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Partial stabilization of input-output contact systems on a Legendre submanifold

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***Abstract*—This paper addresses the structure preserving stabilization by output feedback of conservative input-output contact systems, a class of input-output Hamiltonian systems defined on contact manifolds. In the first instance, achievable contact forms in closed-loop and the associated Legendre submanifolds are analysed. In the second instance the stability properties of a hyperbolic equilibrium point of a strict contact vector field are analysed and it is shown that the stable and unstable manifolds are Legendre submanifolds. In the third instance the consequences for the design of stable structure preserving output feedback are derived: in closed-loop one may achieve stability only relatively to some invariant Legendre submanifold of the closed-loop contact form and furthermore this Legendre submanifold may be used as a control design parameter. The results are illustrated along the paper on the example of heat transfer between two compartments and a controlled thermostat.**

I. INTRODUCTION

In this brief paper we address the stabilization by output-feedback of a class of *input-output contact systems* [22] which are the generalization of input-output Hamiltonian systems defined on contact manifolds. Actually we shall consider a class of these systems, called *conservative input-output contact systems*, in the sense that they leave invariant a Legendre submanifold, often associated with the thermodynamic properties of the system. These nonlinear control systems have been introduced in the context of modelling of open irreversible thermodynamical systems [4] and further developed in [5], [23] [20]. Finally, the analysis of the feedback equivalence to a contact system, in general with respect to a different closed-loop contact form, has led to the definition of input-output contact systems [22].

The main result presented in this paper consists in the *global* stability analysis of the closed-loop contact system obtained by the structure preserving output feedback suggested in [22], yielding the proof that one may achieve only *partial* stability with respect to some Legendre submanifold rendered invariant by the feedback. Therefore it will be proven that a hyperbolic equilibrium point of a strict contact vector field has a stable and unstable submanifold which are Legendre submanifolds, extending the local results in [6], [21], [13]. The paradigmatical example of the 2 cells exchanging heat, the second one connected to a thermostat, is used to illustrate the results along the paper.

The paper is organized as follows. In Section II we recall the definition of conservative input-output contact systems and present the example. The section III consists firstly in a reminder of the family of feedback control which preserve the contact structure along [22], secondly in an analysis of the closed-loop contact form and the associated Lagrangian submanifolds. Section IV presents the main results of the paper, the stability analysis of a hyperbolic equilibrium point of a strict contact vector field and the consequence for the control design. Finally in Section V some closing remarks and lines of future work are given. In the appendix the main definitions related to contact manifolds and contact vector fields are given.

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II. CONSERVATIVE INPUT-OUTPUT CONTACT SYSTEMS

In this section we shall briefly recall the definition and main properties of *input-output contact systems*. Their state space are contact manifolds which arise from the formalization of the geometric structure of the Thermodynamic Phase Space [1], [10] elaborated from Gibbs' geometric definition of Equilibrium Thermodynamics [7]. On this contact manifold reversible and irreversible thermodynamic processes may be described by a dynamical system defined by a contact vector field, leaving invariant the contact geometry [15], [8]. Nonlinear control systems which express the dynamics of open Irreversible Thermodynamic Systems and preserve the contact structure, have been defined as Hamiltonian control systems on contact manifolds first in [4] and further elaborated in [5], [23], [13, sec.3].[1] Input-output contact Hamiltonian systems, which differ from the previous ones by the definition of the output, have been defined in the context of the feedback equivalence preserving the contact structure in [22][2]

A. Input-output contact systems

Consider a differentiable manifold $\mathcal{M}$ equipped with a contact form θ and denote by $\tilde{x} = [x_0, x, p]^\top \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n$ a set of canonical coordinates (the reader is refereed to the Appendix and the references therein for the reminders on contact manifolds and vector fields). On this contact manifold, we shall consider the following class of nonlinear control systems adapted to the contact structure.

Definition 2.1: [20], [22] A *(single) input - (single) output contact system* on the contact manifold $(\mathcal{M}, \theta)$, affine in the scalar input $u \in L_1^{\text{loc}}(\mathbb{R}_+)$ is defined by the two functions $K_0 \in C^\infty(\mathcal{M})$, called the *internal contact Hamiltonian*, $K_c \in C^\infty(\mathcal{M})$ called the *interaction (or control) contact Hamiltonian*, and the state and output equations

$$\frac{d\tilde{x}}{dt} = X_{K_0} + X_{K_c} u, \qquad y = K_c(\tilde{x}) \tag{1}$$

where X_{K_0} and X_{K_c} are the contact vector fields of $(\mathcal{M}, \theta)$ generated by the contact Hamiltonians K_0 and K_c respectively.

Models of irreversible thermodynamic systems belong to a subclass of contact systems [4], [5], [23], called *conservative input-output contact systems*.

Definition 2.2: [4] A *conservative* input-output contact system with respect to the Legendre submanifold $\mathcal{L}$ is an input-output contact system with the internal, respectively control, contact Hamiltonians K_0, respectively K_c, satisfying the two conditions:

(i) they are invariants of the Reeb vector field, satisfying (19)

$$i_E dK_0 = i_E dK_c = 0 \tag{2}$$

(ii) they satisfy the invariance condition (23)

$$K_0\big|_{\mathcal{L}} = 0, \quad K_c\big|_{\mathcal{L}} = 0 \tag{3}$$

The reader may find the detailed justification in the context of Irreversible Thermodynamics and examples in [4], [5], [23]. For the sake of clarity, let us recall some interpretation of these definitions.

