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Feedback equivalence of input–output contact systems

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ABSTRACT

Control contact systems represent controlled (or open) irreversible processes which allow us to represent *simultaneously* the energy conservation and the irreversible creation of entropy. Such systems systematically arise in models established in Chemical Engineering. The differential-geometric of these systems is a contact form in the same manner as the symplectic 2-form is associated to Hamiltonian models of mechanics. In this paper we study the feedback preserving the geometric structure of controlled contact systems and render the closed-loop system again as a contact system. It is shown that only a constant control preserves the canonical contact form, hence a state feedback necessarily changes the closed-loop contact form. For strict contact systems, arising from the modelling of thermodynamic systems, a class of state feedback that shapes the closed-loop contact form and contact Hamiltonian function is proposed. The state feedback is given by the composition of an arbitrary function and the control contact Hamiltonian function. The similarity with structure preserving feedback of input–output Hamiltonian systems leads to the definition of input–output contact systems and to the characterization of the feedback equivalence of input–output contact systems. An irreversible thermodynamic process, namely the heat exchanger, is used to illustrate the results.

1. Introduction

Control contact systems [1,2] have been introduced for the representation of controlled (or open) irreversible processes. They allow us to represent *simultaneously* the energy conservation and the irreversible creation of entropy, the fundamental principles of Irreversible Thermodynamics. Such systems are defined on the Thermodynamic Phase Space which is endowed with a contact structure (or a contact form) which is canonically associated with Gibbs' relation defining the Thermodynamic Equilibrium properties of physical systems [3–6]. Extending the work on reversible thermodynamical transformations in [7] to irreversible transformation of open thermodynamical systems leads to the definition of control contact systems [1,2] which are a strict extension of control Hamiltonian and port-Hamiltonian systems [8], and to the analysis of some of their dynamic properties [2,9].

In this paper we consider the state feedback of controlled contact systems and analyse under which conditions the closed-loop system again is a contact system, more precisely when it leaves invariant some contact structure. This problem is precisely in the line with a similar problem of feedback controls preserving the symplectic structure of input–output Hamiltonian systems treated in [10,11].

The paper is organized as follows. In Section 3 we give conditions under which a state feedback leads to a closed-loop system which is a contact system with respect to some closed-loop contact form in terms of a matching equation between the feedback and the closed-loop contact form. In Section 4 we restrict the problem to control contact systems defined by strict contact vector fields, that is that leave invariant the contact form itself, and the difference between the open-loop and the closed-loop contact form is an exact 1-form. These assumptions allow us to define the class of admissible feedback equations as well as a matching equation for the added exact 1-form defining the closed-loop contact form. In Section 5 we shall deduce a natural output for controlled contact systems and define input–output contact systems. Then we deduce the conditions for the feedback equivalence between input–output contact systems. Some final remarks and perspectives of future work are given in Section 6.

2. On controlled contact systems

In this section we shall briefly recall the definition and main properties of a class of nonlinear control systems, called *control*

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contact systems, that arise when modelling control systems in chemical engineering or any process where the internal energy (or entropy) balance equation is written. They may be considered as the analogue of Lagrangian or Hamiltonian control systems associated with mechanical systems and defined on the state space of configuration–momentum which is endowed with a natural symplectic structure [12,11,13]. Controlled contact systems are defined on the Thermodynamic Phase Space consisting of $n+1$ extensive variables and n intensive variables and endowed with a contact structure associated with Gibbs' relation defining the thermodynamic properties of the system. On the Thermodynamic Phase Space, one may then define controlled contact systems which are the analogue of control Hamiltonian systems and have been introduced in [1] and further developed in [2,9,8]. After the introductory example of a 2-compartment heat exchange system we shall recall the precise definitions needed in this paper.

2.1. The example of the heat exchanger

Consider a system consisting of two compartments exchanging heat flow through a heat conducting wall and one of the compartments exchanging heat flow with the environment and called, for simplicity, a *heat exchanger*. It consists of two entropy balance equations one for each compartment and is the paradigmatic example for irreversible systems, in the same way as the mass–spring system is for mechanical systems.

The thermodynamic perspective to this system consists in considering two simple thermodynamic systems, indexed by 1 and 2 (for instance two ideal gases), which may interact only through a heat conducting wall and compartment 2 exchanging a heat flow with the environment. In the first instance the Thermodynamic properties are described in the Thermodynamic Phase Space as follows. The thermodynamic phase space is $\mathbb{R}^5 \ni (x_0,\ x_1,\ x_2,\ p_1,\ p_2)^\top$ with the first coordinate x_0 corresponding to the total internal energy, the coordinates x_1 and x_2 corresponding to the entropies of subsystem 1 and 2, the coordinates p_1 and p_2 corresponding to the temperatures, the intensive variables conjugated to the entropies x_1 and x_2. The thermodynamic properties are defined by *Gibbs' equation*:

$$dx_0 - \sum_{i=1}^{n} p_i dx_i = 0 \tag{1}$$

and are practically defined by a thermodynamic potential being the sum of the internal energy function of each compartment $U(x_1, x_2) = U_1(x_1) + U_2(x_2)$. The gradient of the total internal energy $\frac{\partial U}{\partial x_i} = T_i(x_i)$ is composed of the temperatures of each compartment with $T_i(x_i) = T_0 \exp\left(\frac{x_i}{c_i}\right)$, where T_0 and c_i are constants [14]. The state space of the heat exchanger is then defined as the following submanifold $\mathcal{L}_U$ of the Thermodynamic Phase Space where Gibbs' equation is satisfied

$$\mathcal{L}_U : \begin{Bmatrix} x_0 = U\,(x_1,\ x_2) \\ x = [x_1, x_2]^\top \\ p = \left[\dfrac{\partial U}{\partial x_1}, \dfrac{\partial U}{\partial x_2}\right]^\top = T(x) = [T_1\,(x_1)\,, T_2\,(x_2)]^\top \end{Bmatrix}.$$

