

Putting Energy Back in Control

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Energy is one of the fundamental concepts in science and engineering practice, where it is common to view dynamical systems as energy-transformation devices. This perspective is particularly useful in studying complex nonlinear systems by decomposing them into simpler subsystems that, upon interconnection, add up their energies to determine the full system's behavior. The action of a controller may also be understood in energy terms as another dynamical system—typically implemented in a computer—interconnected with the process to modify its behavior. The control problem can then be recast as finding a dynamical system and an interconnection pattern such that the overall energy function takes the desired form. This “energy-shaping” approach is the essence of passivity-based control (PBC), a controller design technique that is very well known in mechanical systems.

Our objectives in this article are threefold:

- First, to call attention to the fact that PBC does not rely on some particular structural properties of mechanical systems, but hinges on the more fundamental (and universal) property of energy balancing.
- Second, to identify the physical obstacles that hamper the use of “standard” PBC in applications other than mechanical systems. In particular, we will show that “standard” PBC is stymied by the presence of unbounded energy dissipation, hence it is applicable only to systems that are stabilizable with passive controllers.
- Third, to revisit a PBC theory that has been recently developed to overcome the dissipation obstacle as well as to make the incorporation of process prior knowledge more systematic. These two important features allow us to design energy-based controllers for a wide range of physical systems.

Intelligent Control Paradigm

Control design problems have traditionally been approached from a signal-processing viewpoint; that is, the plant to be controlled and the controller are viewed as signal-processing devices that transform certain input signals into outputs. The control objectives are expressed in terms of keeping some error signals small and reducing the effect of certain disturbance inputs on the given regulated outputs, despite the presence of some unmodeled dynamics. To make the problem mathematically tractable, the admissi-

ble disturbances and unmodeled dynamics are assumed to be norm bounded, and consequently, the indicators of performance are the size of the gains of the operators that map these various signals. In the case of linear time-invariant systems, this “intelligent control paradigm” (paraphrasing Willems [1]) has been very successful, essentially because disturbances and unmodeled dynamics can be discriminated, via filtering, using frequency-domain considerations. The problem of reducing the gains of nonlinear operators can also be expressed in a clear, analytical way [2]. There are two fundamental differences, however, with respect to the linear time-invariant case: first, the solution involves some far from obvious computations. Second, and perhaps more important, since nonlinear systems “mix” the frequencies, it is not clear how to select the most likely disturbances, and we have to “crank up” the gain to quench the (large set of) undesirable signals and meet the specifications. Injecting high gains in the loop, besides being intrinsically conservative—hence yielding below-par performance—brings along a series of well-known undesirable features (e.g., noise amplification, actuator wear, and high energy consumption).

There are many practical control problems where we have available structural information about the plant. In these cases, it is reasonable to expect that the conservatism mentioned above could be reduced if we could incorporate this prior information in the controller design. Unfortunately, a procedure to systematically carry out this objective does not seem to be available. (The typical approach is to classify the nonlinearities according to the role they play in the derivative of a Lyapunov function candidate. This test has very little to do with the physics of the system. It is obviously tied up with the particular choice of the Lyapunov function, which, stemming from our linear inheritance, is systematically taken to be a quadratic function in the “errors.”) It is our contention that the inability to incorporate prior knowledge is inherent to the signal-processing viewpoint of the intelligent control paradigm and is therefore independent of the particular design technique. In the authors' opinion, this situation has stymied communication between practitioners and control theorists, seriously jeopardizing the future of modern model-based nonlinear control systems design.

The purpose of this article is to contribute, if modestly, to the reversal of this trend by calling attention to the impor-

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tance of incorporating energy principles in control. To achieve our objective, we propose to abandon the intelligent control paradigm and instead adopt the behavioral framework proposed by Willems [1]. In Willems's far-reaching interpretation of control, we start from a mathematical model obtained from first principles, say, a set of higher order differential equations and some algebraic equations. Among the vector of time trajectories satisfying these equations are components that are available for interconnection. The controller design then reduces to defining an additional set of equations for these interconnection variables to impose a desired behavior on the controlled system. We are interested here in the incorporation into this paradigm of the essential energy component. Therefore, we view dynamical systems (plants and controllers) as energy-transformation devices, which we interconnect (in a power-preserving manner) to achieve the desired behavior. More precisely, we are interested in lumped-parameter systems that satisfy an energy-balancing principle, where the interconnection with the environment is established through power port variables. The power port variables are conjugated, and their product has units of power, for instance, currents and voltages in electrical circuits or forces and velocities in mechanical systems. This is the scenario that arises from any form of physical network modeling.

Our first control objective is to regulate the static behavior (i.e., the equilibria), which is determined by the shape of the energy function. It is therefore natural to recast our control problem in terms of finding a dynamical system and an interconnection pattern such that the overall energy function takes the desired form. There are at least two important advantages of adopting such an "energy-shaping" perspective of control:

- 1) The energy function determines not just the static behavior, but also, via the energy transfer between subsystems, its transient behavior. Focusing our attention on the system energy, we can then aim, not just at stabilization, but also at performance objectives that can, in principle, be expressed in terms of "optimal" energy transfer. Performance and not stability is, of course, the main concern in applications.
- 2) Practitioners are familiar with energy concepts, which can serve as a *lingua franca* to facilitate communication with control theorists, incorporating prior knowledge and providing physical interpretations of the control action.

Background

The idea of energy shaping has its roots in the work of Takegaki and Arimoto [3] in robot manipulator control, a field where it is very well known and highly successful. Simultaneously and independently of [3], the utilization of these ideas for a large class of Euler-Lagrange systems was

suggested in [4]. (See also Slotine's innovative paper [5] and the related view on the control of physical systems by Hogan [6].) Using the fundamental notion of passivity, the principle was later formalized in [7], where the term PBC was coined to define a controller design methodology whose aim is to render the closed-loop system passive with a given storage function. The importance of linking passivity to energy shaping can hardly be overestimated. On the one hand, viewing the control action in terms of interconnections of passive systems provides an energy-balancing interpretation of the stabilization mechanism. More precisely, we have defined in [8] a class of systems (which includes mechanical systems) such that the application of PBC yields a closed-loop energy that is equal to the difference between the stored and the supplied energies. This special class of PBC energy is called energy-balancing PBC. On the other hand, showing that the approach does not rely on some particular structural properties of mechanical systems, but hinges instead on the more fundamental (and universal) property of passivity, it can be extended to cover a wide range of applications.

In carrying out this extension, two approaches have been pursued:

- The first approach is similar to classical Lyapunov-based design, where we first select the storage function to be assigned and then design the controller that ensures this objective. Extensive applications of this line of research may be found in [9] (see also [10]–[15]) and are not reviewed here. (It should be noted that in this approach, the desired storage function—typically quadratic in the increments—does not qualify as an energy function in any meaningful physical sense. Actually, it has been shown that the stabilization mechanism is akin to system inversion instead of energy shaping [9], hence a stable invertibility assumption is usually required.)
- The second, newer approach stems from the energy-balancing view of mechanical systems discussed above. The closed-loop storage function—which is now a *bona fide* energy function—is not postulated *a priori*, but is instead obtained as a result of our choice of desired subsystems interconnections and damping. This idea was first advanced for stability analysis in [16]; the extension for controller design was then reported in [17] and [8]; since then many successful applications, including mass-balance systems [18], electrical machines [19], power systems [20], magnetic levitation systems [21], and power converters [22], have been reported.

The aim of this article is to provide a new energy-balancing perspective of PBC that embraces and unifies its classical and modern versions. To enhance readability and widen our target audience, we strip away as much as possible the mathematical details and concentrate instead on the basic

underlying principles and limitations. To underscore the fact that the principles are universal, we present them in a very general circuit-theoretic framework, without any additional mathematical structure attached to the system models. Particular emphasis is given to exhibiting the physical interpretation of the concepts, for instance, the central role played by dissipation. Toward this end, we illustrate our main points with simple physical examples.

