sigma)$  given in (5.16). With a Metzler matrix of the form

$$\Pi = \begin{bmatrix} -p & q \\ p & -q \end{bmatrix} \in \mathcal{M}_c \quad (5.23)$$

we have determined from Theorem 13 and Theorem 14 lower and upper bounds for  $60 \leq p \leq 100$  and  $10 \leq q \leq 100$ . Figure 2.4 shows that the lower bound is almost insensitive to the particular value of the Metzler matrix. The same, of course, does not hold for the upper bound. Notice also that a convenient choice of the Metzler matrix provides precise estimation of the interval where the optimal solution of  $\inf_{\sigma \in \mathbb{N}} J(\sigma)$  belongs to. For instance, for  $p = 100$  and  $q = 20$  we obtain  $J_{inf} = 4.2500$  and  $J_{sup} = 4.7158$  which corresponds approximately to a gap between the lower and upper bound of about 10%.

### 5.3 Output Feedback Control

In this section the main control problem reported in this paper is solved. It consists on the design of a stabilizing full order output feedback controller which minimizes the upper bound of the cost function  $J(\sigma)$  introduced in the previous section. To this end, the model (5.1) given again for convenience, is considered

$$\dot{x}(t) = A_{\sigma(t)}x(t) + Bw(t) \quad (5.24a)$$

$$y(t) = C_{\sigma(t)}x(t) + Dw(t) \quad (5.24b)$$

$$z(t) = E_{\sigma(t)}x(t) \quad (5.24c)$$

where the switching policy is of the form (5.4) since the switching strategy must be dependent only on the available measurements. The function  $u(\cdot)$  is indeed a functional of  $y(\cdot)$  in the sense that  $y(t)$  is viewed as the input of the following switched linear filter that rules out the change of the switching index. Introducing the full order switched filter

$$\dot{\hat{x}}(t) = \hat{A}_{\sigma(t)}\hat{x}(t) + \hat{B}_{\sigma(t)}y(t) \quad (5.25)$$

with zero initial condition, where  $(\hat{A}_i, \hat{B}_i)$ ,  $i = 1, 2, \dots, N$  are matrices to be determined, putting (5.24) and (5.25) together we obtain

$$\dot{\hat{x}}(t) = \tilde{A}_{\sigma(t)}\tilde{x}(t) + \tilde{B}_{\sigma(t)}w(t) \quad (5.26a)$$

$$z(t) = \tilde{E}_{\sigma(t)}\tilde{x}(t) \quad (5.26b)$$

where  $\tilde{x}' = [x' \ \hat{x}'] \in \mathbb{R}^{2n}$  and

$$\tilde{A}_i = \begin{bmatrix} A_i & 0 \\ \hat{B}_i C_i & \hat{A}_i \end{bmatrix}, \quad \tilde{B}_i = \begin{bmatrix} B \\ \hat{B}_i D \end{bmatrix}, \quad \tilde{E}_i = [E_i \ 0] \quad (5.27)$$

which evolves from zero initial condition. Therefore the solution of the stated output feedback switching control design problem requires the determination of the switched filter matrices  $\hat{A}_i$  and  $\hat{B}_i$  for all  $i \in \mathbb{N}$  and a switching policy, such that the enlarged switched linear system (5.26) is asymptotically stable. However, in doing so, only switching rules that depend exclusively on  $\hat{x}(\cdot)$  are permitted. In order to apply the results of the previous section, we limit the search for a solution of the Lyapunov-Metzler inequalities with a prescribed structure so as to structurally incorporate switching rules that depends only on the available information. Therefore, let

$$\tilde{P}_i = \begin{bmatrix} X & V \\ V' & \hat{X}_i \end{bmatrix}, \quad \det V \neq 0 \quad (5.28)$$

for all  $i \in \mathbb{N}$  and notice that  $\arg \min_{i \in \mathbb{N}} \tilde{x}' \tilde{P}_i \tilde{x} = \arg \min_{i \in \mathbb{N}} \hat{x}' \hat{X}_i \hat{x}$ . Hence, to fulfill our purposes, we need to find a stabilizing rule of the form  $\sigma(t) = u(\hat{x}(t))$  where

$$u(\hat{x}) = \arg \min_{i \in \mathbb{N}} \hat{x}' \hat{X}_i \hat{x} \quad (5.29)$$

In the sequel the goal is to determine a filter and a switching policy of the form (5.29) such that the upper bound of cost functional  $J(\sigma)$  provided by Theorem 13 is minimized. To ease the presentation we denote by  $\tilde{Q}_i := \tilde{E}_i' \tilde{E}_i \in \mathbb{R}^{2n \times 2n}$  and  $Q_i := E_i' E_i \in \mathbb{R}^{n \times n}$  for all  $i \in \mathbb{N}$ .