In a second instance, one completes the thermodynamic properties by irreversible phenomena, in this example the heat conduction through the internal wall given by Fourier's law with heat conduction coefficient λ. The dynamics of the thermodynamic variables may be shown to leave the submanifold $\mathcal{L}_U$ invariant *and restricted to the submanifold* $\mathcal{L}_U$, defining the following control system

$$\frac{d}{dt}\begin{bmatrix} U \\ x_1 \\ x_2 \\ T_1 \\ T_2 \end{bmatrix} = \begin{bmatrix} u \\ -\dfrac{\lambda(T_1 - T_2)}{T_1} \\ \dfrac{\lambda(T_1 - T_2)}{T_2} + \dfrac{u}{T_2} \\ -{C_V}_1^{-1}\,\lambda(T_1 - T_2) \\ {C_V}_2^{-1}\,[\lambda(T_1 - T_2) + u] \end{bmatrix} \tag{2}$$

where $C_{Vi} = \frac{\partial U_i}{\partial T_i}$ are the calorific capacitances and the input $u(t)$ is the heat flow delivered by the external heat source. This control system expresses the total energy balance in the first coordinate, the entropy balance equations in the second and third coordinates and the partial energy balance equations (written in terms of the temperatures and using the calorimetric relations) for each compartment in the fourth and fifth coordinates.

Hence the Thermodynamic perspective to this heat exchanger is to obtain a redundant dynamical representation where the dynamics of all intensive and extensive thermodynamic variables are expressed.

2.2. Contact manifold and contact systems

The Thermodynamic Phase Space is structured by Gibbs' equation which endows it with a canonical differential-geometric called *contact structure*. In the sequel we shall recall briefly the main definitions and properties of contact geometry used in this paper; the reader is referred to the following textbooks for a detailed justification [15, app. 4.], [5] and to [2,8,9] for the application to controlled irreversible thermodynamic systems.

The contact form corresponds to the definition of Gibbs' equation (1) and is defined as follows.

Definition 2.1. A contact structure on a $2n+1$-dimensional differentiable manifold $\mathcal{M}$ is defined by a 1-form θ of constant class $(2n+1)$ satisfying

$$\theta \wedge (d\theta)^n \neq 0, \tag{3}$$

where $\wedge$ denotes the wedge product, d the exterior derivative and $(\cdot)^n$ the n-th exterior power. The pair $(\mathcal{M}, \theta)$ is then called a *contact manifold*, and θ a *contact form*.

Note that condition (3) represents a non-degeneracy condition [15]. According to Darboux's theorem there exists a set of *canonical coordinates* $\tilde{x} = (x_0, x, p) \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n$ of $\mathcal{M}$ where the contact form θ is given by

$$\theta = dx_0 - \sum_{i=1}^{n} p_i dx_i.$$

There exists a particular vector field, characteristic of the contact form, called the Reeb vector field.

Definition 2.2. The *Reeb vector field E* associated with the contact form θ is the unique vector field satisfying

$$i_E\theta = 1 \quad \text{and} \quad i_E d\theta = 0 \tag{4}$$

where $i_E\cdot$ denotes the contraction of a differential form by the vector field E. In canonical coordinates the Reeb vector field is expressed as $E = \frac{\partial}{\partial x_0}$.

Notice that $i.\cdot$ is also known as the interior product or interior derivative of a differential form by a vector field [15]. The irreversible thermodynamic phenomena leads to dynamical systems which are defined by *contact vector fields*.

Proposition 2.1 ([16]). *A (smooth) vector field X on the contact manifold $\mathcal{M}$ is a* contact vector field *with respect to a contact form*

θ if and only if there exists a smooth function $\rho \in C^\infty(\mathcal{M})$ such that

$$L_X\theta = \rho\theta, \tag{5}$$

where $L_X\cdot$ denotes the Lie derivative with respect to the vector field X.

It may be shown that contact vector fields are uniquely defined by smooth real functions.

Proposition 2.2 ([16]). *The map $\Omega(X) = i_X\theta$ defines an isomorphism from the vector space of contact vector fields in the space of smooth real functions on the contact manifold.*

The real function K generating a contact vector field X is obtained by

$$K = \Omega(X) = i_X\theta \tag{6}$$

and is called the *contact Hamiltonian*. The contact vector field generated by the function K is denoted by $X_K = \Omega^{-1}(K)$, where Ω^{-1} is the inverse isomorphism. Finally the function ρ of (5) is given by

$$\rho = i_E dK \tag{7}$$

where E is the Reeb vector field. A contact vector field, in any set of canonical coordinates, is expressed by

$$X_K = \begin{bmatrix} K \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & -p^\top \\ 0 & 0 & -I_n \\ p & I_n & 0 \end{bmatrix} \begin{bmatrix} \dfrac{\partial K}{\partial x_0} \\ \dfrac{\partial K}{\partial x} \\ \dfrac{\partial K}{\partial p} \end{bmatrix}, \tag{8}$$

where I_n denotes the identity matrix of order n.