[1]The formulation of Thermodynamic systems has led to an extensive publication activity which is not discussed here and in the context of system's theory, we refer the reader to the discussion in [5], [23, sec.3].

[2]They are the analogue of the input-output Hamiltonian systems [2], [26], [24], [12] developed for mechanical systems and defined on *symplectic* manifolds associated with the conjugated position-momentum pairs.

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The condition (2) amounts, in a set of canonical coordinates, to assume that the Hamiltonian functions do not depend on the x_0 coordinate which is the coordinate of the *total* energy (or entropy) of the system and is satisfied for physical models. The condition (3) says that the drift and control vector fields should leave invariant some distinguished Legendre submanifold representing the thermodynamic properties of the physical system and defined by the total energy of the system (or any Legendre transformation of it)[3].

Finally it may be noted that the contact Hamiltonian functions may be constructed as the virtual power associated with the reversible and irreversible phenomena inducing the systems dynamics [4], [23][4].

B. Paradigmatic example: two compartments exchanging heat

We shall illustrate the results of this paper with the example of two compartments only exchanging a heat flow through a heat conducting wall and one of them exchanging a heat flow with the environment (see the detailed presentation in [23]). The Thermodynamic Phase Space $R^5 \ni (x_0, x_1, x_2, p_1, p_2)^\top$, consists of the coordinates of the total internal energy x_0, the coordinates of the entropies $x = (x_1, x_2)^\top$ and the coordinates $p = (p_1, p_2)^\top$ of the temperatures of the two compartments. The thermodynamic properties of the system are described by the total internal energy $U(x_1, x_2) = U_1(x_1) + U_2(x_2)$, sum of the internal energies of each subsystem and satisfied on the Legendre submanifold $\mathcal{L}_U$ generated by U,

$$\mathcal{L}_U : \left\{ \begin{array}{ll} x_0 & = U\left(x_1,\, x_2\right) \\ x & = \left[x_1, x_2\right]^\top \\ p & = \left[\frac{\partial U}{\partial x_1}, \frac{\partial U}{\partial x_2}\right]^\top = \left[T_1\left(x_1\right), T_2\left(x_2\right)\right]^\top \end{array} \right\} \tag{4}$$

where $\frac{\partial U}{\partial x_i} = T_i(x_i)$ are the temperatures of each compartment with $T_i(x_i) = T_0 \exp\left(\frac{x_i}{c_i}\right)$, where T_0 and c_i are some constants [3]. The dynamics of the two compartments may be described by the input-output contact system defined by the internal and control contact Hamiltonians

$$K_0 = -Rp^\top JT - (T_2 - p_2)\,\lambda_e \frac{p_2}{T_2}, \quad K_c = e^{-\lambda_e\left(\frac{p_2}{T_2}-1\right)} - 1 \tag{5}$$

where $\lambda, \lambda_e > 0$ denote Fourier's heat conduction coefficients of the internal and external walls respectively, the controlled input $u(t) \in \mathbb{R}_+$ is the temperature of the external heat source and $R(x,p) = \lambda\left(\frac{p_1 - p_2}{T_1 T_2}\right)$ and $J = \left[\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right]$. The actual physical systems' dynamic is described by the *restriction to the Legendre submanifold* $\mathcal{L}_U$ of the global input-output contact system, where it reduces to the 2-dimensional nonlinear control system consisting of the two entropy balance equations

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \lambda \left(\frac{1}{\frac{\partial U}{\partial x_2}} - \frac{1}{\frac{\partial U}{\partial x_1}} \right) \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{\partial U}{\partial x_1} \\ \frac{\partial U}{\partial x_2} \end{bmatrix} + \lambda_e \begin{bmatrix} 0 \\ \frac{1}{\frac{\partial U}{\partial x_2}} - \frac{1}{u} \end{bmatrix} u. \tag{6}$$

III. Structure preserving feedback and closed-loop invariant Legendre submanifolds

In this section we firstly recall the class of structure preserving feedback of input-output contact systems according to [20], [22] and secondly, we shall analyse some properties of the closed-loop system regarding the closed-loop contact form as well as its Legendre submanifolds. We conclude by illustrating some results on the example of the two cells exchanging heat flows.

A. Structure preserving feedback of input-output contact systems

Let recall briefly the conditions under which the closed-loop system obtained by state-feedback remains a contact vector field defined with respect to a contact form [20], [22].