The remainder of the article is organized as follows. First, we review the basic notions of passivity and stabilization via energy shaping. Next, we define the concept of energy-balancing PBC and prove that this principle is applicable to all mechanical systems. Later we show that systems which extract an infinite amount of energy from the controller (i.e., systems with unbounded dissipation) cannot be stabilized with energy-balancing PBCs. To characterize the class of systems that are stabilizable with energy-balancing PBCs and eventually extend PBC to systems with unbounded dissipation, we propose to adopt Willems's "control-as-interconnection" viewpoint, a perspective that naturally provides a geometric interpretation to the notion of energy shaping. Then, after identifying a class of "admissible dissipations," we view the control action as the interconnection of the system with a passive controller. To stabilize systems with unbounded dissipations, we propose to model the action of the control as a state-modulated power-preserving interconnection of the plant with an infinite energy source system. These developments, which lead to the definition of a new class of PBCs called interconnection and damping assignment PBC, are presented for the so-called port-controlled Hamiltonian systems. Finally, we detail the application of interconnection and damping assignment PBC to a physical example, and then we present some concluding remarks.

Notation

All vectors in the article, including the gradient, are defined as column vectors. Also, we use κ throughout to denote a generic positive constant.

Passivity and Energy Shaping

We are interested here in lumped-parameter systems interconnected to the external environment through some port power variables $u \in \mathbb{R}^m$ and $y \in \mathbb{R}^m$, which are conjugated in the sense that their product has units of power (e.g., currents and voltages in electrical circuits, or forces and velocities in mechanical systems). We assume the system satisfies the energy-balance equation

$$\underbrace{H[x(t)] - H[x(0)]}_{\text{stored energy}} = \underbrace{\int_0^t u^T(s)y(s)ds}_{\text{supplied}} - \underbrace{d(t)}_{\text{dissipated}} \quad (1)$$

where $x \in \mathbb{R}^n$ is the state vector, $H(x)$ is the total energy function, and $d(t)$ is a nonnegative function that captures the dissipation effects (e.g., due to resistances and frictions). Energy balancing is, of course, a universal property of physical systems; therefore, our class, which is nothing other than the well-known passive systems, captures a very broad range of applications that include nonlinear and time-varying dynamics.

Two important corollaries follow from (1):

- The energy of the uncontrolled system (i.e., with $u=0$) is nonincreasing (that is, $H[x(t)] \leq H[x(0)]$), and it will actually decrease in the presence of dissipation. If the energy function is bounded from below, the system will eventually stop at a point of minimum energy. Also, as expected, the rate of convergence of the energy function is increased if we extract energy from the system, for instance, setting $u = -K_{di}y$, with $K_{di} = K_{di}^T > 0$ a so-called damping injection gain.
- If $H(x)$ is nonnegative, we have that

$$-\int_0^t u^T(s)y(s)ds \leq H[x(0)] < \infty. \quad (2)$$

That is, the total amount of energy that can be extracted from a passive system is bounded. [This property, which (somehow misleadingly) is often stated with the inequality inverted, will be instrumental in identifying the class of systems that are stabilizable with energy-balancing PBC.]

Standard Formulation of PBC

The point where the open-loop energy is minimal (which typically coincides with the zero state) is usually not the one of practical interest, and control is introduced to operate the system around some nonzero equilibrium point, say x_* . In the standard formulation of PBC, we label the port variables as inputs and outputs (say u and y , respectively) and pose the stabilization problem in a classical way. (We consider first static state feedback control laws and postpone the case of dynamic controllers to the section on admissible dissipations. Also, we refer the reader to [8] and references therein for further details on the dynamic and output feedback cases.)

- Select a control action $u = \beta(x) + v$ so that the closed-loop dynamics satisfies the new energy-balancing equation

$$H_d[x(t)] - H_d[x(0)] = \int_0^t v^T(s)z(s)ds - d_d(t)$$

where $H_d(x)$, the desired total energy function, has a strict minimum at x_* , z (which may be equal to y) is the new passive output, and we have replaced the natural

dissipation term by some function $d_d(t) \geq 0$ to increase the convergence rate. Assigning an energy function with a minimum at the desired value is usually referred to as energy shaping whereas while the modification of the dissipation is called damping injection.

Later, we will show that this classical distinction between inputs and outputs is restrictive, and the “control-as-interconnection” perspective of Willems is needed to cover a wider range of applications.

Discussion

Remark 1: For simplicity, we have treated *all* the components of the vector u as manipulated variables. In many practical cases, this vector contains some external (non-manipulated) variables such as disturbances or sources (see [8] and [19] for some examples). Furthermore, there are some applications where the control action does not enter at all in u , for instance, in switched devices [22]. The analysis we will present in the sequel applies as well—*mutatis mutandi*—to those cases.

Remark 2: The choice of the desired dissipation in the damping injection stage is far from obvious. For instance, contrary to conventional wisdom, and except for the highly unusual case where we can achieve exponential stability, performance is not necessarily improved by adding positive damping, but it can actually be degraded as illustrated in [22], [23]. Furthermore, as shown in [18] and [21], there are cases in which shuffling the damping between the channels can be beneficial for our control objective; this will be illustrated in the last section of this article.

Remark 3: It is well known that solving the stabilization problem via passivation automatically ensures some robustness properties. Namely, stability will be preserved for all passive unmodeled dynamics between the port variables u and z . When $z = y$, these correspond to phenomena such as frictions and parasitic resistances.

Remark 4: It is clear also that if the dissipation is such that the passivity property is strengthened to output strict passivity, that is,

$$\int_0^t v^T(s)z(s)ds \geq \delta \int_0^t |z(s)|^2 ds - \kappa$$

for some $\delta, \kappa > 0$, then we can show (with a simple completion of the squares argument) that the map $v \mapsto z$ has gain smaller than $1/\delta$. Consequently, we can reduce the amplification factor of the energy of the input noise by increasing the damping. See, however, Remark 2.

Remark 5: Passivity can be used for stabilization independently of the notion of energy shaping. In fact, it suffices to find an output $z = h(x)$ such that z square integrable implies $x(t) \rightarrow x_*$ as $t \rightarrow \infty$. Stabilization via passivation for general nonlinear systems, which has its roots in [24] and [25],

is one of the most active current research areas in nonlinear control, and some constructive results are available for systems in special forms [26], [27]. The energy shaping approach is a reasonable way to incorporate the information about the energy functions that is available in physical systems to simplify the passivation problem. Besides making the procedure more systematic, it usually yields physically interpretable controllers, considerably simplifying their commissioning stage. See [9] for an extensive discussion on these issues, including a detailed historical review, and the application of PBC to many practical examples.

Stabilization via Energy Balancing

There is a class of systems, which interestingly enough includes mechanical systems, for which the solution to the problem posed above is very simple, and it reduces to being able to find a function $\beta(x)$ such that the energy supplied by the controller can be expressed as a function of the state. Indeed, from (1) we see that if we can find a function $\beta(x)$ such that

$$-\int_0^t \beta^T[x(s)]y(s)ds = H_a[x(t)] + \kappa \quad (3)$$

for some function $H_a(x)$, then the control $u = \beta(x) + v$ will ensure that the map $v \mapsto y$ is passive with new energy function

$$H_d(x) \triangleq H(x) + H_a(x). \quad (4)$$

If, furthermore, $H_a(x)$ has a minimum at the desired equilibrium x_* , then it will be stable. Notice that the closed-loop energy is equal to the difference between the stored and the supplied energies. Therefore, we refer to this particular class of PBCs as energy-balancing PBCs.