With this definition of contact vector fields, one may define control contact systems according to [1,2] which represent the dynamics of irreversible Thermodynamic systems [8] such as the Continuous Stirred Tank Reactor [17,18].

Definition 2.3. A *controlled contact system* affine in the scalar input $u(t) \in L_1^{\text{loc}}(\mathbb{R}_+)$ is defined by the two functions $K_0 \in C^\infty(\mathcal{M})$, called the *internal contact Hamiltonian* and $K_c \in C^\infty(\mathcal{M})$ called the *interaction (or control) contact Hamiltonian* and the state equation

$$\frac{d\tilde{x}}{dt} = X_{K_0} + X_{K_c}u, \tag{9}$$

where X_{K_0} and X_{K_c} are the contact vector fields generated by K_0 and K_c with respect to the contact form θ.

2.3. The example of the heat exchanger (continued)

Consider the control contact system defined by the internal and control contact Hamiltonians

$$\begin{aligned} K_0(x,p) &= -R(x,p)p^\top JT(x), \\ K_c(x,p) &= \frac{p_1}{T_1}\left(1 - \frac{p_2}{T_2}\right), \end{aligned} \tag{10}$$

with $R(x,p) = \lambda\left(\frac{p_1-p_2}{T_1T_2}\right)$ and $J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. It may be checked that on the Legendre submanifold generated by U the contact Hamiltonian functions vanish, $K_0|_{\mathcal{L}_U} = 0$ and $K_c|_{\mathcal{L}_U} = 0$, and hence the contact vector field $X_{K_0} + X_{K_c}u$ leaves the Legendre submanifold $\mathcal{L}_U$ invariant (i.e. the thermodynamic properties). Using (8) it is computed that its restriction to $\mathcal{L}_U$ is equivalent to the system equations (2).

3. State feedback of controlled contact systems and invariance of contact forms

The main question of this paper is to characterize under which conditions, in a closed-loop, may the system be interpreted again as an irreversible Thermodynamic system, in other words conserving a physical structure. In this section we shall characterize the state feedback $u = \alpha(\tilde{x})$ such that the closed-loop vector field

$$X = X_{K_0} + X_{K_c}\alpha\left(\tilde{x}\right) \tag{11}$$

is a contact vector field with respect to some contact form which may be different from the open-loop one, θ.

3.1. Feedback equivalence with respect to the same contact form

In a first instance let us analyse under which condition the closed-loop vector field (11) is a contact vector field with respect to the contact form θ. Therefore let us make the following assumption.

Assumption 1. The control contact Hamiltonian $K_c \in C^\infty(\mathcal{M})$ vanishes on a submanifold of measure 0 of $\mathcal{M}$.

Proposition 3.1. *Consider the controlled contact system (9) with Assumption 1, and the feedback $u = \alpha(\tilde{x})$ being a smooth function of the state variables. The closed-loop vector field X is a contact vector field with respect to the canonical contact form θ if and only if the state feedback is constant, i.e., $\alpha(\tilde{x}) = \alpha_0 \in \mathbb{R}$.*

Proof. Recall Cartan's formula: $L_X\cdot = i_Xd\cdot + di_X\cdot$. Then one may compute, using (6) and (7),

$$\begin{aligned} L_X\theta &= L_{X_{K_0}+\alpha X_{K_c}}\theta \\ &= L_{X_{K_0}}\theta + \alpha\, i_{X_{K_c}}d\theta + d\left(\alpha K_c\right) \\ &= L_{X_{K_0}}\theta + \alpha L_{X_{K_c}}\theta + K_c d\alpha \\ &= \left(\rho_0 + \alpha\rho_c\right)\theta + K_c d\alpha \end{aligned}$$

where $\rho_0 = i_E dK_0$, $\rho_c = i_E dK_c$. Hence by (5), the vector field $X = X_{K_0} + X_{K_c}\alpha$ is a contact vector field if and only if there exists a function $\phi \in C^\infty(\mathcal{M})$ such that $K_c d\alpha = \phi\theta$. Using Assumption 1 we may rewrite the last expression as

$$d\alpha = \left(\frac{\phi}{K_c}\right)\theta,$$

and using that $d^2\alpha = 0$ one obtains

$$d\left(\frac{\phi}{K_c}\right) \wedge \theta + \left(\frac{\phi}{K_c}\right) d\theta = 0.$$

Taking the wedge product with θ and using that it is a 1-form, hence $\theta \wedge \theta = 0$, one gets

$$\left(\frac{\phi}{K_c}\right) d\theta \wedge \theta = 0.$$

According to Definition 2.1, $d\theta \wedge \theta$ is nonzero at any point, hence $\left(\frac{\phi}{K_c}\right) = 0$ which implies $d\alpha = 0$ and that α is a constant function. $\square$