Considering the augmented switched linear system (5.26), from Theorem 13 it is seen that if there exist a Metzler matrix  $\Pi \in \mathcal{M}_c$ , positive definite matrices  $\tilde{P}_i$  of the form (5.28) and the filter matrices  $\hat{A}_i$  and  $\hat{B}_i$  for all  $i \in \mathbb{N}$  satisfying the Lyapunov-Metzler inequalities

$$\tilde{A}_i' \tilde{P}_i + \tilde{P}_i \tilde{A}_i + \sum_{j=1}^N \pi_{ji} \tilde{P}_j + \tilde{Q}_i < 0 \quad (5.30)$$

for  $i \in \mathbb{N}$  then the switching control (5.29) makes the equilibrium solution  $x = 0$  of (5.26a) globally asymptotically stable with the associated cost

$$J(u) = \min_{i \in \mathbb{N}} \text{Tr}(\tilde{B}_\ell' \tilde{P}_i \tilde{B}_\ell) \quad (5.31)$$

where  $\ell = \sigma(0) \in \mathbb{N}$  is fixed and supposed to be provided by the designer. However, with no great difficulty, it can be determined by minimizing the associated cost whenever desired. The next theorem gives a complete solution to the output feedback switching control design problem stated before.

**Theorem 15** *There exist matrices  $\hat{A}_i$  and  $\hat{B}_i$ ,  $i \in \mathbb{N}$  for which inequalities (5.30) are satisfied for some positive definite matrices  $\tilde{P}_i$  of the form (5.28) if and only if there exist a Metzler matrix  $\Pi \in \mathcal{M}_c$ , a positive definite matrix  $X$ , a set of positive definite matrices  $(Z_i, R_{ij})$  and a set of matrices  $L_i$  for all  $i, j \in \mathbb{N} \times \mathbb{N}$ , such that the following matrix inequalities*

$$A'_i Z_i + Z_i A_i + \sum_{j=1}^N \pi_{ji} R_{ij} + Q_i < 0 \quad (5.32a)$$

$$A'_i X + X A_i + C'_i L'_i + L_i C_i + Q_i < 0 \quad (5.32b)$$

$$R_{ii} < Z_i, \quad \begin{bmatrix} R_{ij} - Z_j & Z_j - Z_i \\ \bullet & X - Z_j \end{bmatrix} > 0, \quad i \neq j \quad (5.32c)$$

*hold. Moreover, assuming that inequalities (5.32a)-(5.32c) are satisfied, the output feedback switching control  $\sigma(t) = u(\hat{x}(t))$  defined by*

$$u(\hat{x}) = \arg \min_{i \in \mathbb{N}} \hat{x}' V' (X - Z_i)^{-1} V \hat{x} \quad (5.33)$$

*where  $V$  is an arbitrary nonsingular matrix, makes the equilibrium solution  $x = 0$  of (5.26a) globally asymptotically stable and the associated cost is given by  $J(u) = \min_{i \in \mathbb{N}} \text{Tr}(W_i)$  where the linear matrix inequality*

$$\begin{bmatrix} W_i & B' Z_i & B' X + D' L'_i \\ \bullet & Z_i & Z_i \\ \bullet & \bullet & X \end{bmatrix} > 0 \quad (5.34)$$

*holds for all  $i \in \mathbb{N}$ .*

**Proof** Consider symmetric matrices  $\tilde{P}_i \in \mathbb{R}^{2n} \times \mathbb{R}^{2n}$  for all  $i \in \mathbb{N}$  of the form (5.28), that is

$$\tilde{P}_i = \begin{bmatrix} X & V \\ V' & \hat{X}_i \end{bmatrix}, \quad \det V \neq 0 \quad (5.35)$$

and define the nonsingular matrices  $\tilde{T}_i \in \mathbb{R}^{2n} \times \mathbb{R}^{2n}$  as

$$\tilde{T}_i = \begin{bmatrix} I_n & I_n \\ -\hat{X}_i^{-1} V' & 0 \end{bmatrix} \quad (5.36)$$

for all  $i \in \mathbb{N}$ . Therefore, there exist positive definite matrices  $\tilde{P}_i$ ,  $i \in \mathbb{N}$  satisfying the Lyapunov-Metzler inequalities (5.30) if and only if

$$\tilde{S}_i := \tilde{T}_i' \left( \tilde{A}'_i \tilde{P}_i + \tilde{P}_i \tilde{A}_i + \sum_{j=1}^N \pi_{ji} \tilde{P}_j + \tilde{Q}_i \right) \tilde{T}_i < 0 \quad (5.37)$$