Mechanical Systems

Let us look at the classical example of position regulation of fully actuated mechanical systems with generalized coordinates $q \in \mathbb{R}^{n/2}$ and total energy

$$H(q, \dot{q}) = \frac{1}{2} \dot{q}^T D(q) \dot{q} + V(q)$$

where $D(q) = D^T(q) > 0$ is the generalized mass matrix and $V(q)$ is the system’s potential energy, which is also bounded from below. It has been shown in [7] that for these systems, the passive outputs are the generalized velocities (that is, $y = \dot{q}$). The simplest way to satisfy condition (3) and shape the energy is by choosing

$$\beta(q) = \frac{\partial V}{\partial q}(q) - K_p(q - q_*)$$

where q_* is the desired constant position and $K_p = K_p^T > 0$ is a proportional gain. Indeed, replacing the expression above and $y = \dot{q}$ in (3) we get

$$-\int_0^t \beta^T [q(s)] \dot{q}(s) ds = -V[q(t)] + \frac{1}{2} [q(t) - q_*]^T K_p [q(t) - q_*] + \kappa,$$

and the new total energy for the passive closed-loop map $v \mapsto \dot{q}$ is

$$H_d(q, \dot{q}) = \frac{1}{2} \dot{q}^T D(q) \dot{q} + \frac{1}{2} (q - q_*)^T K_p (q - q_*),$$

which has a minimum at $(q_*, 0)$, as desired. To ensure that the trajectories actually converge to this minimum (i.e., that the equilibrium is asymptotically stable), we add some damping $v = -K_d \dot{q}$, as discussed above.

Of course, the controller presented above is the very well-known PD + gravity compensation of [3]. The purpose of the exercise is to provide a new interpretation for the action of this controller, underscoring the fact that the storage function that is assigned to the closed loop is (up to an integration constant) precisely the difference between the stored and the supplied energies (i.e., $H(x) - \int_0^t u^T(s) y(s) ds$). Hence application of PBC for position regulation of mechanical systems yields energy-balancing PBCs.

Remark 6: With the elementary procedure described above, it is possible to rederive most of the energy-balancing PBCs (e.g., with saturated inputs and output feedback) reported for position regulation of robot manipulators. This usually requires some ingenuity to find the “right” energy function to be assigned. It is clear, however, that the technique is restricted to potential energy shaping. Later we present a new methodology that allows us also to shape the kinetic energy, which is required for some underactuated mechanical devices (see [28]-[30]).

Remark 7: In the underactuated case, when the number of control actions is smaller than the number of degrees of freedom, we find that $y = M^T \dot{q}$, with M the input matrix for the force (or torque) vector u . As shown in [9], the energy-shaping procedure still applies in these cases, provided a controller solvability and a dissipation propagation condition are satisfied.

General (f, g, h) Systems

Energy-balancing stabilization can, in principle, be applied to general (f, g, h) nonlinear passive systems of the form

$$\Sigma : \begin{cases} \dot{x} = f(x) + g(x)u \\ y = h(x). \end{cases} \quad (5)$$

From the celebrated nonlinear version of the Kalman-Yakubovich-Popov lemma [31], we know that for this class

of systems, passivity is equivalent to the existence of a nonnegative scalar function $H(x)$ such that

$$\left(\frac{\partial H}{\partial x}(x) \right)^T f(x) \leq 0 \quad h(x) = g^T(x) \frac{\partial H}{\partial x}(x).$$

We have the following simple proposition.

Proposition 1: Consider the passive system (5) with storage function $H(x)$ and an admissible equilibrium x_* . If we can find a vector function $\beta(x)$ such that the partial differential equation

$$\left(\frac{\partial H_a}{\partial x}(x) \right)^T [f(x) + g(x)\beta(x)] = -h^T(x)\beta(x) \quad (6)$$

can be solved for $H_a(x)$, and the function $H_a(x)$ defined as (4) has a minimum at x_* , then $u = \beta(x) + v$ is an energy-balancing PBC. Consequently, setting $v = 0$, we have that x_* is a stable equilibrium with the difference between the stored and the supplied energies constituting a Lyapunov function.

The proof follows immediately, noting that the left-hand side of (6) equals \dot{H}_a while the right-hand side is $-y^T u$, and then integrating from 0 to t .

Caveat emptor: This result, although quite general, is of limited interest. First of all, (f, g, h) models do not reveal the role played by the energy function in the system dynamics. Hence it is difficult to incorporate prior information to select a $\beta(x)$ to solve the partial differential equation (PDE) (6). A more practical and systematic result will be presented later for a more suitable class of models, namely, the so-called port-controlled Hamiltonian systems. Second, we will show below that, beyond the realm of mechanical systems, the applicability of energy-balancing PBC is severely restricted by the system’s natural dissipation.

Dissipation Obstacle

To investigate the conditions under which the PDE (6) is solvable we make the following observation.

Fact: A necessary condition for the global solvability of the PDE (6) is that $h^T(x)\beta(x)$ vanishes at all the zeros of $f(x) + g(x)\beta(x)$; that is,

$$f(\bar{x}) + g(\bar{x})\beta(\bar{x}) = 0 \Rightarrow h^T(\bar{x})\beta(\bar{x}) = 0.$$

Now $f(x) + g(x)\beta(x)$ is obviously zero at the equilibrium x_* , hence the right-hand side $-y^T u$, which is the power extracted from the controller, should also be zero at the equilibrium. This means that energy-balancing PBC is applicable only if the energy dissipated by the system is bounded, and consequently, if it can be stabilized extracting a finite amount of energy from the controller. This is indeed the case in the regulation of mechanical systems where the extracted power is the product of force and velocity and we want to drive the ve-

locity to zero. Unfortunately, it is no longer the case for most electrical or electromechanical systems where power involves the product of voltages and currents and the latter may be nonzero for nonzero equilibria.

Let us illustrate this point with simple linear time-invariant RLC circuits. First, we prove that the series RLC circuit is stabilizable with an energy-balancing PBC. Then we move the resistance to a parallel connection and show that, since for this circuit the power at any nonzero equilibrium is nonzero, energy-balancing stabilization is no longer possible.

Finite Dissipation Example

Consider the series RLC circuit of Fig. 1, where the port power variables are the input voltage and the current. The “natural” state variables for this circuit are the charge in the capacitor and the flux in the inductance $x \triangleq [q_c, \phi_L]^T$, and the total energy function is

$$H(x) = \frac{1}{2C}x_1^2 + \frac{1}{2L}x_2^2. \quad (7)$$

The dynamic equations are given by

$$\Sigma : \begin{cases} \dot{x}_1 = \frac{1}{L}x_2 \\ \dot{x}_2 = -\frac{1}{C}x_1 - \frac{R}{L}x_2 + u \\ y = \frac{1}{L}x_2. \end{cases} \quad (8)$$

The circuit clearly satisfies (1) with $d(t) = R \int_0^t [(1/L)x_2(s)]^2 ds$ (i.e., the energy dissipated in the resistor).

We are given an equilibrium x_* that we want to stabilize. It is clear from (8) that the admissible equilibria are of the form $x_* = [x_1, 0]^T$. It is important to note that the extracted power at any admissible equilibrium is zero.

To design our energy-balancing PBC, we look for a solution of the PDE (6), which in this case takes the form

$$\left(\frac{1}{L}x_2\right) \frac{\partial H_a}{\partial x_1}(x) - \left[\frac{1}{C}x_1 + \frac{R}{L}x_2 - \beta(x)\right] \frac{\partial H_a}{\partial x_2}(x) = -\frac{1}{L}x_2\beta(x).$$

Notice that the energy function $H(x)$ already “has a minimum” at $x_2=0$; thus we only have to “shape” the x_1 component, so we look for a function of the form $H_a = H_a(x_1)$. In this case, the PDE reduces to

$$\beta(x_1) = -\frac{\partial H_a}{\partial x_1}(x_1),$$

which, for any given $H_a(x_1)$, defines the control law as $u = \beta(x_1)$. To shape the energy $H_a(x)$, we add a quadratic

term and complete the squares (in the increments $x - x_*$) by proposing

$$H_a(x_1) = \frac{1}{2C_a}x_1^2 - \left(\frac{1}{C} + \frac{1}{C_a}\right)x_1x_1 + \kappa.$$

(The particular notation for the gain $1/C_a$ will be clarified in the next section.) Replacing in (4) yields

$$H_a(x) = \frac{1}{2}\left(\frac{1}{C} + \frac{1}{C_a}\right)(x_1 - x_1^*)^2 + \frac{1}{2L}x_2^2 + \kappa, \quad (9)$$

which has a minimum at x_* for all gains $C_a > -C$. Summarizing, the control law

$$u = -\frac{x_1}{C_a} + \left(\frac{1}{C} + \frac{1}{C_a}\right)x_1^* \quad (10)$$

with $C_a > -C$ is an energy-balancing PBC that stabilizes x_* with a Lyapunov function equal to the difference between the stored and the supplied energy. Finally, it is easy to verify that the energy supplied by the controller is finite.