3.2. Feedback equivalence with respect to a modified contact form

Proposition 3.1 shows that using non constant state feedback of a controlled contact vector field it is not possible to obtain a contact vector field with respect to *the same contact form*. In this section we develop the feedback conditions under which the closed-loop contact vector field X (9) is again a contact vector field, with respect to a *different contact form* associated with the closed-loop vector field and denoted by θ_d. Therefore it has to be checked that the closed-loop vector field X satisfies condition (5) with respect to θ_d:

$$\begin{aligned} L_X\theta_d &= L_{X_{K_0}+\alpha X_{K_c}}\theta_d \\ &= L_{X_{K_0}}\theta_d + \alpha L_{X_{K_c}}\theta_d + (i_{X_{K_c}}\theta_d)d\alpha \end{aligned}$$

which leads to the following proposition.

Proposition 3.2. *The closed-loop vector field obtained by the feedback $\alpha \in C^\infty(\mathcal{M})$ in (9) is a contact system with respect to a contact form θ_d if and only if there exist a function $\rho_d \in C^\infty(\mathcal{M})$ such that the following matching equation is satisfied*

$$\rho_d\theta_d = L_{X_{K_0}}\theta_d + \alpha L_{X_{K_c}}\theta_d + (i_{X_{K_c}}\theta_d)d\alpha. \tag{12}$$

In the following we proceed to simplify the problem by assuming that the open and closed-loop contact vector fields are *strict contact vector fields*.

Assumption 2. The internal and control contact Hamiltonian functions K_0 and K_c are invariants of the Reeb vector field E of the contact form θ and the closed-loop vector field X is a strict contact vector field with respect to the contact form θ_d (that is, $\rho_d = \rho_0 = \rho_c = 0$).

Assumption 2 expresses that X, and respectively X_{K_0} and X_{K_c}, leave invariant the contact form itself, θ_d respectively θ. For contact systems arising from the modelling of physical systems, this is not restrictive since this is equivalent to assuming that the contact Hamiltonians are invariants of the Reeb vector field. In canonical coordinates this means that they do not depend on the x_0 coordinate associated with the Reeb vector field. For models of physical systems where the x_0 coordinate represents the generating potential of the thermodynamic system (the total energy or the total entropy), this is in general the case [2,8]. Under Assumption 2, the matching equation (12) is reduced to a relation on the feedback α and the closed-loop contact form θ_d

$$L_{X_{K_0}}\theta_d + \alpha L_{X_{K_c}}\theta_d + (i_{X_{K_c}}\theta_d)d\alpha = 0. \tag{13}$$

4. Solutions of the matching equations

4.1. Matching to the contact form $\theta_d = \theta + dF$

In order to facilitate the computation of a solution to the matching equation we shall make in the sequel the following assumption.

Assumption 3. The closed-loop contact form θ_d is defined as

$$\theta_d = \theta + dF, \tag{14}$$

with $F \in C^\infty(\mathcal{M})$ satisfying $i_E dF = 0$.

Note that the condition $i_E dF = 0$ means that F is an invariant of the Reeb vector field E and is equivalent in canonical coordinates to assume that the function F depends only on (x, p) and not on x_0. The following proposition proves that the 1-form θ_d defined in Assumption 3 is actually a contact form for any choice of invariant F of the Reeb vector field E.

Proposition 4.1. *The* 1*-form* (14) *is a contact form.*

Proof. Recall that θ_d is a contact form if it is a Pfaffian form of class $2n + 1$, satisfying [16],

$$\theta_d \wedge (d\theta_d)^n \neq 0.$$

Note that using $d^2F = 0$ one has that

$$\begin{aligned}\theta_d \wedge (d\theta_d)^n &= (\theta + dF) \wedge (d(\theta + dF))^n\\ &= (\theta + dF) \wedge (d\theta)^n.\end{aligned}$$

Proceed by contradiction and assume that $\theta_d \wedge (d\theta_d)^n = 0$. Then, using the fact that i_E is a $\wedge$ antiderivation and the properties (4) of the Reeb vector field:

$$\begin{aligned}i_E\left[\theta_d \wedge (d\theta_d)^n\right] &= i_E\left[(\theta + dF) \wedge (d\theta)^n\right]\\ &= i_E(\theta + dF) \wedge (d\theta)^n\\ &\quad + (-1)(\theta + dF) \wedge i_E\left((d\theta)^n\right)\\ &= (1 + i_E dF) \wedge (d\theta)^n\end{aligned}$$

and $i_E dF = 0$, implies that $(d\theta)^n = 0$ which contradicts the fact that θ is of class $2n + 1$. □

Note that it has been assumed that F satisfies $i_E dF = 0$. However, from the proof of Proposition 4.1 it is clear that it is only required that $i_E dF \neq -1$. In this sense the assumption $i_E dF = 0$ is restrictive, however it may be related to some method of energy shaping as is commented now. Firstly it may be observed that this assumption allows us to derive some canonical coordinates for the closed-loop contact form θ_d. In some set of canonical coordinates (x_0, x, p) of θ, the closed-loop contact form (14) is given by

$$\begin{aligned}\theta_d &= \theta + dF = \left(dx_0 - \sum_{i=1}^{n} p_i dx_i\right) + dF(x, p),\\ &= d(x_0 + F(x, p)) - \sum_{i=1}^{n} p_i dx_i,\\ &= dx_0' - \sum_{i=1}^{n} p_i dx_i.\end{aligned}$$