for all  $i \in \mathbb{N}$ . Introducing a new one-to-one set of variables, namely

$$Z_i := X - V \hat{X}_i^{-1} V' \quad (5.38)$$

$$L_i := V \hat{B}_i \quad (5.39)$$

$$M_i := V \hat{A}_i V^{-1} (X - Z_i) \quad (5.40)$$

each term of the matrix sum appearing in the left hand side of inequality (5.37) can be expressed as follows

$$\tilde{T}_i' \tilde{P}_i \tilde{A}_i \tilde{T}_i = \begin{bmatrix} Z_i A_i & Z_i A_i \\ X A_i + L_i C_i - M_i & X A_i + L_i C_i \end{bmatrix} \quad (5.41)$$

$$\tilde{T}_i' \left( \sum_{j=1}^N \pi_{ji} \tilde{P}_j \right) \tilde{T}_i = \sum_{j=1}^N \pi_{ji} \begin{bmatrix} Z_j + (Z_j - Z_i)(X - Z_j)^{-1}(Z_j - Z_i) & 0 \\ 0 & 0 \end{bmatrix} \quad (5.42)$$

$$\tilde{T}_i' \tilde{Q}_i \tilde{T}_i = \begin{bmatrix} Q_i & Q_i \\ Q_i & Q_i \end{bmatrix} \quad (5.43)$$

where we have used the fact that, from (5.38) we have  $\hat{X}_i = V'(X - Z_i)^{-1}V$  and that  $\Pi \in \mathcal{M}_c$ . Using the Schur Complement to (5.35) it is seen that  $\tilde{P}_i > 0$  if and only if  $X > Z_i > 0$  which indicates that  $\hat{X}_i$  is well defined. Hence defining  $Y_{ij} := Z_j + (Z_j - Z_i)(X - Z_j)^{-1}(Z_j - Z_i)$  for all  $i, j \in \mathbb{N} \times \mathbb{N}$  we obtain

$$\tilde{S}_i = \begin{bmatrix} A_i' Z_i + Z_i A_i + \sum_{j=1}^N \pi_{ji} Y_{ij} + Q_i & \bullet \\ A_i' Z_i + X A_i + L_i C_i + Q_i - M_i & A_i' X + X A_i + L_i C_i + C_i' L_i' + Q_i \end{bmatrix} \quad (5.44)$$

Let us assume that inequalities (5.32a)-(5.32c) are satisfied. Since the linear matrix inequalities (5.32c) imply that  $X > Z_j > 0$  for all  $j \in \mathbb{N}$ , selecting any nonsingular matrix  $V \in \mathbb{R}^{n \times n}$  and setting  $\hat{X}_i = V'(X - Z_i)^{-1}V$  we get  $\tilde{P}_i > 0$  for all  $i \in \mathbb{N}$ . In addition, applying the Schur Complement to (5.32c) it is immediately verified that  $\sum_{j=1}^N \pi_{ji} R_{ij} > \sum_{j=1}^N \pi_{ji} Y_{ij}$  so that the first block diagonal element of  $\tilde{S}_i$  is negative definite as a consequence of (5.32a). Due to (5.32b), the second block diagonal element of matrix  $\tilde{S}_i$  is also negative. Consequently, imposing  $M_i = A_i' Z_i + X A_i + L_i C_i + Q_i$  we conclude that  $\tilde{S}_i < 0$ . Hence, determining the switched filter matrices  $\hat{B}_i$  and  $\hat{A}_i$  from (5.39) and (5.40) the augmented Lyapunov-Metzler inequalities (5.30) hold.

Vice-versa, assume that the inequalities (5.30) hold for some positive definite matrix  $\tilde{P}_i$  of the form (5.28) and matrices  $\hat{B}_i, \hat{A}_i$  of the switched filter. Adopting the change of variables introduced in equations (5.38)-(5.40) it is immediately verified that  $\tilde{S}_i < 0$  for all  $i \in \mathbb{N}$ . As a consequence, the linear matrix inequalities (5.32b) are verified. On the other hand, letting  $R_{ii} = Y_{ii} - \epsilon I_n$  and  $R_{ij} = Y_{ij} + \epsilon I_n$  with  $\epsilon > 0$  small enough, the linear matrix inequalities (5.32c) are verified and inequalities (5.32a) hold due to the fact that the first block diagonal element of  $\tilde{S}_i$  is negative definite.

