Decentralized Control Strategy for the Implementation of Cooperative Dynamic Behaviors in Networked Systems

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Abstract— **Decentralized control of networked systems has been widely investigated in the literature, with the aim of obtaining coordinated emerging behaviors (e.g. synchronization, swarming, coverage, formation control) by means of local interaction. In this paper we consider the possibility of injecting external inputs into the networked system, in order to obtain more complex cooperative behaviors. Specifically, we introduce a strategy that makes it possible to control the overall state of the networked system by directly controlling only a subset of the networked agents, namely the leaders. Exploiting local interaction rules, it is possible to define the inputs for the leaders in such a way that each follower is forced to track a desired periodic setpoint.**

I. INTRODUCTION

This paper introduces a methodology to implement dynamic complex behaviors in a networked system. The main objective is to have a subset of agents, called *leaders*, that are in charge of controlling the overall state of the networked system, in a completely decentralized manner.

Generally speaking, the aim of decentralized control strategies is implementing local interaction rules to obtain a coordinated emerging behavior. Mainly investigated coordinated behaviors include aggregation, swarming, formation control, coverage and synchronization [1]–[4].

The idea of implementing more complex cooperative behaviors have recently appeared in the literature. For instance, [5], [6] present decentralized strategies for the coordination of groups of mobile robots moving along non–trivial paths. A decentralized strategy is presented in [7] that extends the standard consensus protocol to obtain periodic geometric patterns.

Recently a few works appeared that investigate the possibility of *interacting* with a networked system, in order to obtain a desired behavior [8]. The idea is that of having a set of agents, interconnected by means of a graph: a subset of those agents, namely the *leaders*, may be directly controlled, while the others, namely the *followers*, are indirectly controlled through the underlying interconnection graph.

As shown in [9], it is possible to model a networked system in such a way that the classical notions of controllability and observability of LTI systems are applicable. Specifically, it turns out that in the case of networked systems these properties are heavily influenced by the topology of the underlying communication graph. It is possible to demonstrate

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that weighted graphs can be opportunely defined to guarantee the controllability of the networked system almost surely, provided that the graph is connected. This property can be ensured, in a decentralized manner, exploiting connectivity maintenance algorithms [10]–[12]. Duality principle can then be invoked to show that a networked system is controllable if and only if it is observable.

In this paper we exploit the well known *regulator equations* to design a control law that makes a networked system follow a predefined periodic setpoint.

II. PRELIMINARIES

A. Matrix operators

In this section we define some matrix operators that will be used throughout the paper.

Let $\Omega \in \mathbb{R}^{\rho \times \sigma}$ be a generic matrix. Then, we define

- $\Omega[i, :] \in \mathbb{R}^{\sigma}$ as the row vector containing the i–th row of Ω.
- $\Omega[:, j] \in \mathbb{R}^{\rho}$ as the column vector containing the j–th column of Ω.
- $\Omega[i, j] \in \mathbb{R}$ as the element (i, j) of Ω.
- $\Omega[i : k, j] \in \mathbb{R}^{k-i}$, with $k > i$, as the column vector containing those entries of the j–th column of Ω which are included between the indices i and k.
- $\Omega[i, j : k] \in \mathbb{R}^{k-j}$, with $k > j$, as the row vector containing those entries of the i–th row of Ω which are included between the indices j and k.

On the same lines, let $\omega \in \mathbb{R}^{\rho}$ be a generic vector. Then, we define:

- $\omega[i : j] \in \mathbb{R}^{j-i}$, with $j > i$ as the vector containing those entries of ω which are included between the indices i and j.

Let now $\omega \in \mathbb{R}^{\rho}$ be a generic vector. We define the operator $\mathrm{mat}_{\cdot \times \cdot}(\cdot)$ as follows:

$$\mathrm{mat}_{\sigma \times \zeta}(\omega) = \Omega \in \mathbb{R}^{\sigma \times \zeta} \qquad (1)$$

such that

$$\Omega[i, j] = \omega[i + \sigma(j - 1)]$$

$\forall i = 1, \ldots, \sigma, \, \forall j = 1, \ldots, \zeta$, with $\rho = \sigma\zeta$.

B. Model of the system

Consider a group of N agents, namely mobile robots, sensors or other entities, whose interconnection structure is modeled by means of an undirected graph $\mathscr{G}$. Let $\mathsf{V}(\mathscr{G})$ and $\mathsf{E}(\mathscr{G})$ be the vertex set and the edge set of the graph $\mathscr{G}$, respectively. Moreover, let N be the cardinality of $\mathsf{V}(\mathscr{G})$ (i.e. the number of vertices, or nodes, of the graph), and let

M be the cardinality of $\mathsf{E}\left(\mathscr{G}\right)$ (i.e. the number of edges, or links, of the graph). Clearly, $\mathsf{E} \subseteq \mathsf{V} \times \mathsf{V}$.