A set of canonical coordinates for θ_d is now given by (x_0', x, p) with $x_0' = x_0 + F(x, p)$. Secondly one may interpret this as the feedback changing the direction of the Reeb vector field in the closed loop. Recall that the contact structure appears in the differential-geometric representation of thermodynamic systems [5,6,4], where x_0 is the coordinate of a thermodynamic potential, such as the energy U or the entropy S. Given some thermodynamic properties defined for instance by the internal energy, changing the Reeb vector field amounts to changing the energy: $U' = U + F$. This interpretation is in accordance to the one provided in [5, chap. 6] and [6, chap. 9] for the isothermal interaction of thermodynamic system using contact geometry.

Let us now express the matching equation (13) with θ_d defined by (14) in terms of a matching equation in the function F and the feedback α. The Lie derivatives in (13) may be developed as

$$L_{X_{K_0}}(\theta + dF) = L_{X_{K_0}}\theta + L_{X_{K_0}}dF = \rho\theta + L_{X_{K_0}}dF$$

with

$$L_{X_{K_0}}dF = i_{X_{K_0}}d(dF) + d(i_{X_{K_0}}dF) = d\left(X_{K_0}(F)\right).$$

Using Assumption 2 and $i_{X_{K_c}}\theta_d = i_{X_{K_c}}(\theta + dF) = K_c + X_{K_c}(F)$, (13) becomes

$$d\left(X_{K_0}(F)\right) + \alpha d\left(X_{K_c}(F)\right) + \left[K_c + X_{K_c}(F)\right]d\alpha = 0. \tag{15}$$

Since $X = X_{K_0} + X_{K_c}\alpha$, it follows that

$$d(X(F)) = d(X_{K_0}(F)) + \alpha d(X_{K_c}(F)) + X_{K_c}(F)d\alpha.$$

Finally, (15) may be rewritten as the following matching equation in the feedback α and the function F

$$d\left(X(F)\right) + K_c d\alpha = 0. \tag{16}$$

Remark 4.1. Notice that if $d\alpha = 0$ (i.e. α is constant), then (15) (or (16)) is satisfied if $d\left(X(F)\right) = 0$, or equivalently if $X(F)$ is constant. This in turn is satisfied if $dF \in \text{ann}\left(\text{Span}\left\{X_{K_0}, X_{K_c}\right\}\right)$, i.e. $X(F) = 0$. Two special cases may be identified, namely when $dF = 0$ i.e. $\theta_d = \theta$ (Proposition 3.1) and when F is an invariant of X.

4.2. Admissible state feedback

In order to solve the matching equation (15) we shall make the following assumption.

Assumption 4. The differential dK_c of the control contact Hamiltonian $K_c \in C^\infty(\mathcal{M})$ vanishes on a submanifold of measure 0 of $\mathcal{M}$.

Observe that by taking the exterior derivative of (16) we get

$$dK_c \wedge d\alpha = 0.$$

This leads us to consider a candidate feedback function of the interaction contact Hamiltonian function K_c

$$\alpha = \Phi' \circ K_c,$$

where $\Phi' \in C^\infty(\mathbb{R})$ is the derivative of a smooth function $\Phi : \mathbb{R} \to \mathbb{R}$.

Proposition 4.2. *Let $\mathcal{M}$ be a contact manifold with contact form θ with associated Reeb vector field E and consider the smooth real functions $K_0, K_c, F \in C^\infty(\mathcal{M})$, such that $i_E K_0 = i_E K_c = i_E F = 0$. Then the closed-loop vector field $X = X_{K_0} + \alpha X_{K_c}$, with $\alpha \in C^\infty(\mathcal{M})$, is a strict contact vector field with respect to the shaped contact form θ_d and the shaped contact Hamiltonian K, respectively,*

$$\theta_d = \theta + dF \quad \text{and} \quad K = K_0 + \Phi \circ K_c + c_F,$$

where $\Phi \in C^\infty(\mathbb{R})$, if and only if

$$\alpha = \Phi' \circ K_c(x, p),$$

and the matching equation

$$X_{K_0}(F) + (\Phi' \circ K_c)[K_c + X_{K_c}(F)] - \Phi \circ K_c = c_F \tag{17}$$

is satisfied. The closed-loop vector field is then denoted by $X = \hat{X}_K$, where $\hat{X}_K$ denotes the contact vector field generated by K with respect to the contact form θ_d.