To conclude the proof notice that the stabilizing property of the output feedback switching rule (5.33) is a consequence of Theorem 13 and the determination of matrices  $\hat{X}_i$  for all  $i \in \mathbb{N}$ , as indicated before. Once again, from Theorem 13 the cost associated to this control policy is  $J(u) = \min_{i \in \mathbb{N}} \text{Tr}(\tilde{B}_\ell' \tilde{P}_i \tilde{B}_\ell)$  which can be rewritten as  $J(u) = \min_{i \in \mathbb{N}} \text{Tr}(W_i)$  with the additional matrix variable  $W_i$  satisfying  $W_i > \tilde{B}_\ell' \tilde{P}_i \tilde{B}_\ell$  for all  $i \in \mathbb{N}$ . Using the Schur Complement, the equivalent inequalities

$$\begin{bmatrix} W_i & \tilde{B}_\ell' \tilde{P}_i \tilde{T}_i \\ \bullet & \tilde{T}_i' \tilde{P}_i \tilde{T}_i \end{bmatrix} > 0 \quad (5.45)$$

for all  $i \in \mathbb{N}$  provide (5.34). This concludes the proof of the proposed theorem.  $\blacksquare$

Whenever  $\Pi \in \mathcal{M}_c$  is fixed, the matrix inequalities (5.32) and (5.34) reduces to LMIs and so can be solved with no difficulty by the machinery available in the literature to date. Another possibility is to restrict the set of Metzler matrices to those with the same diagonal elements. In this case, Theorem 8 applies from which a simplified version of Theorem 15, expressed by LMIs and an additional scalar, follows. Calling  $\Phi(\Pi)$  the set of all variables satisfying the LMIs (5.32) and (5.34), the determination of the best output feedback switching control is done from the solution of the optimization problem

$$\min_{i \in \mathbb{N}} \left\{ \min_{Z_i, R_{ij}, L_i, W_i, X \in \Phi(\Pi)} \text{Tr}(W_i) \right\} \quad (5.46)$$

where the inner problem is convex. Once it is solved for each  $i \in \mathbb{N}$ , the global (discrete) minimization with respect to  $i \in \mathbb{N}$  is then performed. Since the index  $\ell = \sigma(0)$  may be defined by the designer, it can be involved in the optimization process. However, keeping in mind problem (5.46) it appears that a good choice would be  $\ell = i \in \mathbb{N}$  being thus determined by the outer optimization problem.

After the determination of the involved matrix variables, the filter matrices are readily calculated from the simple formulas

$$\hat{B}_i = V^{-1}L_i \quad (5.47a)$$

$$\hat{A}_i = V^{-1}M_i(X - Z_i)^{-1}V \quad (5.47b)$$

where  $M_i := A_i'Z_i + XA_i + L_iC_i + Q_i$  for all  $i \in \mathbb{N}$ . At this point it is clear that the nonsingular matrix  $V$  defines a particular state space realization of the switched linear filter making invariant the output feedback switching rule. In other words, Theorem 15 provides a parametrization of all feasible filters with  $\hat{P}_i$  for all  $i \in \mathbb{N}$  presenting the prescribed block structure (5.28).

The full-order filter is not in the observer form, i.e.  $\hat{A}_i \neq A_i - \hat{B}_iC_i$ . To recover this condition, an additional constraint, unfortunately non linear, has to be added (the simple check is left to the reader)

$$\begin{aligned} M_i &= (VA_i - L_iC_i)V^{-1}(X - Z_i) \\ &= A_i'Z_i + XA_i + L_iC_i + Q_i \end{aligned} \quad (5.48)$$

A notable exception can be devised by letting  $Q_i = 0$ , so overlooking the cost associated to the controlled output variable  $z(t), \forall t \geq 0$ . Indeed, in this particular but important case, we have the following result.

**Corollary 2** *Assume that there exist a Metzler matrix  $\Pi \in \mathcal{M}_c$ , a positive definite matrix  $X$ , a set of positive matrices  $Z_i$  and a set of matrices  $L_i$  for all  $i \in \mathbb{N}$ , such that the following matrix inequalities*

$$A_i'Z_i + Z_iA_i + \sum_{j=1}^N \pi_{ji}Z_j < 0 \quad (5.49a)$$

$$A_i'X + XA_i + C_i'L_i' + L_iC_i < 0 \quad (5.49b)$$

are satisfied. The output feedback switching control  $\sigma(t) = u(\hat{x}(t))$  defined by

$$u(\hat{x}) = \arg \min_i \hat{x}'Z_i\hat{x} \quad (5.50)$$

makes the equilibrium solution  $x = 0$  of (5.26a) globally asymptotically stable where  $\hat{x}(t)$  satisfies the differential equation of the filter (5.25) in observer form with