Let $\mathcal{L}\left(\mathscr{G}\right) \in \mathbb{R}^{N \times N}$ be the (unweighted) Laplacian matrix of the graph $\mathscr{G}$. Moreover, considering an edge–weighted graph, let $\mathcal{L}_\mathcal{W}\left(\mathscr{G}\right) \in \mathbb{R}^{N \times N}$ be the weighted Laplacian matrix of the graph $\mathscr{G}$, for an opportune set of edge–weights.

Let $x_i \in \mathbb{R}^m$ be the state of the i–th agent: without loss of generality, we will hereafter consider the case where the state corresponds to each agent's position. Then, let the agents be interconnected according to the well known (weighted) consensus protocol [3]:

$$\dot{x}_i = - \sum_{j \in \mathsf{N}_i} w_{ij} \left(x_i - x_j\right) \qquad (2)$$

where $w_{ij} > 0$ is the edge weight, and $\mathsf{N}_i \subseteq \mathsf{V}\left(\mathscr{G}\right)$ is the neighborhood of the i–th agent, defined as the set of the agents that are interconnected to the i–th one, namely:

$$\mathsf{N}_i = \left\{ j \in \mathsf{V}\left(\mathscr{G}\right) \text{ such that } (v_i, v_j) \in \mathsf{E}\left(\mathscr{G}\right) \right\} \qquad (3)$$

Without loss of generality, we will hereafter refer to the scalar case, namely $x_i \in \mathbb{R}$. It is however possible to extend all the results to the multi–dimensional case, considering each component independently.

Hence, let $x = \left[x_1, \ldots, x_N\right]^T \in \mathbb{R}^N$ be the state of the multi–agent system. The interaction rule defined in Eq. (2) can be rewritten as follows:

$$\dot{x} = -\mathcal{L}_\mathcal{W}\left(\mathscr{G}\right) x \qquad (4)$$

As is well known [3], under the consensus protocol the states of the agents converge to a common value. Assume now that the goal is to control the states of the networked agents: for this purpose, define a few leader agents, to whom it is possible to inject a control action. The state of the other agents, referred to as the followers, evolves according to the consensus protocol.

More specifically, let $\mathsf{V}_L\left(\mathscr{G}\right) \subset \mathsf{V}\left(\mathscr{G}\right)$ be the set of the leader agents, and let $\mathsf{V}_F\left(\mathscr{G}\right) = \mathsf{V}\left(\mathscr{G}\right) - \mathsf{V}_L\left(\mathscr{G}\right)$ be the set of the follower agents. Then, as shown in [9] for unweighted graphs, the interaction rule introduced in Eq. (2) is modified as follows:

$$\begin{cases} \dot{x}_i = - \sum_{j \in \mathsf{N}_i} w_{ij} \left(x_i - x_j\right) & \text{if } v_i \in \mathsf{V}_F\left(\mathscr{G}\right) \\ x_i = u_i & \text{if } v_i \in \mathsf{V}_L\left(\mathscr{G}\right) \end{cases} \qquad (5)$$

where $u_i = u_i\left(t\right) \in \mathbb{R}$ is a control input.

Let N_L be the number of leaders. It is always possible to index the agents such that the last N_L agents are the leaders, and the first $N_F = N - N_L$ are the followers. Then, as shown in [9], it is possible to decompose the Laplacian matrix $\mathcal{L}_\mathcal{W}\left(\mathscr{G}\right)$ as follows:

$$\mathcal{L}_\mathcal{W}\left(\mathscr{G}\right) = - \left[\begin{array}{c|c} \mathcal{A} & \mathcal{B} \\ \hline \mathcal{B}^T & \mathcal{L}_L \end{array} \right] \qquad (6)$$

where $\mathcal{A} = \mathcal{A}^T \in \mathbb{R}^{N_F \times N_F}$ is the Laplacian matrix of the subgraph of the followers, $\mathcal{B} \in \mathbb{R}^{N_F \times N_L}$ represents the interconnection among leaders and followers, and

$\mathcal{L}_L = \mathcal{L}_L^T \in \mathbb{R}^{N_L \times N_L}$ is the Laplacian matrix of the subgraph of the leaders.

Define now $x_F \in \mathbb{R}^{N_F}$ as the state vector of the followers, namely $x_F = \left[x_1, \ldots, x_{N_F}\right]^T$. Define also $u \in \mathbb{R}^{N_L}$ as the input vector, namely $u = \left[u_{N_F+1}, \ldots, u_N\right]^T$. Moreover, let $y \in \mathbb{R}^{N_L}$ be the output vector, that is the vector containing the state variables that are measurable by the leaders: it is reasonable to assume that each leader is able to measure the state of its neighbors.