Proof. Note that the control law solves the equation $dK_c \wedge d(\Phi' \circ K_c) = dK_c \wedge (\Phi'' \circ K_c) dK_c = 0$. Using the definition of the feedback, (16) becomes

$$d\left[X_{K_0}(F) + (\Phi' \circ K_c) X_{K_c}(F)\right] + K_c(\Phi'' \circ K_c) dK_c = 0,$$

and by defining $\Psi(\lambda) = \int_0^\lambda \chi \Phi''(\chi) d\chi$ it may be written as

$$d\left[X_{K_0}(F) + (\Phi' \circ K_c) X_{K_c}(F) + \Psi \circ K_c\right] = 0.$$

If $\Psi(\lambda)$ is integrated by parts the following is obtained

$$d\left[X_{K_0}(F) + (\Phi' \circ K_c) X_{K_c}(F) + K_c(\Phi' \circ K_c) - \Phi \circ K_c\right] = 0,$$

where $\Phi(\lambda) = \int_0^\lambda \Phi'(\chi) d\chi$. This means that there is a constant $c_F \in \mathbb{R}$ such that $X = X_{K_0} + \alpha X_{K_c}$ is invariant with respect to θ_d if and only if (17) is satisfied. By taking the exterior derivative of (16) we get $dK_c \wedge d\alpha = 0$ which is a necessary condition for $K_c d\alpha$ to be closed and, by Assumption 4, $d\alpha = \mu dK_c$ for some function μ. However observing that (16) implies that $K_c \mu dK_c$ is an *exact* 1-form and using Assumptions 1 and 4, one obtains that μ is a function of the interaction contact Hamiltonian K_c. Finally by integration one obtains that the feedback α may be written $\alpha = \Phi' \circ K_c$. Now, the closed-loop contact Hamiltonian function is given by the contraction of the closed-loop contact vector field and the closed-loop contact form: $K = i_X \theta_d$. Computing this last expression yields

$$\begin{aligned} K &= i_{X_{K_0}}(\theta + dF) + \alpha i_{X_{K_c}}(\theta + dF), \\ &= K_0 + i_{X_{K_0}} dF + \alpha(K_c + i_{X_{K_c}} dF), \\ &= K_0 + X_{K_0}(F) + \alpha(K_c + X_{K_c}(F)). \end{aligned}$$

Replacing the control law in this expression, and since $F(x, p)$ and $\Phi' \circ K_c$ verify (17), $K = K_0 + \Phi \circ K_c + c_F$ is obtained. Finally, since X is a contact vector field with respect to θ_d, it may be written as

$$X = X_{K_0} + \alpha X_{K_c} = \hat{X}_K,$$

where $\hat{X}_K$ is the contact vector field generated by K with respect to the contact form θ_d. □

Remark 4.2. It is also possible to obtain the expression of the closed-loop contact Hamiltonian by using the representation in coordinates of the closed-loop contact form and vector field. Indeed, the closed-loop contact form is given by

$$\begin{aligned} \theta_d &= d(x_0 + F(x, p)) - p^\top dx \\ &= dx_0 - \left(p - \frac{\partial F}{\partial x}\right)^\top dx + \frac{\partial F}{\partial p}^\top dp. \end{aligned}$$

The closed-loop vector field in local coordinates is $X = X_{K_0} + X_{K_c}\alpha$ and K is given by the contraction of the 1-form θ_d by this vector field. Recalling (8),

$$\begin{aligned} K &= i_X \theta_d = K_0 + \left(\frac{\partial K_0}{\partial x}^\top \frac{\partial F}{\partial p} - \frac{\partial K_0}{\partial p}^\top \frac{\partial F}{\partial x}\right) + \frac{\partial F}{\partial p}^\top p \frac{\partial K_0}{\partial x_0} \\ &\quad + \left[K_c + \left(\frac{\partial K_c}{\partial x}^\top \frac{\partial F}{\partial p} - \frac{\partial K_c}{\partial p}^\top \frac{\partial F}{\partial x}\right) + \frac{\partial F}{\partial p}^\top p \frac{\partial K_c}{\partial x_0}\right]\alpha. \end{aligned}$$

Since $\alpha = \Phi' \circ K_c$ and $\frac{\partial K_0}{\partial x_0} = \frac{\partial K_c}{\partial x_0} = 0$, and using the coordinate expression (8) of a contact vector field, we obtain by identification of the terms

$$\begin{aligned} K &= K_0 + \left(\frac{\partial K_0}{\partial x}^\top \frac{\partial F}{\partial p} - \frac{\partial K_0}{\partial p}^\top \frac{\partial F}{\partial x}\right) \\ &\quad + \left[K_c + \left(\frac{\partial K_c}{\partial x}^\top \frac{\partial F}{\partial p} - \frac{\partial K_c}{\partial p}^\top \frac{\partial F}{\partial x}\right)\right](\Phi' \circ K_c) \\ &= K_0 + X_{K_0}(F) + (\Phi' \circ K_c)[K_c + X_{K_c}(F)]. \end{aligned}$$

Finally replacing (17) in this equation we obtain $K = K_0 + \Phi \circ K_c + c_F$.

The previous development shows that the matching condition (17) is characterized by the state feedback and the function F, which leads to a characterization of the closed-loop contact Hamiltonian function and vector field in terms of the state feedback.

5. Input–output contact systems and their feedback equivalence

5.1. Natural output of controlled contact systems

The result of Proposition 4.2 is similar to the one obtained when investigating the feedback equivalence of input–output Hamiltonian systems [10,11], with the difference that in this frame the Poisson bracket is the same in the open and closed loop whereas for control contact systems the contact form in the open loop is different from that in the closed loop. However in both cases the feedback is defined as the composition of some function with the control Hamiltonian, respectively the control contact Hamiltonian. For input–output Hamiltonian systems the control Hamiltonian defines a natural output. In this section we follow this line and define the natural output of a contact Hamiltonian system in a similar manner.