Proposition 3.1: [20], [22] Consider an input-output contact system according to the Definition 2.1. The closed-loop system $\frac{d\tilde{x}}{dt} = X_{K_0} + \alpha\left(\tilde{x}\right) X_{K_c}$ with state-feedback $\alpha \in C^\infty(\mathcal{M})$, is a contact system defined with respect to the closed-loop contact form θ_d

$$\theta_d = \theta + dF \tag{7}$$

with $F \in C^\infty(\mathcal{M})$ such that $i_E F = 0$, if and only if there exist a real function $\Phi \in C^\infty(\mathbb{R})$ such that the feedback is an output feedback

$$\alpha = \Phi'(y) \tag{8}$$

where Φ' is the derivative of Φ and the following matching equation between the function Φ and F is satisfied[5]

$$X_{K_0}(F) + \Phi'(y)[K_c + X_{K_c}(F)] - \Phi(y) = c_F, \quad c_F \in \mathbb{R} \tag{9}$$

Then the closed-loop vector field $X = X_{K_0} + \Phi'(y) X_{K_c}$ is a strict contact vector field with respect to θ_d generated by the Hamiltonian

$$K = K_0 + \Phi(y) + c_F \tag{10}$$

and will be denoted hereafter by $\hat{X}_K$.

Remark 3.1: In canonical coordinates, the assumption that F is an invariant of the Reeb vector field E, means that, in a set of canonical coordinates, the function F depends only on the variables (x,p) and the closed-loop contact form θ_d admits then canonical coordinates [22] (x_0', x, p) with $x_0' = x_0 + F\left(x, p\right)$.

Notice that, similarly to the case of input-output Hamiltonian systems defined with respect to a Poisson structure [26], [24], the structure preserving feedback is an output feedback, i.e. a function of the control contact Hamiltonian. The (control) function Φ *shapes* the closed-loop contact Hamiltonian in a very similar manner as for the feedback of input-output Hamiltonian systems [26] or the Casimir method for port-Hamiltonian systems [19].

Unlike for input-output Hamiltonian systems, but resembling to the control of port-Hamiltonian systems in the IDA-PBC method [19], [18], the closed-loop vector field is a contact vector field with respect to a *different* geometrical structure, defined by the closed-loop contact form θ_d (7). As in the IDA-PBC method, the geometric structure, i.e. the closed-loop contact structure (7) and the closed-loop contact Hamiltonian (10) are related by a matching equation. This matching equation (9) may equivalently be written as

$$\left\langle X_{K_0} + (\Phi' \circ K_c) X_{K_c}\,,\, dF \right\rangle + (\Phi' \circ K_c) K_c - \Phi \circ K_c = 0, \tag{11}$$

where $\langle\,,\,\rangle$ denotes the pairing between vector fields and 1-forms on $\mathcal{M}$. Choosing a function Φ (that is using it as *the control design parameter*), the matching equation defines a linear first-order PDE in the function F, which may be treated using classical methods such as the method of characteristics [17].

[3]Some authors consider nonlinear control systems which do leave invariant the Legendre submanifold defining the thermodynamic properties of the system but are *not* defined by contact vector fields and do not leave invariant the contact structure [9].

[4]There exist an alternative representation where the total entropy function of the system is used to define the Thermodynamic properties and the contact Hamiltonian has the dimension of a rate of entropy [5], [6].

[5]Note that the constant c_F is actually an integration constant [22] and may be arbitrarily chosen.

B. Characterization of the closed-loop system

In the first instance, we shall prove that, while the open- and closed-loop contact forms are different, their Reeb vector fields are equal.

Corollary 3.2: Under the assumptions of Proposition 3.1, the Reeb vector field E associated with the open loop contact form θ is equal to the Reeb vector field associated with the closed-loop contact form θ_d defined in (7).

Proof: Remind that it has been proven in [22] that if F is an invariant of the Reeb vector field E of θ, then the closed-loop contact form θ_d, defined in (7), is a contact form. Note firstly that the fact that the difference of the contact form is exact: $\theta_d - \theta = dF$, implies that $d\theta_d = d\theta$ and hence $i_E d\theta_d = i_E d\theta = 0$ by (17). Secondly note that, as F is an invariant of the Reeb vector field E (i.e. satisfies (19)), one has $i_E \theta_d = i_E (\theta + dF) = i_E \theta = 1$, hence, according to Definition A.2, the vector field E is also the Reeb vector field of the closed-loop contact form θ_d. ∎

In the second instance, we shall show that, under the assumptions of the Proposition 3.1, the Legendre submanifolds for the open- and the closed-loop contact forms may be different. Therefore we use the Definition A.4 which defines Legendre submanifolds using the field of contact elements.

Proposition 3.3: Under the conditions of Proposition 3.1 and assuming that $\ker dF$ is of dimension $2n$ almost everywhere, the fields of contact elements[6] are different in open- and closed-loop: $\ker \theta_d \neq \ker \theta$.

Proof: Indeed, from the definition of the closed-loop contact form (7), for any vector field X of $\mathcal{M}$: $\theta_d(X) = \theta(X) + i_X dF$ Hence, $\ker \theta_d = \ker \theta$ implies $\ker \theta \subset \ker dF$. With the regularity condition on the function F that $\ker dF$ is of dimension $2n$ almost everywhere, then $\ker \theta = \ker dF$ which contradicts the fact that the Reeb vector field satisfies $i_E \theta = 1$ according to the Definition A.2 and that $i_E dF = 0$ by the assumptions of the Proposition 3.1. ∎

On the other hand, the intersection of the distribution $\ker \theta \cap \ker \theta_d$ is a distribution of dimension $(2n-1)$ at least, and from the preceding proposition it is hence exactly of dimension $(2n-1)$. As a consequence, it is the generic case that the open-loop Legendre submanifold $\mathcal{L}$ of a conservative contact system is also a Legendre submanifold of the closed-loop contact form (7)[7]. In conclusion, according to the Proposition 3.1, a structure preserving feedback control (8) is an output feedback of the input-output contact system and is parametrized by a real function Φ which shapes the closed-loop contact Hamiltonian (10). If moreover, one desires that the closed-loop system is a *conservative* contact system (as physical models are), then the function Φ should be such that the closed-loop contact Hamiltonian (10) vanishes on some desired Legendre submanifold with respect to the closed-loop contact form (7).