We assume that the leader nodes are able to directly exchange information among each other. Namely, the following assumption is made:

Assumption 1 *A complete communication graph exists among the leader nodes.*

Therefore, the dynamics of the networked system can then be rewritten as a standard LTI system, namely:

$$\begin{cases} \dot{x}_F & = \mathcal{A} x_F + \mathcal{B} u \\ y & = \mathcal{B}^T x_F \end{cases} \qquad (7)$$

Hence, the classical notions of controllability and observability can be applied to the networked system itself. In particular, the following property can be derived:

Property 1 *A networked system whose dynamics are written according to Eq.* (7) *is observable if and only if it is controllable.*

Hence, we will hereafter suppose that the (possibly weighted) communication graph $\mathscr{G}$ is designed in such a way that the corresponding LTI system defined as in Eq. (7) is controllable and observable.

Therefore, once observability is guaranteed, a standard Luenberger state observer [13] can be designed. Specifically, let $\hat{x} \in \mathbb{R}^{N_F}$ be the estimate of x_F, and let $\mathcal{K}_l \in \mathbb{R}^{N_F \times N_L}$ be an opportunely chosen gain matrix. The following update law may be defined for the state observer:

$$\dot{\hat{x}} = \mathcal{A}\hat{x} + \mathcal{B}u - \mathcal{K}_l \left(y - \mathcal{B}^T \hat{x}\right) \qquad (8)$$

It is worth remarking that, under Assumption 1, this state observer can be implemented in a decentralized manner, that is exploiting only information locally available to the leaders. The estimation error $\hat{e} \in \mathbb{R}^{N_F}$ can be defined as

$$\hat{e} = x_F - \hat{x} \qquad (9)$$

III. DEFINITION OF THE CONTROL LAW FOR SETPOINT TRACKING

In this Section we introduce a methodology to define a control law that makes the followers track a periodic setpoint, defined by means of an exosystem [14], that is an autonomous system whose state vector $\xi \in \mathbb{R}^{2n+1}$ evolves according to the following dynamics:

$$\dot{\xi} = \mathcal{G}\xi \qquad (10)$$

where the matrix $\mathcal{G} \in \mathbb{R}^{(2n+1)\times(2n+1)}$ is a block diagonal matrix, namely:

$$\mathcal{G} = \begin{bmatrix} \tilde{\mathcal{G}}_0 & 0 & \dots & 0 \\ 0 & \tilde{\mathcal{G}}_1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \tilde{\mathcal{G}}_n \end{bmatrix} \tag{11}$$

Let each block $\tilde{\mathcal{G}}_p$ be defined as follows:

$$\tilde{\mathcal{G}}_p = \begin{cases} 0 & \text{if } p = 0 \\ \begin{bmatrix} 0 & p\dfrac{2\pi}{T} \\ -p\dfrac{2\pi}{T} & 0 \end{bmatrix} & \text{if } p = 1,\dots,n \end{cases} \tag{12}$$

The exosystem exhibits the following property:

Property 2 *Any periodic function with period T may be approximated with a linear combination of the elements of the vector ξ.*

In fact, defining the initial state of the exosystem $\xi_0 \in \mathbb{R}^{2n+1}$ as follows:

$$\xi_0 = [1, 0, 1, 0, \dots, 1]^T \tag{13}$$

the solution of the differential equation in Eq. (10) is the following:

$$\xi(t) = e^{\mathcal{G}t}\xi_0 = \begin{bmatrix} 1 \\ \sin\left(\dfrac{2\pi}{T}t\right) \\ \cos\left(\dfrac{2\pi}{T}t\right) \\ \vdots \\ \sin\left(n\dfrac{2\pi}{T}t\right) \\ \cos\left(n\dfrac{2\pi}{T}t\right) \end{bmatrix} \tag{14}$$

Consider then a generic periodic setpoint $x_s(t) \in \mathbb{R}^{N_F}$, that has to be tracked by the state $x_F(t) \in \mathbb{R}^{N_F}$ of the followers. According to Property 2, it is possible to define a matrix $\mathcal{J} \in \mathbb{R}^{N_F \times (2n+1)}$ such that

$$x_s(t) = \mathcal{J}\xi(t) \tag{15}$$

Therefore, once the period T and the number of harmonics n have been defined, then the choice of the matrix $\mathcal{J}$ leads to the definition of a particular periodic setpoint x_s.

Define then the regulation error $e(t) \in \mathbb{R}^{N_F}$ as follows:

$$e(t) = x_F(t) - \mathcal{J}\xi(t) \tag{16}$$

Then, considering Eqs. (7), (10) and (16), the dynamics of the networked system can be rewritten as follows:

$$\begin{cases} \dot{x}_F & = \mathcal{A}x_F + \mathcal{B}u \\ y & = \mathcal{B}^T x_F \\ e & = x_F - \mathcal{J}\xi \\ \dot{\xi} & = \mathcal{G}\xi \end{cases} \tag{17}$$