Definition 5.1. An *(single) input–(single) output contact system* is an affine control contact system, according to Definition 2.3, augmented with the output relation

$$y = K_c(\tilde{x}).$$

One may note immediately that this definition of output also coincides with the more general definition suggested in [12] for control Hamiltonian systems nonlinear in the inputs: $y = \frac{\partial K}{\partial u}(\tilde{x}, u) = K_c(\tilde{x})$ with the definition of the contact Hamiltonian $K = K_0 + uK_c + c_F$. One may also note that this output is quite different from V-conjugated outputs for conservative contact systems introduced in [2,19], defined with respect to an arbitrary smooth function $V \in C^{\infty}(\mathcal{M})$ and the interaction contact Hamiltonian function K_c. Using Definition 5.1 the state feedback of Proposition 4.2 may be expressed as an output feedback

$$\alpha = \Phi'(y), \tag{18}$$

and the closed-loop contact Hamiltonian as a function of the natural output

$$K = K_0 + \Phi(y) + c_F. \tag{19}$$

5.2. Feedback equivalence of input–output systems

Having defined input–output contact systems, we may now follow similar questions as for input–output Hamiltonian systems [20], and look for the feedback equivalence of these input–output contact systems. This means that we look for a control

$$u\left(t, \tilde{x}\right) = \alpha\left(\tilde{x}\right) + v(t) \tag{20}$$

such that the closed-loop system

$$\frac{d\tilde{x}}{dt} = \left(X_{K_0} + X_{K_c}\alpha\left(\tilde{x}\right)\right) + X_{K_c}v \tag{21}$$

is again an input–output contact system. From Section 4 we know that the closed-loop drift vector field of (21) is a contact vector field when Proposition 4.2 is satisfied. In order to have an input–output contact system it remains to check that its input vector field X_{K_c} is also a strict contact vector field with respect to the closed-loop contact form θ_d. This is true if $L_{X_{K_c}}\theta_d = 0$ which by,

$$\begin{aligned} L_{X_{K_c}}\theta_d &= L_{X_{K_c}}\left(\theta + dF\right) \\ &= L_{X_{K_c}} dF \\ &= dX_{K_c}(F) = 0. \end{aligned}$$

As a consequence, the feedback equivalence of input–output contact systems is summarized in the following proposition.

Proposition 5.1. *An input–output contact system, according to Definition 5.1, on some contact manifold $\mathcal{M}$ endowed with the contact form θ, with internal contact Hamiltonian K_0 and control Hamiltonian K_c, is feedback equivalent using (20) to an input–output contact system with respect to the contact form $\theta_d = \theta + dF$, defined in Assumption 3, if and only if there exist two real numbers c_1 and c_F as well as a real function $\Phi \in C^{\infty}(\mathcal{M})$ such that the following system of linear PDE's is satisfied*

$$X_{K_c}(F) = c_1, \tag{22}$$

$$X_{K_0}(F) + (\Phi' \circ K_c)[K_c + c_1] - \Phi \circ K_c = c_F. \tag{23}$$

5.3. Some remarks on control synthesis

From the expressions of the closed-loop contact Hamiltonian (19) and the output feedback (18) it is clear that the function Φ is a control design parameter. *A choice of Φ shapes the closed-loop contact Hamiltonian* (19) in a very similar manner as the feedback of input–output Hamiltonian systems [10] or the Casimir method for port-Hamiltonian systems [21].

However there is an additional condition that there should exist a real function $F \in C^{\infty}(\mathcal{M})$ satisfying the matching condition (17), which may equivalently be written

$$\left\langle X_{K_0} + (\Phi' \circ K_c)X_{K_c},\ dF\right\rangle + (\Phi' \circ K_c)K_c - \Phi \circ K_c = 0,$$

where $\langle\ ,\ \rangle$ denotes the pairing between vector fields and 1-forms on $\mathcal{M}$. It then appears clearly that the matching equation defines a linear first-order PDE in the function F defining the modified contact form θ_d in (14). In the canonical coordinates of θ this PDE may be written as

$$\begin{bmatrix} \dfrac{\partial F}{\partial x} \\ \dfrac{\partial F}{\partial p} \end{bmatrix}^{\top} \begin{bmatrix} -\dfrac{\partial K_0}{\partial p} - (\Phi' \circ K_c)\dfrac{\partial K_c}{\partial p} \\ \dfrac{\partial K_0}{\partial x} + (\Phi' \circ K_c)\dfrac{\partial K_c}{\partial x} \end{bmatrix} + (\Phi' \circ K_c)K_c - \Phi \circ K_c = 0.$$

This linear PDE may then be solved by using classical methods such as the method of characteristics [22–24]. If one looks for the feedback equivalence to an input–output contact system, according to Proposition 5.1, this function F should moreover satisfy the linear first-order PDE (22) which however does *not* depend on the feedback (that is on the function Φ).