C. Paradigmatic example: structure preserving output feedback

Following the Proposition 3.1, any structure preserving feedback is expressed as the output feedback (8). Using the expression of the control contact Hamiltonian in (5), it may be seen that any structure preserving output feedback $\alpha = \Phi'(y)$ is a function of (x_2, p_2) only and is hence only a function of the co-state p_2 and the temperature of the second compartment which is the only one in direct contact with the environment. In the sequel we shall express the output feedback as follows:

$$\alpha(x_2, p_2) = \Phi' \circ K_c(x_2, p_2) \doteq \beta\left(\lambda_e \frac{p_2 - T_2}{T_2}\right) \tag{12}$$

[6]see the Definition A.4

[7]Then, according to the expression of the closed-loop contact Hamiltonian (10), the invariance condition (23) implies that $\Phi(0) = -c_F$.

with the real function $\beta(\xi) = \Phi'\left(e^{\xi} - 1\right)$. Assume moreover that there exists a function Φ which satisfies the conditions of Proposition 3.1 and such that the closed-loop system leaves invariant some Legendre submanifold $\mathcal{L}_{U_d}$ defined with respect to some generating function $U_d(x)$ in some canonical coordinates of the closed-loop contact form (7) (see remark 3.1). Then, considering the *restriction of the closed-loop system to the invariant Legendre submanifold* $\mathcal{L}_{U_d}$, the feedback (12) becomes a function of the two extensive variables (the entropy variables) only:

$$u(x_1, x_2) = \alpha(x_2, p_2)\big|_{p_2 = \frac{\partial U_d}{\partial x_2}(x_1, x_2)} = \beta\left(\lambda_e \frac{\frac{\partial U_d}{\partial x_2}(x_1, x_2) - T_2(x_2)}{T_2(x_2)}\right) \tag{13}$$

which may be interpreted as a nonlinear function of a "virtual" entropy flux into the compartment 2 induced by a control temperature $\frac{\partial U_d}{\partial x_2}(x_1, x_2)$ defined by the closed-loop Legendre submanifold. Assigning the closed-loop Legendre submanifold may be interpreted as shaping the *apparent* thermodynamic properties of the system composed of the 2 compartments and the thermostat with control temperature *in closed-loop with* the state-feedback (13).

IV. Partial stabilization by feedback equivalence to a conservative contact system

This section presents the main result of this paper, namely the mathematical justification of the control objectives presented in [6], [21], that in closed-loop the system is a conservative system which is stable on some invariant Legendre submanifold. The section mainly consists in deriving some mathematical results on the stability of equilibria of strict contact vector fields, needed to determine *stabilizing* structure preserving controllers of input-output contact systems. In the first instance we recall and precise some results on the equilibrium points of a contact system and in the second instance we present novel results on their stability properties. Finally we conclude with the consequences for the control objectives for structure preserving control of input-output contact systems.

A. Equilibria of a contact system and restriction of the contact system

Let us first recall the conditions for the existence of an equilibrium point of a strict contact vector field.

Proposition 4.1: [11, p. 322] Consider a contact manifold $(\mathcal{M}, \theta)$ and a strict contact vector field X_K generated by the contact Hamiltonian $K \in C^\infty(\mathcal{M})$. Then a point $\tilde{x}^* \in \mathcal{M}$ is an equilibrium point of the contact system defined by

$$\frac{d}{dt}\tilde{x} = X_K(\tilde{x}) \tag{14}$$

if and only if it satisfies $K(\tilde{x}^*) = 0$ and $dK|_{\tilde{x}^*} = 0$.

Remark 4.1: In [6] the Proposition 4.1 was expressed in canonical coordinates using the expression (20). In a set of canonical coordinates $(x_0, x, p) \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n$, a point (x_0^*, x^*, p^*) is an equilibrium point of a *strict* contact vector field if and only if it is a zero and a critical point of the contact Hamiltonian K, that is, satisfies: $K(x_0^*, x^*, p^*) = 0$ and $\frac{\partial K}{\partial x}(x_0^*, x^*, p^*) = \frac{\partial K}{\partial p}(x_0^*, x^*, p^*) = 0$.

Now assuming that there exists a zero of the vector field, implies that the set $S = K^{-1}(0)$ is not empty. Furthermore all equilibria of the contact system (14) belong to S. In the sequel we shall make the following regularity assumption on the contact Hamiltonian K.

Assumption 1: The set $S = K^{-1}(0)$ is a differentiable manifold of constant dimension $2n$.