Infinite Dissipation Example

Even though in the previous example we could find a very simple energy-balancing solution to our stabilization problem, it is easy to find systems that are not stabilizable with energy-balancing PBCs. For instance, consider a parallel RLC circuit. With the same definitions as before, the dynamic equations are now

$$\Sigma : \begin{cases} \dot{x}_1 = -\frac{1}{RC}x_1 + \frac{1}{L}x_2 \\ \dot{x}_2 = -\frac{1}{C}x_1 + u \\ y = \frac{1}{L}x_2. \end{cases} \quad (11)$$

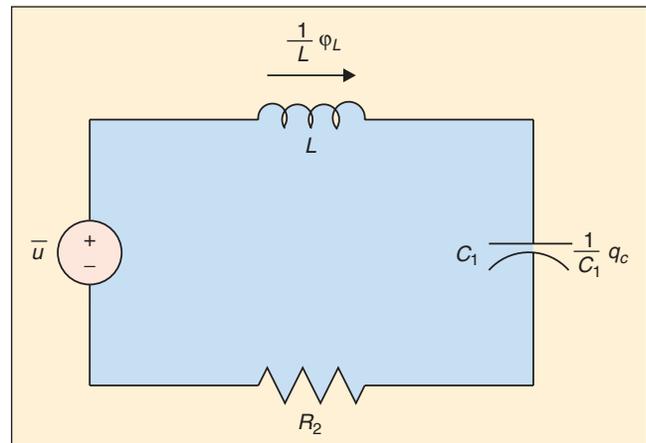


Figure 1. Series RLC circuit.

Notice that only the dissipation structure has changed, but the admissible equilibria are now of the form $x_* = [Cu_*, (L/R)u_*]^T$ for any u_* . The problem is that the power at any equilibrium except the trivial one is nonzero, and consequently, any stabilizing controller will yield $\lim_{t \rightarrow \infty} \left| \int_0^t u(s)y(s)ds \right| = \infty$ (we will eventually run down the battery!).

We will not elaborate further here on the infinite dissipation problem. A precise characterization, within the context of port-controlled Hamiltonian systems, will be given in the next section.

Remark 8: The well-known analogies between electrical and mechanical systems might lead us to conclude that with another choice of states, we could overcome the infinite dissipation obstacle for energy-balancing PBC. The obstacle is, however, “coordinate free.” The point is that in the mechanical case, the dissipation only comes in at the momentum level, where the input is also appearing. This eliminates the possibility of infinite dissipation.

Remark 9: In the linear time-invariant case, we can design an energy-balancing PBC working on incremental states. This procedure is usually not feasible, however, and despite its popularity is actually unnatural, for the general nonlinear case. The PBC design procedure we will present later to handle the infinite dissipation case does not rely on the generation of incremental dynamics. Furthermore, except for the linear case, the resulting energy functions will be nonquadratic.

Admissible Dissipations for Energy-Balancing PBC

In the previous section we showed that energy-balancing PBC is applicable only to systems with finite dissipation—obviously including conservative systems that have no dissipation at all. We have also shown that this class of systems contains all mechanical systems, as well as some electrical circuits with dissipation. A natural question then is how to characterize the “admissible dissipations” for energy-balancing PBC. To provide an answer to this question, we find it convenient to adopt a variation of Willems’s “control as interconnection” viewpoint. This perspective is also used in the next section, where viewing the action of the controller as an infinite energy source with a state modulated interconnection to the plant, we extend PBC to systems with infinite dissipation.

Passive Controllers

As shown in Fig. 2, we view the controller, Σ_c , as a one-port system that will be coupled with the plant to be controlled, Σ , via a two-port interconnection subsystem, Σ_l . We need the following definition.

Definition 1: The interconnection of Fig. 2 is said to be power preserving if the two-port subsystem Σ_l is lossless; that is, if it satisfies

$$\int_0^t [y^T(s), y_c^T(s)] \begin{bmatrix} u(s) \\ u_c(s) \end{bmatrix} ds = 0.$$

We now make the following important observation.

Proposition 2: Consider the interconnection of Fig. 2 with some external inputs (v, v_c) as

$$\begin{bmatrix} u \\ u_c \end{bmatrix} = \Sigma_l \begin{bmatrix} y \\ y_c \end{bmatrix} + \begin{bmatrix} v \\ v_c \end{bmatrix}.$$

Assume Σ_l is power preserving and Σ, Σ_c are passive systems with states $x \in \mathbb{R}^n, \zeta \in \mathbb{R}^{n_c}$ and energy functions $H(x), H_c(\zeta)$, respectively. Then $[v^T, v_c^T]^T \mapsto [y^T, y_c^T]^T$ is also a passive system with new energy function $H(x) + H_c(\zeta)$.

This fundamental property is proven with the following simple calculations:

$$\begin{aligned} \int_0^t [v^T(s), v_c^T(s)] \begin{bmatrix} y(s) \\ y_c(s) \end{bmatrix} ds &= \int_0^t u^T(s)y(s)ds + \int_0^t u_c^T(s)y_c(s)ds \\ &\geq H[x(t)] + H_c[\zeta(t)] - H[x(0)] - H_c[\zeta(0)] \end{aligned} \quad (12)$$

where the first equation follows from the lossless property of Σ_l and the last inequality is obtained from the passivity of each subsystem.

Invariant Functions Method

From Proposition 2, we conclude that passive controllers and power-preserving interconnections can, in principle, be used to “shape” the closed-loop total energy. However, although $H_c(\zeta)$ can be freely assigned, the system energy function $H(x)$ is given, and it is not clear how we can effectively shape the overall energy. The central idea of the invariant functions method [32], [33] is to restrict the motion of the closed-loop system to a certain subspace of the extended state-space (x, ζ) , say

$$\Omega \triangleq \{(x, \zeta) \mid \zeta = F(x) + \kappa\}. \quad (13)$$

In this way, we have a functional relationship between x and ζ , and we can express the closed-loop total energy as a function of x only, namely

$$H_d(x) \triangleq H(x) + H_c[F(x) + \kappa]. \quad (14)$$

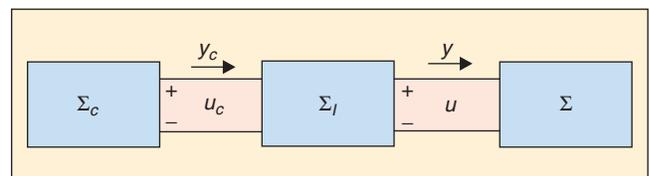


Figure 2. Control as interconnection.

(Notice that $H_c[F(x)+\kappa]$ plays the same role as $H_a(x)$ in (4).) This function can now be shaped with a suitable selection of the controller energy $H_c(\zeta)$. The problem then translates into finding a function $F(\cdot)$ that renders Ω invariant. (Recall that a set $\Omega \subset \mathbb{R}^n$ is invariant if the following implication holds: $x(0) \in \Omega \Rightarrow x(t) \in \Omega, \forall t \geq 0$.)

Let us illustrate this idea of generation of invariant subspaces to design stabilizing PBCs with the simple series RLC circuit example described by (8). Following Proposition 2, we consider passive controllers with state ζ and energy function $H_c(\zeta)$ to be defined. Since, as discussed above, we only need to modify the first coordinate, we propose to take ζ a scalar. Furthermore, for simplicity, we choose the dynamics of the controller to be a simple integrator; that is

$$\Sigma_c : \begin{cases} \dot{\zeta} = u_c \\ y_c = \frac{\partial H_c}{\partial \zeta}(\zeta). \end{cases} \quad (15)$$

Notice that if $H_c(\zeta)$ is bounded from below, then $u_c \mapsto y_c$ is indeed passive.