5.4. The example of the heat exchanger (continued)

Consider the example of the heat exchanger presented in Section 2.1. We shall briefly illustrate Proposition 5.1 by giving a particular solution to the matching equations (22) and (23), corresponding to some choice of feedback. We consider the control contact system defined by the internal and control contact Hamiltonians (10)

$$K_0(x, p) = -R(x, p)p^{\top}J_sT(x),$$

$$K_c(x, p) = \frac{p_1}{T_1}\left(1 - \frac{p_2}{T_2}\right).$$

It appears that for the solution of the matching equation it eases the computations, and the interpretations of the results, to use another lift of the entropy balance equations (2) and modify the internal contact Hamiltonian K_0 by adding the following auxiliary contact Hamiltonian

$$K_a = \lambda T_1\left(\frac{p_1}{T_1} - \frac{p_2}{T_2}\right)^2 + \frac{\lambda_e^2}{2}\frac{p_1^2}{T_1}\left(1 - \frac{p_2}{T_2}\right)^2,$$

and model the heat exchanger with the contact vector field $X_{K_0+K_a} + X_{K_c}u$. The function K_a has been chosen such that it vanishes on $\mathcal{L}_U$ and that $X_{K_a}|_{\mathcal{L}_U} = 0$. As a consequence the restrictions of both contact vector fields $X_{K_0+K_a} + X_{K_c}u$ and $X_{K_0} + X_{K_c}u$ to the Legendre submanifold $\mathcal{L}_U$ are equal and both define admissible lifts of the entropy balance equations of the heat exchanger (see Section 2.1). Let us choose $\Phi\left(\chi\right) = -\frac{1}{2}\chi^2$, from which the following control law is obtained

$$u\left(t, \tilde{x}\right) = \Phi'(K_c)\left(\tilde{x}\right) + v(t) = -\lambda_e\frac{p_1}{T_1}\left(1 - \frac{p_2}{T_2}\right) + v(t).$$

A solution of (23) is then given by the function $F = \left(\frac{p_1}{T_1} + \frac{p_2}{T_2}\right)$ which moreover is an invariant of X_{K_c}, i.e., $X_{K_c}F = 0$ and satisfies (22). According to Proposition 5.1, the closed-loop contact system is an input–output contact system with contact form

$$\theta_d = dx_0' - p^{\top}dx = d\left(x_0 + \frac{p_1}{T_1} + \frac{p_2}{T_2}\right) - p^{\top}dx,$$

and closed-loop contact Hamiltonian

$$K = K_0 + K_a - \frac{1}{2}K_c^2 + vK_c.$$

Remark 5.1. The stability of the closed-loop system is not discussed in this paper. However it is possible to define a restriction of the control law to some desired Legendre submanifold $\mathcal{L}_{U_d}$, where U_d is a desired generating function, such that the closed-loop contact vector field is stable restricted to $\mathcal{L}_{U_d}$. This has been presented in [25]. For this particular example, an invariant Legendre submanifold with $p_1 = p_2 = \frac{\partial U_d}{\partial x_1} = \frac{\partial U_d}{\partial x_2} = T^* > 0$, where T^* is a desired temperature, stabilizes the closed-loop contact vector field restricted to $\mathcal{L}_{U_d}$ at T^*.

6. Conclusions

In this paper the feedback equivalence of input–output contact systems have been analysed extending preliminary results of [26].

In Section 3 we have shown that the only state feedback preserving the contact structure of a control contact system is the constant one. This result is different than for the control of Hamiltonian systems [10,11], despite the formal similarity between the two classes of systems. This leads us to look for a state feedback which results in a closed-loop system which leaves a different contact form invariant. This is a problem quite similar to the IDA-PBC method for port-Hamiltonian systems, where the closed-loop system is port-Hamiltonian with respect to different structure matrices (or Leibniz brackets) [27]. We have then established a matching condition between the closed-loop contact form and the state feedback.

In Section 4 we restrict the problem to control contact systems defined by strict contact vector fields, that is that leave invariant the contact form itself, and where the difference between the open-loop and the closed-loop contact form is an exact 1-form. This allows to show that the admissible feedbacks are the composition of an arbitrary function with the control contact Hamiltonian, a result completely similar to input–output Hamiltonian systems [10,11]. However there is an additional condition to be satisfied which consist in a linear first-order PDE in the function whose differential is the added exact 1-form defining the closed-loop contact form, and which guarantees the existence of a closed-loop contact form.

In Section 5, based on the definition of the admissible feedback, the natural output of a control contact system is defined as the control contact Hamiltonian. From this follows the definition of input–output contact systems, completely analogous to input–output Hamiltonian systems. It is shown that the conditions for feedback equivalence of input–output contact systems consist in adding to the previous matching PDE, the condition that the function whose differential is the added exact 1-form, is an invariant of the control contact vector field.

A logical extension of this work is to consider multi-input and output contact systems, but more interesting is the problem of finding *stabilizing* structure-preserving feedback controls. Preliminary work [25] has considered a subclass of control contact systems, called conservative contact systems, which leave invariant some Legendre submanifold in closed-loop. In this case the closed-loop system may be interpreted as a thermodynamic system and the control law may be expressed as a state-feedback of the base manifold of extensive variables of the system. Finally it should be observed that contact systems have been contextualized in this paper as irreversible thermodynamic systems expressed in the Thermodynamic Phase Space. However contact systems also appear to represent time-dependent Hamiltonian systems [16, Chap. V] and in this context, the present work could eventually also be used for the stabilization of time-dependent port-Hamiltonian systems [28].

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