In the sequel we shall prove that the contact vector field X_K leaves invariant the submanifold $S = K^{-1}(0)$, as a particular case of the following result.

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Lemma 4.2: The contact vector field X_K with contact Hamiltonian K being an invariant of the Reeb vector field, leaves any submanifold $K^{-1}(c)$, $c \in \mathbb{R}$ invariant.

Proof: Indeed using the Definition A.3 of the Jacobi bracket and its expressions in (21), one has

$$L_{X_K} K = i_{X_K} dK = [K, K]_\theta + K\, i_E dK = K\, i_E dK$$

the contact Hamiltonian is an invariant of the Reeb vector field, that is: $i_E dK = 0$, hence $L_{X_K} K = 0$. The contact Hamiltonian is an invariant of the contact system, hence the submanifold $S = K^{-1}(c)$ too. ■

Remark 4.2: Note also that according to the definition of conservative contact vector fields (i.e. leaving invariant a Legendre submanifold $\mathcal{L}_U$) the Proposition A.4 states that $\mathcal{L}_U \subset S$.

It follows that an equilibrium point of a strict contact vector field (14) cannot be asymptotically stable. Therefore in the sequel we shall discuss its stability relatively to the restriction of the contact vector field to the submanifold $S = K^{-1}(0)$.

B. About the stability of equilibria points relatively to $S = K^{-1}(0)$

In this subsection, we shall analyze the stability properties of the restriction, denoted by $X_K|_S = \bar{X}_K$, of the contact vector field X_K to the submanifold $S = K^{-1}(0)$.

Proposition 4.3: Let $\tilde{x}^* \in S$ be an hyperbolic critical point of the restriction $\bar{X}_K$ of the strict contact vector field on the submanifold $S = K^{-1}(0)$. The stable manifold $S^+(\{\tilde{x}^*\})$ and the unstable manifold $S^-(\{\tilde{x}^*\})$ are Legendre submanifolds of $(\mathcal{M}, \theta)$.

Proof: By assumption, the vector field $\bar{X}_K$ is complete. Therefore denoting by φ_t the integral flow of $\bar{X}_K$ one has

$$\begin{aligned} &\theta(s)\left(\bar{X}_K(s)\right) = \theta(\varphi_t(s))\left(\varphi_t(s)_* \bar{X}_K(s)\right) \\ &\forall s \in S^+(\{\tilde{x}^*\}),\ t \in \mathbb{R}_+. \end{aligned}$$

As the vector field $\bar{X}_K$ generates an orbit converging to the equilibrium point $\tilde{x}^*$, $\lim_{t\to+\infty} \varphi_t(s)_* \bar{X}_K(s) = 0$, hence

$$\theta(s)\left(\bar{X}_K(s)\right) = 0,\ \forall s \in S^+(\{\tilde{x}^*\}),\ \bar{X}_K(s) \in T_s S^+(\{\tilde{x}^*\}) \tag{15}$$

As a consequence, the stable manifold $S^+(\{\tilde{x}^*\})$ is an integral manifold of θ and, according to Proposition A.3, is of dimension less or equal than n. It may be shown with similar arguments (but reversing the time limit) that the unstable manifold $S^-(\{\tilde{x}^*\})$ also satisfy (15), hence is an integral manifold of θ and has dimension less or equal than n. As the equilibrium point is assumed to be hyperbolic, the stable and unstable submanifolds have complementary dimensions in $S = K^{-1}(0)$ which by assumption has dimension $2n$. Hence both the stable and unstable submanifolds have the maximal dimension n and, according to Proposition A.3, are Legendre submanifolds. ■

This proposition is the global version of the local stability results given in [6], [21] by considering the Jacobian DX_K of the strict contact field X_K at an equilibrium point expressed in some canonical coordinates.

Lemma 4.4: Under the conditions of Proposition 4.1, let us consider an equilibrium state (x_0^*, x^*, p^*) of the contact vector field X_K. Then zero is eigenvalue of DX_K and the remaining $2n$ eigenvalues are symmetrical with respect to the imaginary axis.

Proof: Since $\frac{\partial K}{\partial x_0}(x_0, x, p) = 0$, the Jacobian is given by

$$DX_K = \begin{bmatrix} 0 & \left(\frac{\partial^\top K}{\partial x} - p^{*\top} \frac{\partial^2 K}{\partial x \partial p}\right) & -p^{*\top} \frac{\partial^2 K}{\partial p^2} \\ 0 & -\frac{\partial^2 K}{\partial x \partial p} & -\frac{\partial^2 K}{\partial p^2} \\ 0 & \frac{\partial^2 K}{\partial x^2} & \left(\frac{\partial^2 K}{\partial x \partial p}\right)^\top \end{bmatrix}. \tag{16}$$

According to Lemma 4.1, $\frac{\partial K}{\partial x}(x_0^*, x^*, p^*) = 0$ and denoting $A = \frac{\partial^2 K}{\partial x \partial p}(x_0^*, x^*, p^*)$, $B = B^\top = \frac{\partial^2 K}{\partial p^2}(x_0^*, x^*, p^*)$, $C = C^\top = \frac{\partial^2 K}{\partial x^2}(x_0^*, x^*, p^*)$, the characteristic polynomial of DX_K may be evaluated by using cofactor expansion with respect to the first column and the properties of the determinant of block matrices [14].