We already know that this system is stabilizable with an energy-balancing PBC; therefore, we interconnect the circuit and the controller with the standard negative feedback interconnection

$$\begin{bmatrix} u \\ u_c \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} y \\ y_c \end{bmatrix}. \quad (16)$$

To establish a relationship between x_1 and ζ , of the form $\zeta = F(x_1) + \kappa$, we define an invariant function candidate

$$C(x_1, \zeta) \triangleq F(x_1) - \zeta \quad (17)$$

and look for an $F(\cdot)$ such that $(d/dt)C \equiv 0$. Some simple calculations with (8), (15), (16), and (17) yield

$$\frac{d}{dt}C = \frac{1}{L}x_2 \left(\frac{\partial F}{\partial x_1}(x_1) - 1 \right),$$

from which we conclude that we should take $F(x_1) = x_1$, and the invariant subspaces are the linear spaces $\Omega = \{(x_1, x_2, \zeta) \mid \zeta = x_1 + \kappa\}$.

We now have to select the energy function of the controller such that in these invariant subspaces, the total energy function $H(x) + H_c(\zeta)$ has a minimum at x . Following the same rationale as in the previous section, we aim at a quadratic function (in the increments $x - x$); hence we fix

$$H_c(\zeta) = \frac{1}{2C_a} \zeta^2 - \left(\frac{1}{C} + \frac{1}{C_a} \right) x_1 \zeta$$

where C_a is a design parameter. As expected, the closed-loop energy, which results from (14) with $F(x_1) = x_1$, coincides with (9). (Notice that we have taken $\kappa = 0$. This is without loss of generality, because κ is determined by the controller's initial conditions.)

Remark 10: One important feature of PBC is that we can usually give a physical interpretation to the action of the controller. Indeed, a physical realization of the energy-balancing PBC (15) consists of a constant voltage source in series with a capacitor C_a . Notice, however, that the control action can be implemented without the addition of dynamics. Indeed, the input-output relationship of the controller dynamics (15), together with the interconnection (16), reduces to the static state feedback (10).

Remark 11: For simplicity, we have assumed above that $n_c = n$. In [17] we consider the more general case when $n_c \neq n$ and not all the controller states are related with the plant state variables.

Remark 12: Even though stabilization is ensured for all values of the added capacitance such that $C_a > -C$, it is clear that the system Σ_c is passive only for positive values of C_a .

Remark 13: The problem of finding a function $F(\cdot)$ that renders Ω invariant involves, of course, the solution of a PDE that is, in general, difficult to find. (In the simple case above, this is the trivial equation $(\partial F / \partial x_1)(x_1) = 1$.) One of the main messages we want to convey in this article is that the search for a solution of the PDE can be made systematic by the incorporation of additional structure to the problem—starting with the choice of a suitable system representation. We will further elaborate this point in the next subsection.

Energy-Balancing PBC of Port-Controlled Hamiltonian Systems

To characterize a class of (finite dissipation) systems stabilizable with energy-balancing PBC and simplify the solution of the PDE discussed above, we need to incorporate more structure into the system dynamics, in particular, making explicit the damping terms and the dependence on the energy function. Toward this end, we consider port-controlled Hamiltonian models that encompass a very large class of physical nonlinear systems. They result from the network modeling of energy-conserving lumped-parameter physical systems with independent storage elements and have been advocated as an alternative to more classical Euler-Lagrange (or standard Hamiltonian) models in a series of recent papers (see [2] for a list of references). These models are of the form

$$\Sigma : \begin{cases} \dot{x} = [J(x) - R(x)] \frac{\partial H}{\partial x}(x) + g(x)u \\ y = g^T(x) \frac{\partial H}{\partial x}(x) \end{cases} \quad (18)$$

where $H(x)$ is the energy function, $J(x)=-J^T(x)$ captures the interconnection structure, and $R(x)=R^T(x)\geq 0$ is the dissipation matrix. Clearly these systems satisfy the energy-balancing equation (1).

Motivated by Proposition 2, we consider port-controlled Hamiltonian controllers of the form

$$\Sigma_c : \begin{cases} \dot{\zeta} = [J_c(\zeta) - R_c(\zeta)] \frac{\partial H_c}{\partial \zeta}(\zeta) + g_c(\zeta) u_c \\ y_c = g_c^T(\zeta) \frac{\partial H_c}{\partial \zeta}(\zeta) \end{cases}$$

for any skew-symmetric matrix $J_c(\zeta)$, any positive-semidefinite matrix $R_c(\zeta)$, and any function $g_c(\zeta)$. The interconnection constraints are given by (16). The overall interconnected system is defined in the extended state-space (x, ζ) and can be written as

$$\begin{bmatrix} \dot{x} \\ \dot{\zeta} \end{bmatrix} = \begin{bmatrix} J(x) - R(x) & -g(x)g_c^T(\zeta) \\ g_c(\zeta)g^T(x) & J_c(\zeta) - R_c(\zeta) \end{bmatrix} \begin{bmatrix} \frac{\partial H}{\partial x}(x) \\ \frac{\partial H_c}{\partial \zeta}(\zeta) \end{bmatrix}. \quad (19)$$

Notice that it still belongs to the class of port-controlled Hamiltonian models with total energy $H(x) + H_c(\zeta)$.

We introduce at this point the concept of Casimir functions [2], [33], which are conserved quantities of the system for any choice of the Hamiltonian, and so are completely determined by the geometry (i.e., the interconnection structure) of the system. For ease of presentation, we keep the same notation we used in the previous subsection and look for Casimir functions of the form

$$C(x, \zeta) = F(x) - \zeta. \quad (20)$$

Since the time derivative of these functions should be zero along the closed-loop dynamics for all Hamiltonians $H(x)$, this means that we are looking for solutions of the PDEs

$$\left[\left(\frac{\partial F}{\partial x}(x) \right)^T \vdots -I_m \right] \begin{bmatrix} J(x) - R(x) & -g(x)g_c^T(\zeta) \\ g_c(\zeta)g^T(x) & J_c(\zeta) - R_c(\zeta) \end{bmatrix} = 0. \quad (21)$$

The following proposition was established in [17].

Proposition 3: The vector function (20) satisfies (21) (and thus is a Casimir function for the interconnected system (19)) if and only if $F(x)$ satisfies

$$\left(\frac{\partial F}{\partial x}(x) \right)^T J(x) \frac{\partial F}{\partial x}(x) = J_c(\zeta) \quad (22)$$

$$R(x) \frac{\partial F}{\partial x}(x) = 0 \quad (23)$$

$$R_c(\zeta) = 0 \quad (24)$$

$$\left(\frac{\partial F}{\partial x}(x) \right)^T J(x) = g_c(\zeta) g_c^T(x). \quad (25)$$

In this case, the dynamics reduced to the set Ω (13) is a port-controlled Hamiltonian system of the form

$$\dot{x} = [J(x) - R(x)] \frac{\partial H_d}{\partial x}(x) \quad (26)$$

with the shaped energy function $H_d(x) = H(x) + H_c[F(x) + \kappa]$.

Admissible Dissipation

Condition (23) of Proposition 3 characterizes the admissible dissipations for energy-balancing PBC in terms of the coordinates where energy can be shaped. Indeed, if (23) holds, then

$$R(x) \frac{\partial H_c(F)}{\partial x}(x) = 0$$

for any controller energy function H_c . Roughly speaking, this means that H_c should not depend on the coordinates where there is natural damping. The latter restriction can then be interpreted as: Dissipation in energy-balancing PBC is admissible only on the coordinates that do not require “shaping of the energy.”

Recall that in mechanical systems, where the state consists of position and velocities, damping is associated with the latter; hence it appears in the lower right corner of the matrix $R(x)$. On the other hand, in position regulation, we are only concerned with potential energy shaping; thus the condition (23) will be satisfied. In the case of the series RLC circuit of the previous section, the resistance appears in a coordinate that did not need to be modified (i.e., the current x_2), whereas in the parallel RLC circuit, both coordinates have to be shaped.