As is well known [14] the regulation problem can be solved defining the input u as follows:

$$u = \mathcal{F}x_F + (\Gamma - \mathcal{F}\Pi)\xi \tag{18}$$

where $\mathcal{F}$ is an arbitrary matrix, chosen such that $(\mathcal{A} + \mathcal{B}\mathcal{F})$ is Hurwitz stable, and Π and Γ are the solution of the *regulator equations* that, considering the dynamical system described in Eq. (17), can be written as follows:

$$\begin{cases} \mathcal{A}\Pi + \mathcal{B}\Gamma = \Pi\mathcal{G} & \text{(19a)} \\ \Pi - \mathcal{J} = 0 & \text{(19b)} \end{cases}$$

It is worth noting that Eq. (19a) defines a generalized Sylvester equation [15], while Eq. (19b) can be solved satisfying the following equality:

$$\mathcal{J} = \Pi \tag{20}$$

We will hereafter introduce some quantities that will be exploited to provide a solution for the regulator equations. Hence, define the matrix $\bar{\mathcal{G}} \in \mathbb{R}^{N_F(2n+1)\times(2n+1)}$ as follows:

$$\bar{\mathcal{G}} = \begin{bmatrix} \mathcal{G}^0 \\ \mathcal{G}^1 \\ \vdots \\ \mathcal{G}^{N_F-1} \end{bmatrix} \tag{21}$$

Inspired by [15], define $\Xi \in \mathbb{R}^{N_F \times (N_L N_F)}$ as follows:

$$\Xi = \begin{bmatrix} \mathcal{R}_{(N_F-1),0}\mathcal{B} \mid \mathcal{R}_{(N_F-1),1}\mathcal{B} \mid \dots \mid \mathcal{R}_{(N_F-1),(N_F-1)}\mathcal{B} \end{bmatrix} \tag{22}$$

Each matrix $\mathcal{R}_{(N_F-1),p} \in \mathbb{R}^{N_F \times N_F}$, $p = 0,\dots,N_F - 1$ is computed according to the following recursive definition:

$$\mathcal{R}_{q+1,p} = \begin{cases} \mathcal{A}\mathcal{R}_{q,p} + \chi_{q+1,p}\mathbb{I} & \text{if } p = 0 \\ -\mathcal{R}_{q,p-1} + \chi_{q+1,p}\mathbb{I} & \text{if } p = q + 1 \\ \mathcal{A}\mathcal{R}_{q,p} - \mathcal{R}_{q,p-1} + \chi_{q+1,p}\mathbb{I} & \text{otherwise} \end{cases}$$

$$\chi_{q+1,p} = \begin{cases} -\dfrac{1}{q+1}\,\mathrm{tr}\left(\mathcal{A}\mathcal{R}_{q,p}\right) & \text{if } p = 0 \\ \dfrac{1}{q+1}\,\mathrm{tr}\left(\mathcal{R}_{q,p-1}\right) & \text{if } p = q + 1 \\ -\dfrac{1}{q+1}\,\mathrm{tr}\left(\mathcal{A}\mathcal{R}_{q,p} - \mathcal{R}_{q,p-1}\right) & \text{otherwise} \end{cases}$$

with $\mathcal{R}_{0,0} = \mathbb{I}$

$$\tag{23}$$

where $\mathbb{I}$ is the identity matrix of opportune dimension, and $\mathrm{tr}(\cdot)$ is the trace operator.

Define then $\mathcal{Z} \in \mathbb{R}^{N_L \times (2n+1)}$ as an arbitrarily chosen parameter matrix. Considering the definition of $\mathcal{G}$ given in Eq. (11), the matrix $\mathcal{Q} \in \mathbb{R}^{(N_F N_L)\times(2n+1)}$ may be defined as follows:

$$\mathcal{Q} = \begin{bmatrix} \mathcal{Z} \\ \mathcal{Z}\mathcal{G} \\ \vdots \\ \mathcal{Z}\mathcal{G}^{N_F-1} \end{bmatrix} \tag{24}$$

Moreover, considering the definition of $\chi_{i,j}$ given in Eq. (23), define the matrix $\mathcal{X} \in \mathbb{R}^{(2n+1) \times (2n+1)}$ as follows:

$$\mathcal{X} = \sum_{p=0}^{N_F} \chi_{N_F,p} \mathcal{G}^p \tag{25}$$

As shown in [15, Theorem 1], assuming that matrices $\mathcal{A}$ and $\mathbb{I}$ have no eigenvalues in common with $\mathcal{G}$, then a solution for the generalized Sylvester equation in Eq. (19a) is given by the matrices Π and Γ defined as follows:

$$\begin{cases} \Pi = \Xi \mathcal{Q} & (26a) \\ \Gamma = \mathcal{Z} \mathcal{X} & (26b) \end{cases}$$

A. Admissible setpoint functions

The equality in Eq. (20) represents a constraint on matrix $\mathcal{J}$, that, according to Eq. (15), implies a constraint on the choice of the setpoint.

Considering then the definition of the setpoint functions $x_s(t)$ given in Eq. (15), and considering the regulator equations in Eq. (19), we now introduce the following definition of *admissible setpoint functions*.