$$\begin{aligned} \det(DX_K - \lambda I) &= -\lambda \det\left(\begin{bmatrix} -(A + \lambda I) & -B \\ C & (A - \lambda I)^\top \end{bmatrix}\right), \\ &= \lambda \det(A - \lambda I)^\top \det\left((A + \lambda I) - B(A - \lambda I)^{-\top} C\right), \\ &= \lambda \det(A + \lambda I) \det\left((A - \lambda I)^\top - C(A + \lambda I)^{-1} B\right), \end{aligned}$$

where the inverse matrices $(A - \lambda I)^{-1}$ and $(A + \lambda I)^{-1}$ are computed for values of λ different than the eigenvalues of A or their opposite. It follows that $\lambda = 0$ is always an eigenvalue (corresponding to the invariance of the manifold S) and that the remaining $2n$ eigenvalues are symmetrical with respect to the imaginary axis. ■

It may be observed that these results are the extension to contact manifolds of the stability properties of Hamiltonian systems defined on symplectic manifolds [27, Lemma 1], [25, chap. 8].

C. Consequences for structure preserving stabilizing control

The first consequence of the Proposition 4.3 is the a posteriori justification of the stabilization objectives presented in [6], [21]. Indeed it shows that, under the assumptions of Proposition 3.1, nothing more may be achieved in closed-loop by a structure preserving feedback, than that the system is a conservative contact system which is stable only on some invariant Legendre submanifold. Furthermore it shows the existence of such a stable invariant Legendre submanifold for any hyperbolic equilibrium of the closed-lop systems obtained by structure preserving feedback. Note that the restriction of the output feeedback (8) to the stable invariant Legendre submanifold in closed-loop, may then be expressed as a state-feedback of any coordinates of the stable closed-loop submanifold being interpretable as for instance the intensive variables or the extensive variables of the system.

The second and most important consequence is that the control problem is completely parametrized by *the stable Legendre submanifold which may serve as design parameter* for the structure preserving stabilizing control and that *it is totally equivalent to the use of the function* Φ *as design parameter*. This leads to formulate the following control problem.

Definition 4.1: An output feedback of an input-output contact system of Definition 2.1 which is structure preserving according to Proposition 3.1, is said *partially stabilizing* the point $\tilde{x}^* \in \mathcal{M}$ with respect to some Legendre submanifold $\mathcal{L}_d \subset \mathcal{M}$ (with respect to the closed-loop contact form θ_d) and containing $\tilde{x}^*$, if the closed-loop system admits the point $\tilde{x}^*$ as hyperbolic equilibrium point and $\mathcal{L}_d$ is its stable submanifold.

One way to proceed for the control design is the following. Once the family of structure preserving output feedbacks is characterized according to the Proposition 3.1, one chooses some Legendre submanifold $\mathcal{L}_d$ for the closed-loop system and then use the invariance condition as additional conditions on the function Φ characterizing the closed-loop contact form (7) and the output feedback. Finally it remains to choose a solution Φ such that the stability on the Legendre submanifold is satisfied [20, sec. 4.3].

D. Paradigmatic example: stable feedback

Consider again the example of the two cells exchanging heat and let us select a stabilizing controller among the structure preserving stabilizing controllers defined in the subsection III-C: $\Phi(y)$ with $y = K_c(T_2, p_2)$ according to (5). In the first instance, let us choose the most simple desired closed-loop submanifold $\mathcal{L}_d$ generated by the function

$$U_d(x_1, x_2) = (x_1 + x_2)\, T^*$$

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where T^* is a desired temperature. It follows, according to the Definition (22), that the closed-loop intensive variables, which may be interpreted as virtual temperatures, are equal and constant: $p^d = \frac{\partial U_d}{\partial x}(x) = \begin{bmatrix} T^* & T^* \end{bmatrix}^\top$. In the second instance, using the expression of the contact Hamiltonians (5), we check the invariance condition of the Legendre submanifold $\mathcal{L}_d$: $K_0\big|_{\mathcal{L}_d} + \Phi(y)\big|_{\mathcal{L}_d} = -c_F$ and we obtain the condition

$$-\lambda_e(T_2 - T^*)\frac{T^*}{T_2} + \Phi(y)\big|_{\mathcal{L}_{U_d}} = 0.$$

with $c_F = 0$. This implies that on $\mathcal{L}_{U_d}$, $\Phi(y)\big|_{\mathcal{L}_{U_d}} = \lambda_e(T_2 - T^*)\frac{T^*}{T_2}$. And one may find a Φ that satisfies the invariance condition as follows

$$\Phi(y) = T^* \ln(y+1) = \lambda_e(T_2 - p_2)\frac{T^*}{T_2}$$

with the restriction to the closed-loop Legendre submanifold being $\Phi(y)\big|_{\mathcal{L}_d} = \lambda_e(T_2 - T^*)\frac{T^*}{T_2}$. The state feedback on the whole Thermodynamic Phase Space is, according to Proposition 3.1: $\alpha(x_2, p_2) = \Phi'(y) = \frac{T^*}{y+1} = T^* e^{\lambda_e\left(\frac{p_2}{T_2} - 1\right)}$ and the actual control is its restriction to the closed-loop Legendre submanifold $\mathcal{L}_d$:

$$u(x_2) = \Phi'(y)\big|_{\mathcal{L}_d} = T^* e^{\lambda_e\left(\frac{T^*}{T_2} - 1\right)}.$$