Overcoming the Dissipation Obstacle

In Proposition 3 we have shown that under certain conditions, the interconnection of a port-controlled Hamiltonian plant with a port-controlled Hamiltonian controller leads to a reduced dynamics given by another port-controlled Hamiltonian system (26) with a shaped Hamiltonian. The reduction of the dynamics stems from the existence of Casimir functions that relate the states of the controller with those of the plant. In this section, we will show that, explicitly incorporating information on the systems state, we can shape

the energy function without the need for Casimir functions. This will lead to the definition of a new class of PBCs that we call *interconnection and damping assignment* PBCs.

Control as a State-Modulated Source

To extend PBC to systems with infinite dissipation, we introduce two key modifications. First, since these systems cannot be stabilized by extracting a finite amount of energy from the controller, we consider the latter to be an (infinite energy) source; that is, a system described by

$$\Sigma_c : \begin{cases} \dot{\zeta} = u_c \\ y_c = \frac{\partial H_c}{\partial \zeta}(\zeta) \end{cases} \quad (27)$$

with energy function

$$H_c(\zeta) = -\zeta. \quad (28)$$

Second, the classical unitary feedback interconnection (through the power port variables) imposes some very strict constraints on the plant and controller structures as reflected by the conditions (22)-(25). To provide more design flexibility, we propose to incorporate state information, which is done by coupling the source system with the plant via a state-modulated interconnection of the form

$$\begin{bmatrix} u(s) \\ u_c(s) \end{bmatrix} = \begin{bmatrix} 0 & -\beta(x) \\ \beta(x) & 0 \end{bmatrix} \begin{bmatrix} y(s) \\ y_c(s) \end{bmatrix}. \quad (29)$$

This interconnection is clearly power preserving. The overall interconnected system (18), (27)-(29) can be written as

$$\begin{bmatrix} \dot{x} \\ \dot{\zeta} \end{bmatrix} = \begin{bmatrix} J(x) - R(x) & -g(x)\beta(x) \\ \beta^T(x)g^T(x) & 0 \end{bmatrix} \begin{bmatrix} \frac{\partial H}{\partial x}(x) \\ \frac{\partial H_c}{\partial \zeta}(\zeta) \end{bmatrix} \quad (30)$$

which is still a port-controlled Hamiltonian system with total energy $H(x) + H_c(\zeta)$. It is important to note that the x dynamics above describes the behavior of the system (5) with a static state feedback $u = \beta(x)$; hence our choice of the symbol β for the state-modulation function.

We have shown in [8] that the damping restriction (23) is a necessary condition for the existence of Casimir functions in this case as well. The key point here is that the energy of the x subsystem can be shaped and the port-controlled Hamiltonian structure preserved without generation of Casimir functions. Indeed, if (for the given $J(x), R(x)$, and $g(x)$) we can solve the PDE

$$[J(x) - R(x)] \frac{\partial H_a}{\partial x}(x) = g(x)\beta(x) \quad (31)$$

for some $\beta(x)$, then the plant dynamics will be given by (26) with energy function $H_d(x) = H(x) + H_a(x)$. If we can furthermore ensure that $H_d(x)$ has a minimum at the desired equilibrium, then the static state feedback control $u = \beta(x)$ will stabilize this point. Notice that there is no “finite dissipation” constraint for the solvability of (31); hence the new PBC design is, in principle, applicable to systems with infinite dissipation.

Parallel RLC Circuit Example

Before presenting the main result of this section, which is a systematic procedure for PBC of port-controlled Hamiltonian systems, let us illustrate the new energy-shaping method with the parallel RLC circuit example. The dynamics of this circuit (11) can be written in port-controlled Hamiltonian form (18) with energy function (7) and the matrices

$$J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad R = \begin{bmatrix} 1/R & 0 \\ 0 & 0 \end{bmatrix}, \quad g = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

The PDE (31) becomes

$$\begin{aligned} -\frac{1}{R} \frac{\partial H_a}{\partial x_1}(x) + \frac{\partial H_a}{\partial x_2}(x) &= 0 \\ -\frac{\partial H_a}{\partial x_1}(x) &= \beta(x). \end{aligned}$$

The first equation can be trivially solved as

$$H_a(x) = \Phi(Rx_1 + x_2)$$

where $\Phi(\cdot): \mathbb{R} \rightarrow \mathbb{R}$ is an arbitrary differentiable function, whereas the second equation defines the control law. We now need to choose the function Φ so that $H_d(x)$ has a minimum at the desired equilibrium point $x_* = (Cu_*, (L/R)u_*)$. For simplicity, we choose it to be a quadratic function

$$\Phi(Rx_1 + x_2) = \frac{K_p}{2} [(Rx_1 + x_2) - (Rx_{1*} + x_{2*})]^2 - Ru_*(Rx_1 + x_2)$$

which, as can be easily verified, ensures the desired energy shaping for all

$$K_p > \frac{-1}{(L + CR^2)}. \quad (32)$$

The assigned energy function, as expected, is quadratic in the increments

$$H_d(x) = (x - x^*)^T \begin{bmatrix} \frac{1}{C} + R^2 K_p & RK_p \\ RK_p & \frac{1}{L} + K_p \end{bmatrix} (x - x^*) + \kappa \quad \frac{\partial K}{\partial x}(x) = \left[\frac{\partial K}{\partial x}(x) \right]^T \quad (33)$$

Clearly, (32) is the necessary and sufficient condition for x , to be a unique global minimum of this function. The resulting control law is a simple linear state feedback

$$u = -K_p [R(x_1 - x_{1^*}) + x_2 - x_{2^*}] + u^*$$

Discussion

Remark 14: We should underscore that in the example above we did not need to “guess” candidate functions for $H_a(x)$ or $\beta(x)$. Instead, the solution of the PDE (31) provided a family of “candidates” parametrized in terms of the free function $\Phi(\cdot)$. The PDE, in turn, is uniquely determined by the system’s interconnection, damping, and input matrices; we will show below that to provide more degrees of freedom to the design, we can also change the first two matrices. From this family of solutions, we then have to select one that achieves the energy shaping. Also, once a solution $H_a(x)$ is obtained, we know that the new energy function $H_d(x)$ will be nonincreasing, because \dot{H}_d is nonpositive by construction. This situation should be contrasted with classical Lyapunov-based designs (or “standard” PBC, e.g., [9]), where we fix *a priori* the Lyapunov (energy) function—typically a quadratic in the increments—and then calculate the control law that makes its derivative negative definite. We claim that the proposed approach is more natural because, on the one hand, it is easier to incorporate prior knowledge in the choice of the desired interconnection and damping matrices; on the other hand, the resulting energy (Lyapunov) function will be specifically tailored to the problem.

Remark 15: Of course, stabilization of linear systems is possible using other, much simpler, methods. Our point is that, as we will show in the next subsection, the present procedure applies verbatim to the nonlinear case. Furthermore, even in the linear case, the technique allows us to design nonlinear controllers, which might be of interest to improve performance (e.g., assigning steeper Lyapunov functions for faster convergence or imposing certain shapes of the level sets to handle state or input constraints); see [22] for an example of the latter.

Remark 16: As discussed in [8] (see also Proposition 4), we do not even need to solve the PDE (31) for $H_a(x)$. Indeed, we can look for a solution of the problem directly in terms of $\beta(x)$, as follows. If $J(x) - R(x)$ is invertible (see [8] and Proposition 4 for the noninvertible case) it is well known that (31) has a solution if and only if the integrability conditions

hold, where

$$K(x) \triangleq [J(x) - R(x)]^{-1} g(x) \beta(x). \quad (34)$$

Given $J(x)$, $R(x)$, and $g(x)$, (33) defines a set of PDEs for $\beta(x)$. For instance, for the parallel RLC circuit example, we have that (33) is equivalent to

$$-\frac{1}{R} \frac{\partial \beta}{\partial x_1}(x) + \frac{\partial \beta}{\partial x_2}(x) = 0$$

whose solution yields directly the control law $\beta(x) = \Phi(Rx_1 + x_2)$. Although in this simple linear example both procedures lead to the same PDE, this will not be the case for the general nonlinear case. Furthermore, the importance of determining necessary and sufficient conditions for solvability can hardly be overestimated. We will elaborate further on these issues in the next subsection.