Definition 1 *The set of admissible setpoint functions $\mathsf{S}_a \in \mathbb{R}^{N_F}$ is defined as follows:*

$$\mathsf{S}_a = \{x_s(t) = \mathcal{J}\xi(t) \text{ such that } \mathcal{J} = \Pi\} \tag{27}$$

where Π is a solution of the generalized Sylvester equation in Eq. (19a), that is Π is defined according to Eq. (26a).

We will now characterize the set of admissible setpoint functions, defined according to Definition 1. For this purpose, define the matrix $\mathcal{H} \in \mathbb{R}^{N_F(2n+1) \times N_L(2n+1)}$ according to the following equation:

$$\mathcal{H}[pq,:] = \bar{\mathcal{G}}^T[q,:] \left(\mathrm{mat}_{N_L \times N_F}(\Xi[p,:])\right)^T \otimes \mathbb{I} \tag{28}$$

where $\mathbb{I}$ is the identity matrix of opportune dimension, and the symbol $\otimes$ represents the Kronecker product.

Consider again the arbitrary parameter matrix $\mathcal{Z} \in \mathbb{R}^{N_L \times (2n+1)}$. The vector $\Lambda \in \mathbb{R}^{N_F(2n+1)}$ may be defined as follows:

$$\Lambda = \mathcal{H}\bar{\mathcal{Z}} \tag{29}$$

where the vector $\bar{\mathcal{Z}} \in \mathbb{R}^{N_L(2n+1)}$ is defined as follows:

$$\bar{\mathcal{Z}} = [\mathcal{Z}[1,:] \mid \mathcal{Z}[2,:] \mid \ldots \mid \mathcal{Z}[N_L,:]]^T \tag{30}$$

Being $\mathcal{Z}$ an arbitrary parameter matrix, then the vector Λ is arbitrarily defined in the image of $\mathcal{H}$.

The following Proposition provides the main result of the paper. Namely, we show that the image of $\mathcal{H}$ defines the set of admissible setpoint functions S_a.

Proposition 1 *Consider the definition of the setpoint functions given in Eq. (15), and consider the definition of the set of admissible setpoint functions S_a given in Eq. (27). Consider also a vector Λ defined as in Eq. (29), that is a generic vector in the image of $\mathcal{H}$. Then, a matrix $\mathcal{J}$ defined as follows:*

$$\mathcal{J} = \mathrm{mat}_{N_F \times (2n+1)}(\Lambda) \tag{31}$$

defines the set S_a according to Definition 1.

Proof: Consider the definition of matrix Π given in Eq. (26a). According to the definition of matrix $\mathcal{Q}$ in Eq. (24), then the element (p,q) of Π can be written as follows:

$$\Pi[p,q] = \sum_{k=1}^{N_L} \mathcal{Z}[k,:] \sum_{r=0}^{N_F-1} \Xi[p,(r+1)N_L+k] \mathcal{G}^r[:,q] \tag{32}$$

Considering then the definition of $\bar{\mathcal{G}}$ in Eq. (21), and the definition of $\bar{\mathcal{Z}}$ in Eq. (30), then Eq. (32) can be rewritten as follows:

$$\Pi[p,q] = \bar{\mathcal{Z}}^T \mathrm{mat}_{N_L \times N_F}(\Xi[p,:]) \otimes \mathbb{I} \bar{\mathcal{G}}[:,q] \tag{33}$$

where $\mathbb{I}$ is the identity matrix of opportune dimension.

Since $\Pi[p,q] \in \mathbb{R}$ is a scalar, then it it possible to rewrite Eq. (33) computing the transpose of the right–hand side term, namely:

$$\Pi[p,q] = \bar{\mathcal{G}}^T[q,:] \left(\mathrm{mat}_{N_L \times N_F}(\Xi[p,:])\right)^T \otimes \mathbb{I} \bar{\mathcal{Z}} \tag{34}$$

According to Eq. (28), it is possible to rewrite Eq. (34) as follows:

$$\Pi[p,q] = \mathcal{H}[pq,:] \bar{\mathcal{Z}} \tag{35}$$

Thus, considering the definition of Λ given in Eq. (29), then Eq. (35) can be rewritten in a compact form as follows:

$$\begin{bmatrix} \Pi[:,1] \\ \vdots \\ \Pi[:,N_F] \end{bmatrix} = \mathcal{H}\bar{\mathcal{Z}} = \Lambda \tag{36}$$

Therefore, defining $\mathcal{J}$ according to Eq. (31), then

$$\mathcal{J} = \Pi$$

Namely, the condition in Eq. (20) is satisfied, which proves the statement. ∎

It is worth remarking that the elements of $\mathcal{Z} \in \mathbb{R}^{N_L \times (2n+1)}$ can be arbitrarily defined, and represent then $N_L(2n+1)$ degrees of freedom in the design of the control system.