In this case the function β defining the nonlinear control (12) is $\beta(\zeta) = T^* e^{\zeta}$. In the third instance, it remains to be checked that the submanifold $\mathcal{L}_d$ is indeed the stable submanifold. This may be done using local arguments at the equilibrium point or directly on the entropy balance equations (6) as (x_1, x_2) are coordinates for the open-loop as well as for the closed-loop system. Consider the function $V(x_1, x_2) = \frac{1}{2}\sum_{i=1}^{2}(U_i - U_i^*)^2$ where $U_i^* = U_i(x_i^*)$ with $T^* = \frac{\partial U_i}{\partial x_i}(x_i^*)$: it has a global strict minimum at (x_1^*, x_2^*). Furthermore its differential is: $\begin{bmatrix} \frac{\partial V}{\partial x_1} \\ \frac{\partial V}{\partial x_2} \end{bmatrix} = \begin{bmatrix} (U_1 - U_1^*)T_1(x_1) \\ (U_2 - U_2^*)T_2(x_2) \end{bmatrix}$ and one obtains

$$\begin{aligned} \frac{dV}{dt} &= \lambda\left(\tfrac{1}{T_2} - \tfrac{1}{T_1}\right)\Big(-(U_1 - U_1^*)T_1T_2 + (U_2 - U_2^*)T_2T_1\Big) \\ &\quad - \lambda_e\big(U_2 - U_2^*\big)\left(T_2 - T^* e^{\lambda_e\left(\frac{T^*}{T_2} - 1\right)}\right) \\ &= -c\lambda\,(T_1 - T_2)^2 - c\lambda_e\big(T_2 - T^*\big)\left(T_2 - T^* e^{\lambda_e\left(\frac{T^*}{T_2} - 1\right)}\right) \end{aligned}$$

Where it has been assumed that the two gases have the same properties and that: $U_i = c\exp\left(\frac{x_i}{c}\right) = cT_i$. The Lyapunov stability follows by nothing that $(T_2 - T^*)\left(T_2 - T^*\exp\left(\lambda_e\left(\frac{T^*}{T_2} - 1\right)\right)\right) \geq 0$, with equality only if $T_1 = T_2 = T^*$.

V. Conclusions

Contact systems are Hamiltonian systems defined on contact manifolds and arise from the modelling of open thermodynamical systems in an analogous way as Hamiltonian systems defined on symplectic manifolds arise from the models of mechanical systems. In this paper we have addressed the problem of the design of stabilizing nonlinear control laws for conservative input-output contact systems. These conservative input-output systems are control systems defined on contact structure which leave invariant some Legendre submanifold and often arise from the formulation of the dynamics of irreversible thermodynamic systems on the complete thermodynamic Phase Space [4][5], [23]. We have considered structure preserving nonlinear control of these systems suggested in [22] in the sense it uses the feedback equivalence of the closed-loop system on the complete Thermodynamic Phase Space, which may again be defined as a contact system however with respect to some different contact structure.

In this paper the class of achievable closed-loop contact forms have been characterized with respect to their Reeb vector field which is shown to be equal to the Reeb vector field of the open-loop contact form. Furthermore the set of Legendre submanifolds associated with the achievable closed-lop contact form have been characterized. The main result concerns the stability in closed-loop and therefore it has firstly been shown that a hyperbolic equilibrium point of a strict contact vector field has a stable and unstable submanifold which are Legendre submanifolds. As a consequence, and under some mild regularity assumptions on the closed-loop contact Hamiltonian, it has been shown that the asymptotic stabilizing control laws of input-output contact systems are parametrized by the Legendre submanifolds (actually their generating functions) of the feedback equivalent conservative contact system. This has been illustrated on the paradigmatic example of the two cells exchanging heat and connected to a thermostat where the control laws are parametrized by a potential function which may be interpreted as a virtual internal energy associated with the thermostat.

Future work will address the application of such control design to systems related to applications in Chemical Engineering such as the Continuous Stirred Tank Reactor but also to the generalization to the dynamic control laws where the closed-loop system is no more conservative, that is using the feedback equivalence to more general contact systems defined with respect to vector fields which are not necessarily strict.

Appendix

In this appendix we briefly recall the main definitions and properties of contact geometry used in this paper; the reader is referred to the following textbooks for a detailed exposition [10],[1, app. 4.] and [11, chap. 5] which is our main reference. We shall consider systems defined on state-spaces which are *contact manifolds*, that are $(2n+1)$-dimensional differentiable manifolds $\mathcal{M} \ni \tilde{x}$ equipped with a *contact form.* We denote by $\mathfrak{X}(\mathcal{M})$ its set of vector fields and by $\Lambda(\mathcal{M})$ its set of 1-forms.