$$\hat{B}_i = -X^{-1}L_i \quad (5.51a)$$

$$\hat{A}_i = A_i - \hat{B}_iC_i \quad (5.51b)$$

**Proof** The proof relies to Theorem 15, by letting  $Z_i \rightarrow \epsilon Z_i$  with  $\epsilon > 0$  arbitrarily small and  $V = -X$  yielding  $R_{ij} \rightarrow Z_j$  for all  $i, j \in \mathbb{N} \times \mathbb{N}$ . Indeed, notice that the condition (5.48) for the filter to be in observer form is satisfied for  $\epsilon$  going to zero and that

$$\begin{aligned} &\arg \min_{i \in \mathbb{N}} \hat{x}'V'(X - \epsilon Z_i)^{-1}V\hat{x} = \\ &\arg \min_{i \in \mathbb{N}} \hat{x}'(X + (Z_i^{-1}/\epsilon - X^{-1})^{-1})\hat{x} \sim \\ &\arg \min_{i \in \mathbb{N}} \hat{x}'\epsilon Z_i\hat{x} \sim \arg \min_{i \in \mathbb{N}} \hat{x}'Z_i\hat{x} \end{aligned} \quad (5.52)$$

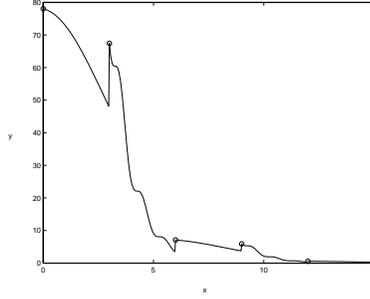


Figure 5.2: Time simulation.

holds.

The conclusion is that if there exist  $N$  gains that make the filter quadratically stable, see equation (5.49b), then the usual solution to the Metzler-Lyapunov inequalities (see the state feedback, equation (5.49a)) provides an output feedback stabilizing switching rule calculated from the state variable of the observer. It is important to keep in mind that if we want to determine a switching strategy by minimizing the cost  $J(u)$  then this solution although stabilizing is not the best that can be done. Moreover, it should be noticed that the output feedback strategies invoked by the theorems presented so far require the existence of state-observer injection matrices  $\hat{L}_i = X^{-1}L_i$ ,  $i \in \mathbb{N}$  that render the set of matrices  $A_i + \hat{L}_i C_i$  quadratically stable (see e.g. equation (5.49a)).

**Remark 2** *It is important to stress that there is no difficulty to get the version of Theorem 15 associated to the modified Lyapunov-Metzler inequalities appearing in Theorem 8. The bilinear matrix inequalities are replaced by LMIs with an additional parameter that can be determined by line search. The results follow the same pattern of each mentioned theorem and corollary, being thus omitted.*

**Example 5** *Consider a continuous time switched linear system (5.24a)-(5.24c) defined by matrices  $A_1$ ,  $A_2$ ,  $Q_1$ ,  $Q_2$  and  $B$  given in Example 4. We have considered  $D = [1 \ 1]$  and different measurements for each one of the two modes defined by  $C_1 = [1 \ -1 \ 0]$  and  $C_2 = [1 \ 0 \ 0]$ . The Metzler matrix has been set as*

$$\Pi = \begin{bmatrix} -100 & 20 \\ 100 & -20 \end{bmatrix} \in \mathcal{M}_c \quad (5.53)$$

*The optimal filter and the associated output feedback switching control have been determined from the solution of the convex programming problem (5.46) with  $\ell = i \in \mathbb{N}$ , yielding  $J(u) = 12.9725$ . Each subplot in Figure 5.2 shows in solid line the time evolution of the state variables of the system and in dashed line the time evolution of the state variables of the filter. From  $t \in [0, 10)$  we have imposed the constant output switching control  $\sigma(t) = 1$ . It is clear that both the system and the filter are unstable. At  $t = 10$  the output feedback switching control is connected and the closed loop system (and the filter) converge to zero, showing that the proposed control is actually effective for stabilization.*

## 5.4 Practical Application

This section discusses a practical application of the output feedback switching control design presented in Section 4.3.3. The problem consists in the design of a switching control strategy for comfort-oriented semi-active suspensions in road vehicles, and is motivated by the paper [43] where the so-called sky-hook (SH) approach is introduced and the recent paper [29], where a new strategy, henceforth referred to as ADD (Acceleration Driven Damper) strategy, is proposed that improves on SH in certain frequency ranges of the road profile disturbance. The model is as follows:

$$\begin{aligned} M\ddot{\xi}(t) &= -c(t)(\dot{\xi}(t) - \dot{\xi}_t(t)) - k(\xi(t) - \xi_t(t)) + k\Delta_s - Mg \\ m\ddot{\xi}_t(t) &= c(t)(\dot{\xi}(t) - \dot{\xi}_t(t)) + k(\xi(t) - \xi_t(t)) - k_t(\xi_t(t) - \xi_r(t)) - k\Delta_s + k_t\Delta_t - mg \\ \dot{c}(t) &= -\beta c(t) + \beta c_{in}(t) \end{aligned}$$