Proposition 1 proves that the elements of matrix Π correspond to those of Λ. According to Eq. (29), Λ is arbitrarily defined in the image of $\mathcal{H}$. Hence, the image of $\mathcal{H}$ completely defines the set of admissible setpoint functions S_a. Specifically, matrix Π is completely determined once the $N_L(2n+1)$ terms of $\mathcal{Z}$ have been defined. In other words, any admissible setpoint function is defined by means of the selection of $N_L(2n+1)$ free parameters. Since N_L is the number of leaders, it is possible to conclude that each leader introduces $(2n+1)$ degrees of freedom for the definition of the setpoint function.

Clearly, the dimension of the image of $\mathcal{H}$ is related to the rank of $\mathcal{H}$ itself. It is worth noting that matrix $\mathcal{H}$ is completely determined once the topology of the networked system has been defined, as well as the number of harmonics n used to represent the setpoint. In fact, considering its definition given in Eq. (28), matrix $\mathcal{H}$ is defined as a function of $\bar{G}$ and Ξ:

- According to Eqs. (11) and (21), matrix $\bar{G}$ is completely determined once n and N_F have been defined.
- According to Eqs. (22) and (23), matrix Ξ is completely determined once matrices $\mathcal{A}$ and $\mathcal{B}$ have been defined, that is once the topology of the graph has been defined.

Therefore, the rank of $\mathcal{H}$ is a function of the topology of the network. Specifically, it is possible to demonstrate that, based on the choice of the leaders, the dimension of the image of $\mathcal{H}$ may be lower than $N_L(2n+1)$. In other words, increasing the number of leaders (i.e. increasing N_L) does not always guarantee increasing the dimension of the image of $\mathcal{H}$. Optimal leader selection is out of the scope of this paper, and then will not be analyzed. However, several strategies can be found in the literature to solve this issue [16], [17].

B. Definition of the control law

The solution of the regulator equations introduced in Eq. (19), namely a pair of matrices (Π, Γ) defined as in Eq. (26), can then be exploited to define a control law to make the system follow an admissible setpoint.

Specifically, the control law in Eq. (18) makes the networked system follow the desired setpoint function, defined as in Eq. (15), where matrix $\mathcal{J}$ is defined in order to satisfy the conditions of Proposition 1.

However, for the control law to be applicable in a decentralized manner, the vector x_F in Eq. (18) has to be replaced with its estimate, computed by means of the state observer introduced in Eq. (8).

Considering the dynamics of the networked system defined in Eq. (7), the estimation error $\hat{e}$ defined in Eq. (9), the state observer defined in Eq. (8), and the control law defined in Eq. (18), the dynamics of the closed loop system can then be summarized as follows:

$$\begin{bmatrix} \dot{x}_F \\ \dot{\hat{e}} \\ \dot{\xi} \end{bmatrix} = \begin{bmatrix} \mathcal{A} + \mathcal{B}\mathcal{F} & -\mathcal{B}\mathcal{F} & \mathcal{B}(\Gamma - \mathcal{F}\Pi) \\ \mathbb{O} & \mathcal{A} + \mathcal{K}_l \mathcal{B}^T & \mathbb{O} \\ \mathbb{O} & \mathbb{O} & \mathcal{G} \end{bmatrix} \begin{bmatrix} x_F \\ \hat{e} \\ \xi \end{bmatrix} \tag{37}$$

where $\mathbb{O}$ is a zero matrix of opportune dimension.

As is well known from basic linear control theory [13], Eq. (37) guarantees that the state vector $x_F(t)$ tracks the desired septoin $x_s(t)$.

IV. SIMULATIONS

Several simulations have been carried out in order to evaluate the performance of the proposed control strategy. In the simulations, we considered single integrator agents moving in a three dimensional environment, namely $x_i \in \mathbb{R}^3$, $\forall i = 1, \dots, N$. Let $(\mathsf{x}, \mathsf{y}, \mathsf{z})$ represent the global reference frame.

Specifically, different graph topologies have been exploited to implement several admissible periodic setpoint functions. Examples are represented in Figs. 1 and 2, which show the results of simulations performed with $N = 8$ and $N = 15$ agents, respectively. In the figures, the topology of the communication graph is depicted (with Li indicating the i–th leader, and Fj indicating the j–th follower), as well

as the periodic setpoint functions defined for each follower. A complete communication graph is assumed among the leaders, but is not depicted for clarity purpose.

In order to evaluate the performance of the proposed control strategy, the regulation error $e(t)$, defined according to Eq. (16), namely

$$e(t) = x_F(t) - x_s(t)$$

was computed for each follower.

Specifically, Figs. 1 and 2 show the evolution of each element of the tracking error, namely $e^i(t)$, $\forall i = 1, \dots, N_F$. Due to space limitations, only the component along the x–axis of the tracking error is depicted.