Assigning Interconnection and Damping Structures

In the previous subsections, we have shown that the success of our PBC design essentially hinges on our ability to solve the PDE (31). It is well known that solving PDEs is not easy. It is our contention that, for the particular PDE that we have to solve here, it is possible to incorporate prior knowledge about the system to simplify the task. More specifically, for port-controlled Hamiltonian models, besides the control law, we have the additional degrees of freedom of selecting the interconnection and damping structures of the closed loop. Indeed, our energy-shaping objective is not modified if, instead of (26), we aim at the closed-loop dynamics

$$\dot{x} = [J_d(x) - R_d(x)] \frac{\partial H_d}{\partial x}(x) \quad (35)$$

for some *new* interconnection $J_d(x) = -J_d^T(x)$ and damping $R_d(x) = R_d^T(x) \geq 0$ matrices. For this so-called interconnection and damping assignment PBC, the PDE (31) becomes

$$[J(x) + J_a(x) - R(x) - R_a(x)] \frac{\partial H_a}{\partial x}(x) = -[J_a(x) - R_a(x)] \frac{\partial H_a}{\partial x}(x) + g(x) \beta(x) \quad (36)$$

where

$$J_a(x) \triangleq J_d(x) - J(x), \quad R_a(x) \triangleq R_d(x) - R(x)$$

are new design parameters that add more degrees of freedom to the solution of the PDE.

The proposition below (established in [8]) follows immediately from the derivations above. It is presented in a form that is particularly suitable for symbolic computations. We refer the interested reader to [8] for additional comments and discussions.

Proposition 4: Given $J(x), R(x), H(x), g(x)$, and the desired equilibrium to be stabilized x_* , assume we can find functions $\beta(x), R_a(x), J_a(x)$ such that

$$\begin{aligned} J(x) + J_a(x) &= -[J(x) + J_a(x)]^T, \\ R(x) + R_a(x) &= [R(x) + R_a(x)]^T \geq 0 \end{aligned}$$

and a vector function $K(x)$ satisfying

$$\begin{aligned} [J(x) + J_a(x) - (R(x) + R_a(x))]K(x) &= \\ -[J_a(x) - R_a(x)] \frac{\partial H}{\partial x}(x) + g(x)\beta(x) \end{aligned} \quad (37)$$

and such that the following conditions occur:

- i) (*Integrability*) $K(x)$ is the gradient of a scalar function; that is, (33) holds.
- ii) (*Equilibrium assignment*) $K(x)$, at x_* , verifies

$$K(x_*) = -\frac{\partial H}{\partial x}(x_*).$$

- iii) (*Lyapunov stability*) The Jacobian of $K(x)$, at x_* , satisfies the bound

$$\frac{\partial K}{\partial x}(x_*) > -\frac{\partial^2 H}{\partial x^2}(x_*).$$

Under these conditions, the closed-loop system $u = \beta(x)$ will be a port-controlled Hamiltonian system with dissipation of the form (35), where $H_d(x) = H(x) + H_a(x)$ and

$$\frac{\partial H_a}{\partial x}(x) = K(x). \quad (38)$$

Furthermore, x_* will be a (locally) stable equilibrium of the closed loop. It will be asymptotically stable if, in addition, the largest invariant set under the closed-loop dynamics contained in

$$\left\{ x \in \mathbb{R}^n \cap \mathcal{B} \left[\left[\frac{\partial H_d}{\partial x}(x) \right]^T R_d(x) \frac{\partial H_d}{\partial x}(x) = 0 \right] \right\}$$

equals $\{x_*\}$.

Remark 17: From the following simple calculations

$$\begin{aligned} \dot{H}_d &= u^T y - \underbrace{\left[\frac{\partial H}{\partial x}(x) \right]^T R(x) \frac{\partial H}{\partial x}(x)}_{\dot{H}} + \dot{H}_a \\ &= -\left[\frac{\partial H_d}{\partial x}(x) \right]^T R_d(x) \frac{\partial H_d}{\partial x}(x) \end{aligned}$$

and the fact that $R_d(x) = R_a(x) + R(x)$, we have that

$$\begin{aligned} \dot{H}_a &= -u^T y - \left[2 \frac{\partial H}{\partial x}(x) + \frac{\partial H_a}{\partial x}(x) \right]^T R(x) \frac{\partial H_a}{\partial x}(x) \\ &\quad - \left[\frac{\partial H_d}{\partial x}(x) \right]^T R_a(x) \frac{\partial H_d}{\partial x}(x). \end{aligned}$$

Consequently, if $R_a(x) = 0$ and the natural damping $R(x)$ satisfies the condition

$$R(x) \frac{\partial H_a}{\partial x}(x) = 0,$$

then the new PBC is an energy-balancing PBC. This is exactly the same condition that we obtained in the previous section.

Remark 18: In a series of papers, we have shown that, in many practical applications, the desired interconnection and damping matrices can be judiciously chosen by invoking physical considerations. The existing applications of interconnection and damping assignment PBC include mass-balance systems [18], electrical motors [19], power systems [20], magnetic levitation systems [21], underactuated mechanical systems [28], and power converters [22]. In the next section we present in detail a magnetic levitation system and refer the reader to the references cited above for additional examples that illustrate the generality of the new approach.

Remark 19: An interesting alternative to the Hamiltonian description of actuated mechanical systems is the Lagrangian description, with the Lagrangian being given by the difference of the kinetic and the potential energy. In this framework it is natural to pose the problem of when and how a state feedback for the actuation inputs can be designed such that the closed-loop system is again a Lagrangian system with a “desired” Lagrangian (as well as a desired damping). This line of research, called the technique of “controlled Lagrangians,” was developed in a series of papers by Bloch et al. (e.g., [29] and [36] and followed up in [37] and [38]). The relation of these approaches to the approach of interconnection and damping assignment for port-controlled Hamiltonian systems taken here is rather straightforward. In particular, it is possible to show that modifying the kinetic energy of a mechanical system without affecting the potential energy or the damping (as done in [29]) is tantamount—in our formulation—to selecting the closed-loop interconnection matrix as

$$J_d(q,p) \begin{bmatrix} 0 & M_d^{-1}(q)M(q) \\ -M(q)M_d^{-1}(q) & Z(q,p) \end{bmatrix}$$

where $M_d(q), M(q)$ are the closed-loop (“modified”) and open-loop inertia matrices, respectively, and the elements of $Z(q,p)$ are computed as

$$Z(q,p)_{i,j} = -p^T M^{-1}(q) M_d(q) \left[(M_d^{-1}M)_{\cdot,i}, (M_d^{-1}M)_{\cdot,j} \right] (q) \quad (39)$$

with $(M_d^{-1}M)_{\cdot,i}$ the i th column of $M_d^{-1}M$ and $[\cdot, \cdot]$ the standard Lie bracket; see [2]. Furthermore, the addition of damping in the Lagrangian framework corresponds to damping assignment in the Hamiltonian case, while shaping the potential energy clearly fits within the shaping of the Hamiltonian. Hence we may conclude that the method of the “controlled Lagrangians” for actuated mechanical systems is a special case of our approach for the port-controlled Hamiltonian description of these systems. For example, in our approach the closed-loop interconnection matrix J_d can be chosen much more generally than in (39). On the other hand, the freedom in choosing J_d may be so overwhelmingly rich that it is useful to have more specific subclasses of possible interconnection structure matrices like the one in (39) at hand. In general, it seems of interest to investigate more deeply the embedding of the technique of controlled Lagrangians within our approach, also in relation to issues of “integrability,” in particular, the satisfaction of Jacobi-identity for J_d .