where  $\xi(t)$ ,  $\xi_t(t)$  and  $\xi_r(t)$  are the vertical position of the body, the unsprung mass and the road profile, respectively. The coefficients  $M$  and  $m$  are the quarter-car body mass and the unsprung mass (tire, wheel, brake, etc...), respectively. The coefficients  $\beta$ ,  $k$  and  $k_t$  are the bandwidth of the active shock absorber, the stiffness of the suspension spring and of the tire, respectively. The coefficients  $\Delta_s$  and  $\Delta_t$  are the length of the unloaded suspension spring and of the tire. Finally,  $c(t)$  and  $c_{in}(t)$  are the actual and requested damping coefficients of the passive shock-absorber. In order to simplify the computations we assume that  $\beta$  is large enough so that  $c(t) \sim c_{in}(t)$ . Moreover we consider a genuine switching strategy, so that  $c(t)$  can assume only two values, namely  $c_{min}$  and  $c_{max}$ , to be specified later on.

The control strategy consists in minimizing the chassis vertical acceleration  $\ddot{\xi}(t)$  by a suitable choice of the control variable  $c(t) \in \{c_{min}, c_{max}\}$ . In the classical two-state SH approach [43], the system is switched according to the sign of  $\dot{\xi}(t)(\dot{\xi}(t) - \dot{\xi}_t(t))$ , whereas in [29] the switching law depends on the sign of  $\ddot{\xi}(t)(\dot{\xi}(t) - \dot{\xi}_t(t))$ .

In order to fit this example in the framework of the present paper, let us take the variations  $\delta\xi(t)$  and  $\delta\xi_t(t)$  of  $\xi(t)$  and  $\xi_t(t)$  around an equilibrium point associated with zero road profile, arriving to the system

$$\begin{aligned} \ddot{\bar{\xi}}(t) &= A_\sigma \bar{\xi}(t) + B_r \xi_r(t) \\ y(t) &= C_\sigma \bar{\xi}(t) + d(t) \\ z(t) &= E_\sigma \bar{\xi}(t) \end{aligned}$$

where  $d(t)$  is the measurement noise and

$$\begin{aligned} A_1 &= \begin{bmatrix} 0 & 1 & 0 & 0 \\ -k/M & -c_{min}/M & k/M & c_{min}/M \\ 0 & 0 & 0 & 1 \\ k/m & c_{min}/m & -(k+k_t)/m & -c_{min}/m \end{bmatrix} \\ A_2 &= \begin{bmatrix} 0 & 1 & 0 & 0 \\ -k/M & -c_{max}/M & k/M & c_{max}/M \\ 0 & 0 & 0 & 1 \\ k/m & c_{max}/m & -(k+k_t)/m & -c_{max}/m \end{bmatrix} \\ E_1 &= \begin{bmatrix} -k/M & -c_{min}/M & k/M & c_{min}/M \end{bmatrix} \\ E_2 &= \begin{bmatrix} -k/M & -c_{max}/M & k/M & c_{max}/M \end{bmatrix} \\ B_r &= \begin{bmatrix} 0 \\ 0 \\ 0 \\ k_t/m \end{bmatrix} \end{aligned}$$

and  $C_\sigma$  depends on the choice of the measured variable. The state vector  $\bar{\xi}(t)$  contains the chassis displacement  $\delta\xi(t)$ , its derivative, the tire displacement  $\delta\xi_t(t)$  and its derivative. Again, the disturbance vector  $\xi_r(t)$  is the road profile. One reasonable set of measurements is given by the stroke  $\xi(t) - \xi_t(t)$  and its derivative, leading two

$$C_1 = C_2 = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix} \quad (5.54)$$

We also consider the alternative choice

$$C_1 = \begin{bmatrix} -k/M & -c_{min}/M & k/M & c_{min}/M \\ 0 & 1 & 0 & -1 \end{bmatrix} \quad (5.55)$$

$$C_2 = \begin{bmatrix} -k/M & -c_{max}/M & k/M & c_{max}/M \\ 0 & 1 & 0 & -1 \end{bmatrix} \quad (5.56)$$

that corresponds to measuring the body acceleration and the stroke derivative.

In the following we apply the state-feedback and output feedback stabilization strategy to the suspension system in order to minimize the  $\mathcal{L}_2$  norm of the chassis acceleration  $\ddot{\xi}(t)$  with respect to impulsive signals on the road profile acceleration  $\ddot{\xi}_r(t)$ . This is indeed a realistic situation including road profiles described by ramps, in the deterministic setting, or double integral of a white noise, in the stochastic case.