As expected, Figs. 1 and 2 show that the tracking error asymptotically vanishes, for all the follower agents. Results of these simulations are also shown in Part 1 and Part 2 of the accompanying video clip[1], respectively. In the video, leaders are represented with green pyramids, while followers are represented with red pyramids.
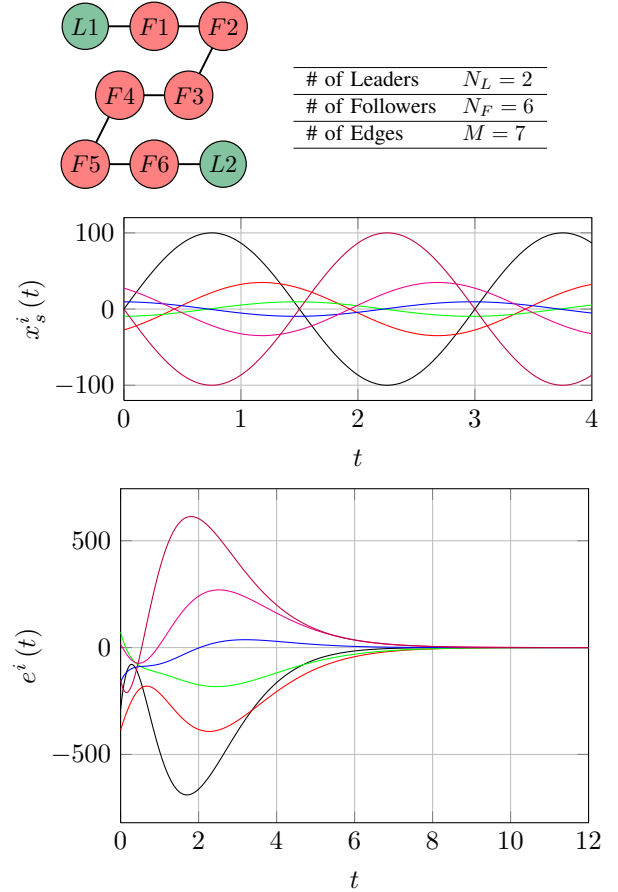

# of Leaders	$N_L = 2$
# of Followers	$N_F = 6$
# of Edges	$M = 7$

Fig. 1. Simulation results, with $N = 8$ agents (setpoint and tracking error along the x–axis)

The proposed control strategy was also validated for decentralized control of groups of quadrotor Unmanned Aerial Vehicles (UAVs). Several strategies can be found in the

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[1]A high quality extended version of the video clip is freely available at http://www.arscontrol.unimore.it/iros13

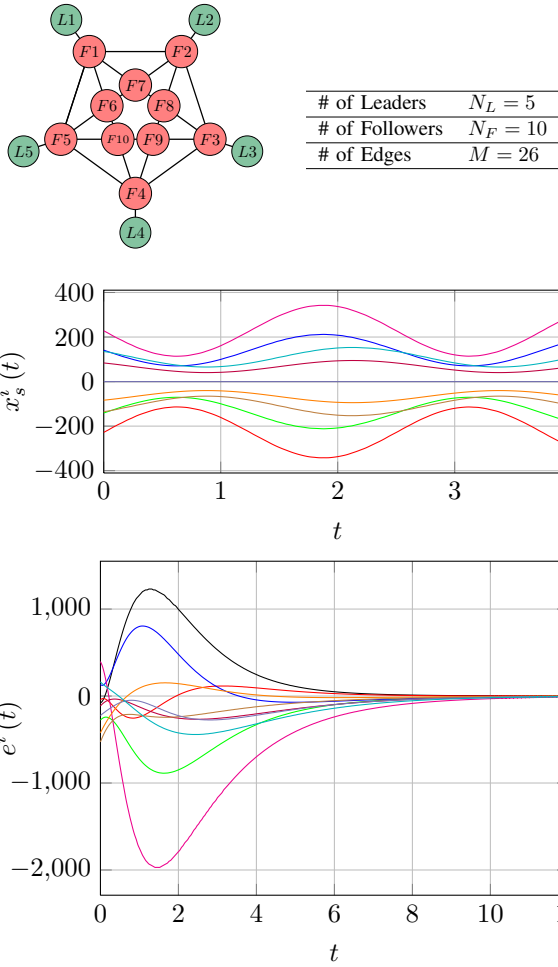| # of Leaders	$N_L = 5$
# of Followers	$N_F = 10$
# of Edges	$M = 26$

Fig. 2. Simulation results, with $N = 15$ agents (setpoint and tracking error along the x–axis)

literature [18] to build a local controller such that the closed loop behavior of each quadrotor UAV can be effectively approximated with that of a single integrator kinematic agent.

Hence, the scenarios previously simulated with Matlab were replicated with a team of UAVs. Specifically, Part 3 and Part 4 of the accompanying video clip show the movements of the follower agents, with the communication graph described as in Fig. 1 and Fig. 2 respectively.