Magnetic Levitation System

Model

Consider the system of Fig. 3 consisting of an iron ball in a vertical magnetic field created by a single electromagnet. Here we adopt the standard assumption of unsaturated flux; that is, $\lambda = L(\theta)i$, where λ is the flux, θ is the difference between the position of the center of the ball and its nominal position, with the θ -axis oriented downward, i is the current, and $L(\theta)$ denotes the value of the inductance. The dynamics of the system is obtained by invoking Kirchoff’s voltage law and Newton’s second law as

$$\begin{aligned} \dot{\lambda} + Ri &= u \\ m\ddot{\theta} &= F - mg \end{aligned}$$

where m is the mass of the ball, R is the coil resistance, and F is the force created by the electromagnet, which is given by

$$F = \frac{1}{2} \frac{\partial L}{\partial \theta} (\theta) i^2.$$

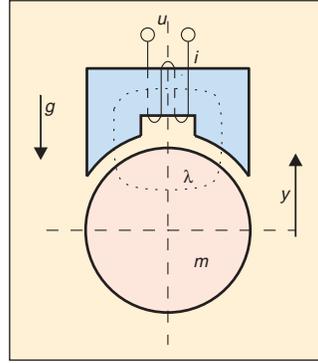


Figure 3. Levitated ball ($y=0$).

A suitable approximation for the inductance (in the domain $-\infty < \theta < 1$) is $L(\theta) = k/(1-\theta)$, where k is some positive constant that depends on the number of coil turns, and we have normalized the nominal gap to one.

To obtain a port-controlled Hamiltonian model, we define the state variables as $x = [\lambda, \theta, m\dot{\theta}]^T$. The Hamiltonian function is given as

$$H(x) = \frac{1}{2k}(1-x_2)x_1^2 + \frac{1}{2m}x_3^2 + mgx_2$$

and the port-controlled Hamiltonian model becomes

$$\dot{x} = \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}}_J - \underbrace{\begin{bmatrix} R & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_K \frac{\partial H}{\partial x}(x) + \underbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}_g u.$$

Given a constant desired position for the ball x_2^* , the equilibrium we want to stabilize is $x_* = [\sqrt{2kmg}, x_2^*, 0]^T$.

Changing the Interconnection

Next we show that, with the natural interconnection matrix of the system J , it is not possible to stabilize the desired equilibrium point with the proposed methodology; hence it is necessary to modify J . Toward this end, we observe that the key PDE to be solved (31) yields

$$(J-R)K(x) = g\beta(x) \Leftrightarrow \begin{cases} -RK_1(x) = \beta(x) \\ K_2(x) = 0 \\ K_3(x) = 0 \end{cases}$$

with $K(x)$ defined as (38). This means that the function $H_a(x)$ can only depend on x_1 . Thus the resulting Lyapunov function would be of the form

$$H_d(x) = \frac{1}{2k}(1-x_2)x_1^2 + \frac{1}{2m}x_3^2 + mgx_2 + H_a(x_1).$$

Even though, with a suitable selection of $H_a(x_1)$, we can satisfy the equilibrium assignment condition of Proposition 4, the Hessian will be defined as

$$\frac{\partial^2 H_d}{\partial x^2}(x) = \begin{bmatrix} \frac{(1-x_2)}{k} + \frac{\partial^2 H_a}{\partial x^2}(x_1) & -\frac{x_1}{k} & 0 \\ -\frac{x_1}{k} & 0 & 0 \\ 0 & 0 & \frac{1}{m} \end{bmatrix}$$

which is sign indefinite for all $H_a(x_1)$. It can actually be shown that the equilibrium is not stable.

The source of the problem is the lack of an *effective coupling* between the electrical and mechanical subsystems. Indeed, the interconnection matrix J only couples position with velocity. To overcome this problem, we propose to enforce a coupling between the flux x_1 and the velocity x_3 ; thus we propose the desired interconnection matrix

$$J_d = \begin{bmatrix} 0 & 0 & -\alpha \\ 0 & 0 & 1 \\ \alpha & -1 & 0 \end{bmatrix}$$

where α is a constant to be defined. Now, the key equation (37) becomes (with $R_a=0$)

$$\begin{aligned} -RK_1(x) &= \frac{\alpha}{m}x_3 + \beta(x) \\ K_3(x) &= 0 \\ \alpha K_1(x) - K_2(x) &= -\frac{\alpha}{k}(1-x_2)x_1. \end{aligned}$$

The first equation defines the control signal, whereas the last one can be readily solved (e.g., using symbolic programming languages) as

$$H_a(x) = \frac{1}{6k\alpha}x_1^3 + \frac{1}{2k}x_1^2(x_2-1) + \Phi\left(x_2 + \frac{1}{\alpha}x_1\right)$$

where $\Phi(\cdot)$ is an arbitrary continuous differentiable function. This function must be chosen to satisfy the equilibrium assignment and Lyapunov stability conditions of Proposition 4; that is, to assign a strict minimum at x , to the new Lyapunov function

$$H_d(x) = \frac{1}{6k\alpha}x_1^3 + \frac{1}{2m}x_3^2 + mgx_2 + \Phi\left(x_2 + \frac{1}{\alpha}x_1\right).$$

It is easy to verify [21] that a suitable choice is given by

$$\Phi\left(x_2 + \frac{1}{\alpha}x_1\right) = mg\left[-\left(\tilde{x}_2 + \frac{1}{\alpha}\tilde{x}_1\right) + \frac{b}{2}\left(\tilde{x}_2 + \frac{1}{\alpha}\tilde{x}_1\right)^2\right]$$

where $\tilde{x}_i \triangleq x_i - x_{i*}$, and $\alpha, b > 0$.

In conclusion, we have shown that the control law

$$u = \frac{R}{k}(1-x_2)x_1 - K_p\left(\frac{1}{\alpha}\tilde{x}_1 + \tilde{x}_2\right) - \frac{\alpha}{m}x_3 - \frac{R}{\alpha}\left(\frac{1}{2k}x_1^2 - mg\right) \quad (40)$$

stabilizes the equilibrium point x , for all $K_p, \alpha > 0$, where we have defined a new constant K_p . It can be further estab-

lished that stability is asymptotic and an estimate of the domain of attraction can be readily determined.

Changing the Damping

A closer inspection of the control law (40) provides further insight that helps in its commissioning and leads to its simplification. The first right-hand term equals Ri ; thus it cancels the voltage drop along the resistance. The second and third right-hand terms are linear proportional and derivative actions, respectively. Finally, the last term, which is proportional to acceleration, contains an undesirable nonlinearity that might saturate the control action. (We should note that the effect of the quadratic nonlinearity cannot be reduced without sacrificing the convergence rate, as can be seen from the dependence of the PD terms on α .) With the intent of removing this term, we propose to shuffle the damping, namely, to remove it from the electrical subsystem and add it up in the position coordinate; that is, we propose the added damping matrix

$$R_a = \begin{bmatrix} -R & 0 & 0 \\ 0 & R_a & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

where R_a is some positive number. Applying again the technique of Proposition 4, we can show that stabilization is possible with the simplified control law

$$u = \frac{R}{k}(1-x_2)x_1 - K_p\left(\frac{1}{\alpha}\tilde{x}_1 + \tilde{x}_2\right) - \left(\frac{\alpha}{m} + K_p R_a\right)x_3$$

where we have now defined $K_p \triangleq b\alpha / R_a$. Compare with (40).

Concluding Remarks

We have given a tutorial presentation of a control design approach for physical systems based on energy considerations that has been developed by the authors of the present article, as well as by some other researchers cited in the references, in the last few years. The main premise of this approach is that the fundamental concept of energy is lost in the signal processing perspective of most modern control techniques; hence we present an alternative viewpoint that focuses on interconnection. The choice of a suitable description of the system is essential for this research; thus we have adopted port-controlled Hamiltonian models that provide a classification of the variables and the equations into those associated with phenomenological properties and those defining the interconnection structure related to the exchanges of energy.

There are many possible extensions and refinements to the theory we have presented in this article. Some of these topics, and the lines of research we are pursuing to address them, may be found in [8]. Central among the various open

issues that need to be clarified one finds, of course, the solvability of the PDE (37). Although we have shown that the added degrees of freedom ($J_a(x), R_a(x)$) can help us in its solution, it would be desirable to have a better understanding of their effect, that would lead to a more systematic procedure in their design. For general port-controlled Hamiltonian systems, this is, we believe, a far-reaching problem. Hence we might want to study it first for specific classes of physically motivated systems.

Solving new problems is, of course, the final test for the usefulness of a new theory. Our list of references is testament to the breadth of application of our approach, hence we tend to believe that this aspect has been amply covered by our work.

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