Consequently, we have to rewrite the model in order to fit in the formulation given in (5.24a)-(5.24c), in which

$$w(t) = \begin{bmatrix} \ddot{\xi}_r(t) \\ d(t) \end{bmatrix}$$

and  $z(t) = \ddot{\xi}(t)$ . To do that, define

$$\begin{aligned} x_1(t) &= \xi(t) - \xi_r(t) \\ x_2(t) &= \dot{\xi}(t) - \dot{\xi}_r(t) \\ x_3(t) &= \xi_t(t) - \xi_r(t) \\ x_4(t) &= \dot{\xi}_t(t) - \dot{\xi}_r(t) \end{aligned}$$

With these new variables, the system can be equivalently rewritten as

$$\dot{x}(t) = A_\sigma x(t) + Bw(t) \quad (5.57)$$

$$y(t) = C_\sigma x(t) + Dw(t) + C_\sigma(\bar{\xi}(t) - x(t)) \quad (5.58)$$

$$z(t) = E_\sigma x(t) + E_\sigma(\bar{\xi}(t) - x(t)) \quad (5.59)$$

where  $A_1, A_2, C_1, C_2, E_1, E_2$  have been already defined and

$$B = \begin{bmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} 0 & r_1 & 0 \\ 0 & 0 & r_2 \end{bmatrix}$$

The parameters  $r_1$  and  $r_2$  reflect the measurements uncertainties and are specified later. Notice now that  $E_\sigma(\bar{\xi}(t) - \xi(t)) = 0$  and  $C_\sigma(\bar{\xi}(t) - \xi(t)) = 0$ , for each  $\sigma = 1, 2$  and both choices of the output matrices indicated in (5.54)-(5.56). Therefore system (5.57)-(5.59) is identical to (5.24a)-(5.24c). The output feedback stabilization problem has been solved by taking the following set of parameters:  $M = 400kg$ ,  $m = 50kg$ ,  $k = 2.0 \times 10^4 N/m$ ,  $k_t = 2.5 \times 10^5 N/m$ ,  $c_{min} = 3.0 \times 10^2 Ns/m$  and  $c_{max} = 3.9 \times 10^3 Ns/m$ . For these parameters the two matrices

	OF <sub>1</sub>	OF <sub>2</sub>	SF	SH	ADD	PS <sub>1</sub>	PS <sub>2</sub>
$\int_0^\infty \ddot{\xi}(t)^2 dt$ for $\ddot{\xi}_r(t) = \delta(t)$	7.767	7.835	7.721	8.288	8.150	26.548	8.307
$\frac{\int_0^T \ddot{\xi}(t)^2 dt}{\int_0^T \ddot{\xi}_r(t)^2 dt}$ for $T = 20$	0.718	0.697	0.643	0.787	0.823	3.558	0.719

Table 5.1: Performance of closed loop strategies

$A_1$  and  $A_2$  are both stable (although with poorly damped oscillating modes) hence, our main scope here is to improve the transient dynamical behavior of the system by minimizing the vertical acceleration of the chassis.

Two sets of simulations have been carried out. The first set refers to the response of  $\ddot{\xi}(t)$  to a unit impulse on the road acceleration  $\ddot{\xi}_r(t)$ . The first row of Table 1 reports the integral of the squared chassis acceleration obtained with different control strategies. The symbols in the table have the following meaning:

- OF<sub>1</sub> : Output feedback switching control designed with the output matrices of equation (5.54).
- OF<sub>2</sub> : Output feedback switching control designed with the output matrices of equations (5.55)-(5.56).
- SF : State-feedback switching control.
- SH : Two-state sky-hook strategy.
- ADD : Acceleration-driven damper strategy with sampling period  $\delta_T = 10^{-3}sec$ .
- PS<sub>1</sub> : Passive suspension with fixed damping coefficient equal to  $c_{min}$ .
- PS<sub>2</sub> : Passive suspension with fixed damping coefficient equal to  $c_{max}$ .