Definition A.1: A contact structure on a $2n+1$-dimensional differentiable manifold $\mathcal{M}$ is defined by a 1-form θ of constant class $(2n+1)$ satisfying $\theta \wedge (d\theta)^n \neq 0$, where $\wedge$ denotes the wedge product, d the exterior derivative and $(\cdot)^n$ the n-th power of the exterior product. The pair $(\mathcal{M}, \theta)$ is then called a (strict) *contact manifold*, and θ a *contact form.*

According to Darboux's Theorem there exists, locally, a set of *canonical coordinates* $\tilde{x} = (x_0, x, p) \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n$ of $\mathcal{M}$ where the contact form θ is given by

$$\theta = dx_0 - \sum_{i=1}^{n} p_i dx_i.$$

Definition A.2: The *Reeb vector field* E associated with the contact form θ is the unique vector field satisfying

$$i_E\theta = 1, \qquad i_E d\theta = 0, \tag{17}$$

called *interior product*) of a differential form by the vector field E. In canonical coordinates the Reeb vector field is expressed as $E = \frac{\partial}{\partial x_0}$.

There is a distinguished set of vector fields associated with the contact structure $(\mathcal{M}, \theta)$, i.e. which are infinitesimal automorphisms of the contact structure, called *contact vector fields*.

Proposition A.1: A (smooth) vector field X on the contact manifold $\mathcal{M}$ is a *contact vector field* with respect to a contact form θ if and only if there exists a smooth function $\rho \in C^\infty(\mathcal{M})$ such that

$$L_X\theta = \rho\theta, \tag{18}$$

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where $L_X \cdot$ denotes the Lie derivative with respect to the vector field X. When the $\rho = 0$, the contact vector field is called a *strict contact vector* field.

It may be shown that contact vector fields are uniquely defined by smooth real functions.

Proposition A.2: The map $\Omega(X) = i_X\theta$ defines an isomorphism from the vector space of contact vector fields in the space of smooth real functions on the contact manifold.

The real function K generating a contact vector field X is obtained by $K = \Omega(X) = i_X\theta$ and is called *contact Hamiltonian*. The contact vector field generated by the function K is denoted in this paper by $X_K = \Omega^{-1}(K)$, where Ω^{-1} is the inverse of the isomorphism defined in Proposition A.2. Finally the function ρ of (18) is given by $\rho = i_E dK$ where E is the Reeb vector field. When the contact vector field X_K is strict then its contact Hamiltonian K satisfies

$$i_E dK = 0 \tag{19}$$

i.e., K is a *first integral of the Reeb vector field* or in canonical coordinates $\frac{\partial K}{\partial x_0} = 0$. The expression of a contact vector field, in any set of canonical coordinates is

$$X_K = \begin{bmatrix} K \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & -p^\top \\ 0 & 0 & -I_n \\ p & I_n & 0 \end{bmatrix} \begin{bmatrix} \frac{\partial K}{\partial x_0} \\ \frac{\partial K}{\partial x} \\ \frac{\partial K}{\partial p} \end{bmatrix}, \tag{20}$$

where I_n denotes the identity matrix of order n. It should be mentioned that the isomorphism Ω also defines a Lie algebraic function, called the *Jacobi bracket*.

Definition A.3: [11, pp. 319-320] The *Jacobi bracket* $[f, g]_\theta$ of two differentiable functions f and g, is defined by

$$\begin{aligned} [f,g]_\theta &= \Omega\left(\left[\Omega^{-1}(f), \Omega^{-1}(g)\right]\right) \\ &= i_{X_f} dg - g\, i_E df \\ &= -i_{X_g} df + f\, i_E dg \end{aligned} \tag{21}$$

Furthermore the contact form defines a set of distinguished submanifolds, the *isotropic* and *Legendre submanifolds* [11, p. 312, 383].

Definition A.4: An *isotropic submanifold* of a $(2n+1)$-dimensional contact manifold $(\mathcal{M}, \theta)$ is an integral submanifold $\mathcal{L} \subset \mathcal{M}$ of $\ker\theta$ called *field of contact elements* [11, p. 322].

We shall use the following Proposition.

Proposition A.3: A Legendre submanifold is an isotropic submanifold of a $(2n+1)$-dimensional contact manifold $(\mathcal{M}, \theta)$, of maximal dimension, equal to n.

In some set of canonical coordinates, the Legendre submanifold is defined by a generating function $U \in C^\infty(\mathbb{R}^n)$ as follows

$$\mathcal{L}_U = \left\{ x_0 = U(x), x = x, p = \frac{\partial U}{\partial x}(x),\ x \in \mathbb{R}^n \right\}. \tag{22}$$

Contact vector fields may satisfy an additional condition, namely that they leave some Legendre submanifold invariant.

Proposition A.4: [16] Let $\mathcal{L}$ be a Legendre submanifold. Then X_K is tangent to $\mathcal{L}$ if and only if K vanishes on $\mathcal{L}$, i.e., $\mathcal{L} \subset K^{-1}(0)$ which may be stated as follows

$$K\big|_{\mathcal{L}} = 0 \tag{23}$$

where $\cdot|_{\mathcal{L}}$ denotes the restriction to $\mathcal{L}$.

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