V. CONCLUSIONS

In this paper we introduced a methodology to obtain complex cooperative behaviors in networked systems. Specifically, we introduced a strategy that makes it possible to control the overall state of the networked system by directly controlling only a subset of the networked agents, namely the leaders. Modeling the dynamics of the networked system as a standard LTI system, we demonstrated that it is possible to exploit standard design methodologies, based on the regulator equations, to make each follower agent track a desired periodic setpoint function.

We demonstrated that, given the topology of the communication graph, it is possible to define a set of admissible setpoint functions. Current work aims at solving the inverse problem, namely finding a suitable graph topology such that a given setpoint function is admissible.

Moreover, the proposed technique assumes that the communication topology is fixed, which is not always reasonable in real application scenarios, when dealing, for instance, with mobile robots with finite communication range. Current work aims at extending the scope of the proposed control strategy to variable topology communication graphs.

REFERENCES

[1] A. Ganguli, J. Cortes, and F. Bullo, "Multirobot rendezvous with visibility sensors in nonconvex environments," *IEEE Transactions on Robotics*, vol. 25, no. 2, pp. 340 –352, april 2009.

[2] J. A. Fax and R. M. Murray, "Information flow and cooperative control of vehicle formations," *IEEE Transactions on Automatic Control*, pp. 1465–1476, 2004.

[3] R. Olfati–Saber, J. A. Fax, and R. M. Murray, "Consensus and cooperation in networked multi–agent systems," *Proceedings of the IEEE*, vol. 95, no. 1, pp. 215–233, 2007.

[4] L. Sabattini, C. Secchi, and C. Fantuzzi, "Arbitrarily shaped formations of mobile robots: artificial potential fields and coordinate transformation," *Autonomous Robots (Springer)*, vol. 30, no. 4, pp. 385–397, may 2011.

[5] L. C. A. Pimenta, G. A. S. Pereira, N. Michael, R. C. Mesquita, M. M. Bosque, L. Chaimowicz, and V. Kumar, "Swarm coordination based on smoothed particle hydrodynamics technique," *IEEE Transactions on Robotics*, vol. PP, no. 99, pp. 1 –17, 2013.

[6] L. Sabattini, C. Secchi, and C. Fantuzzi, "Closed–curve path tracking for decentralized systems of multiple mobile robots," *Journal of Intelligent and Robotic Systems (Springer)*, vol. 71, no. 1, pp. 109–123, July 2013.

[7] P. Tsiotras and L. Castro, "Extended multi-agent consensus protocols for the generation of geometric patterns in the plane," in *Proceedings of the IEEE American Control Conference (ACC)*, 29 2011-july 1 2011, pp. 3850 –3855.

[8] A. Franchi, C. Secchi, H. I. Son, H. H. Bulthoff, and P. Robuffo Giordano, "Bilateral teleoperation of groups of mobile robots with time-varying topology," *IEEE Transactions on Robotics*, vol. 28, no. 5, pp. 1019–1033, 2012.

[9] M. Egerstedt, S. Martini, M. Cao, K. Camlibel, and A. Bicchi, "Interacting with networks: How does structure relate to controllability in single-leader, consensus networks?" *IEEE Control Systems*, vol. 32, no. 4, pp. 66 –73, aug. 2012.

[10] L. Sabattini, N. Chopra, and C. Secchi, "On decentralized connectivity maintenance for mobile robotic systems," in *Proceedings of the IEEE Conference on Decision and Control*, 2011.

[11] ——, "Decentralized connectivity maintenance for cooperative control of mobile robotic systems," *International Journal of Robotics Research*, 2013.

[12] L. Sabattini, C. Secchi, N. Chopra, and A. Gasparri, "Distributed control of multi–robot systems with global connectivity maintenance," *IEEE Transactions on Robotics*, 2013 (In press).

[13] E. D. Sontag, *Mathematical Control Theory: Deterministic Finite Dimensional Systems*. Springer, 1998.

[14] A. Saberi, A. A. Stoorvogel, and P. Sannuti, *Control of Linear Systems with Regulation and Input Constraints*. Springer-Verlag New York, LLC, 2000.

[15] A. G. Wu, G. R. Duan, and B. Zhou, "Solution to generalized sylvester matrix equations," *IEEE Transactions on Automatic Control*, vol. 53, no. 3, pp. 811 –815, april 2008.

[16] A. Franchi, H. H. Bulthoff, and P. Robuffo Giordano, "Distributed online leader selection in the bilateral teleoperation of multiple uavs," in *Proceedings of the IEEE Conference on Decision and Control and European Control Conference (CDC-ECC)*, 2011, pp. 3559–3565.

[17] A. Clark, L. Bushnell, and R. Poovendran, "On leader selection for performance and controllability in multi-agent systems," in *Proceedings of the IEEE Conference on Decision and Control (CDC)*, 2012, pp. 86–93.

[18] W. Ren and R. W. Beard, *Distributed Consensus in Multi-Vehicle Cooperative Control: Theory and Applications*. Springer, 2008.