The design OF<sub>1</sub> and OF<sub>2</sub> depend on the tuning parameters  $r_1$ ,  $r_2$  and  $\Pi$ , that have been optimized after a limited number of trials. The resulting tuning parameters in OF<sub>1</sub> are

$$r_1 = 0.1, r_2 = 0.5, \Pi = \begin{bmatrix} -1000 & 1000 \\ 1000 & -1000 \end{bmatrix}$$

and in OF<sub>2</sub> are

$$r_1 = 2.0, r_2 = 0.5, \Pi = \begin{bmatrix} -100 & 10 \\ 100 & -10 \end{bmatrix}$$

Finally, the parameter  $\Pi$  for SF has been selected as in OF<sub>1</sub>. As apparent from Table 1, the algorithm OF<sub>1</sub> outperforms all other strategies based on incomplete measurements. Remarkably, the difference between the outcomes of OF<sub>1</sub> and SF is relatively small. By the way, the state-feedback performance is quite close to that obtained by applying the theoretical optimal switching strategy corresponding to  $k_t \rightarrow \infty$ , see [42].

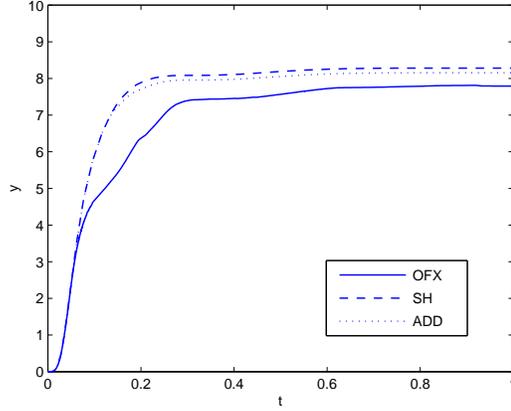


Figure 5.3: Time history of the integral of  $\ddot{\xi}(t)^2$  due to an impulse of  $\ddot{\xi}_r(t)$ .

Figure 5.3 shows the integral of the square of chassis acceleration against time. It can be seen that  $OF_1$  is capable of lowering the acceleration in the transient better than the other methods.

In the second set of simulations the road profile  $\xi_r(t)$  has been generated as the double integral of a sample realization of a white noise process with power  $\chi^2 = 0.1$ . The performance of the seven algorithms above, with the same values of the tuning parameters, has been measured as the power attenuation on the chassis acceleration, namely the ratio

$$\Theta_T = \frac{\int_0^T \ddot{\xi}(t)^2 dt}{\int_0^T \ddot{\xi}_r(t)^2 dt}$$

This value, for  $T = 20$  sec., is reported in the second row of Table 1. The relative ranking of the proposed algorithms is in good agreement with the indices shown before, the only difference being the slight improvement of  $OF_2$  with respect to  $OF_1$ .

Figure 5.4 shows the behavior of the acceleration for the three methods  $OF_2$ , SH and ADD. The plot has been restricted to an interval of 2 seconds, in order to better represent the effects of the commutations in the three methods. The  $OF_2$  strategy outperforms the other two algorithms at the price of faster switching commutation and shorter dwell intervals.

Finally the power attenuation  $\Theta_T$  as a function of  $T$  is plotted in Figure 5.5 to show the effectiveness of the proposed output feedback strategy.

Obviously, the choice of the design parameters (in particular  $\Pi$ ) is still an open issue. As a reasonable guideline, one could exploit the performance bounds discussed in Section 4.3.2 and 5.3. However, it must be stressed that the optimization of the upper bounds with respect to  $\Pi$  does not ensure that the minimum of the real performance is attained.

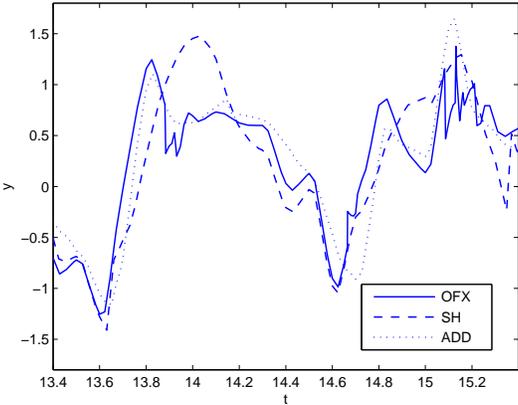


Figure 5.4: Chassis acceleration during a short interval under a random road acceleration.

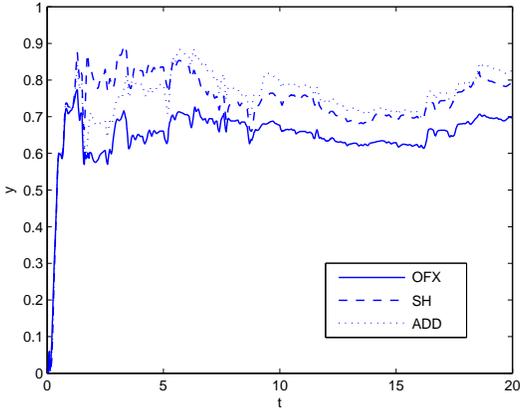


Figure 5.5: Power attenuation under a random road acceleration.



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