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Robust integral control of port-Hamiltonian systems: The case of non-passive outputs with unmatched disturbances

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ABSTRACT

Regulation of passive outputs of nonlinear systems can be easily achieved with an integral control (IC). In many applications, however, the signal of interest is not a passive output and ensuring its regulation remains an open problem. Also, IC of passive systems rejects constant input disturbances, but no similar property can be ensured if the disturbance is not matched. In this paper we address the aforementioned problems and propose a procedure to design robust ICs for port-Hamiltonian models, that characterize the behavior of a large class of physical systems. Necessary and sufficient conditions for the solvability of the problem, in terms of some rank and controllability properties of the linearized system, are provided. For a class of fully actuated mechanical systems, a globally asymptotically stabilizing solution is given. Simulations of the classical pendulum system illustrate the good performance of the scheme.

1. Introduction

One of the central features of passivity-based control (PBC), where the first step is passivation of the system [1], is that the passive output can be easily regulated using integral control (IC)—with arbitrary positive gains. The regulation is, moreover, robust with respect to constant input disturbances. In many applications, however, the signal to be regulated is not a passive output and the disturbances are not matched with the input. Classical examples are mechanical systems and electrical motors, where the passive outputs are velocities and currents, respectively, but the output of interest is often position.

In this paper we propose a procedure to design ICs to regulate non-passive outputs, which are robust to unmatched disturbances. We restrict our attention to port-Hamiltonian (pH) models that, as is widely known, characterize the behavior of a large class of physical systems [2,3]. Another motivation to consider pH systems is that the popular interconnection and damping assignment PBC design technique [4,5]—and the closely related canonical transformation PBC [6]—endow an arbitrary nonlinear system with a pH structure. The aim of the additional IC is then to ensure that output regulation is robust *vis-à-vis* external disturbances.

The controller design is formulated in the paper as a feedback equivalence problem, where a dynamic feedback controller and a change of coordinates such that the transformed closed-loop system takes a desired pH form are sought. To avoid the need to solve partial differential equations, the interconnection and damping matrices of the target system, as well as its energy function, are kept equal to the ones of the original system, and only add to it an integral action in the non-passive output. This construction is largely inspired by the one proposed in [7], but here we explicitly take into account the presence of the disturbances, which significantly complicates the task. An additional contribution of our paper is that necessary and sufficient conditions for feedback equivalence, in terms of some rank and controllability properties of the linearized system, are given. The method is applied to linear and mechanical systems for which robust globally asymptotically stabilizing solutions are obtained, under some reasonable assumptions.

The remaining of the paper is organized as follows. The problem is formulated in Section 2. The output regulation and disturbance rejection properties of IC of the passive output are revisited in Section 3. In Section 4 the feedback equivalence problem is presented, and its solution is given in Section 5. Section 6 contains the main result of the paper, namely the robust stabilization of pH systems via IC of non-passive outputs. In Section 7 the application to linear and mechanical systems is given. Finally, we wrap-up the paper with some concluding remarks in Section 8.

Notation: All vectors defined in the paper are column vectors and all functions are sufficiently smooth. For a scalar function $H : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \times \cdots \times \mathbb{R}^{n_p} \to \mathbb{R}, x \mapsto H$, where $x := \mathrm{col}(x_1, x_2, \ldots, x_p)$, the operators $\nabla_i H(x) := \left(\frac{\partial H(x)}{\partial x_i}\right)^\top$ are defined. The shorthand notation

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$[\nabla_i H](z) := \nabla_i H(x)|_{x=z}$, that is, the evaluation of the function $\nabla_i H(x)$ at $z \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \times \cdots \times \mathbb{R}^{n_p}$ is used. $\nabla^2 H(x)$ is used for the Hessian matrix. For the distinguished constant element $x^\star \in \mathbb{R}^n$ and a mapping $L : \mathbb{R}^n \to \mathbb{R}^{s\times p}$, the abridged notation $L(x^\star) =: L^\star$ is used. Unless stated otherwise, it is assumed that the various properties of the functions—*e.g.*, rank conditions and signs—are satisfied in a neighborhood of an equilibrium point.

2. Perturbed port-Hamiltonian systems and problem formulation

2.1. Class of systems and control objectives

The perturbed pH systems considered in the paper are of the form

$$\begin{aligned}\dot{x} &= F(x)\nabla H(x) + g(x)u + d\\ y &= g^\top(x)\nabla H(x)\end{aligned} \tag{1}$$

where $x \in \mathbb{R}^n, u \in \mathbb{R}^m, g : \mathbb{R}^n \to \mathbb{R}^{n\times m}$ is the full rank input matrix, $d \in \mathbb{R}^n$ is a constant disturbance, $H : \mathbb{R}^n \to \mathbb{R}$ is the energy function and

$$F(x) + F^\top(x) \le 0.$$

As is well-known [2,3], unperturbed pH systems define cyclo-passive operators $u \mapsto y$, with storage function $H(x)$. This property is strengthened to passivity if $H(x)$ is bounded from below.

We are interested in the scenario where the energy-shaping and damping injection stages of PBC, for the unperturbed system, have been accomplished. That is, it is assumed that an output feedback proportional term has already been added[1] and, consequently,

$$\nabla H^\top(x)[F(x) + F^\top(x)]\nabla H(x) \le -\alpha|g^\top(x)\nabla H(x)|^2, \tag{2}$$

for some $\alpha > 0$, where $|\cdot|$ is the Euclidean norm. Furthermore, it is assumed that a suitable energy function $H(x)$ has been assigned. The choice of this function is a delicate point that, as explained below, depends on whether the disturbances are matched or unmatched.

The control objectives are now, to preserve stability of a desired equilibrium and to drive a given output towards zero, in spite of the presence of disturbances. It will be shown below that, for matched disturbances, *i.e.*, those that enter the image of $g(x)$, and the passive output y, an IC around y achieves the objectives. In this paper we are interested in the cases where the disturbance is not matched and the signal to be regulated is not the passive output—but is also zero at the desired equilibrium.

2.2. Notational simplifications

In writing the paper we have decided to sacrifice generality for clarity of presentation. Consequently, two assumptions that, without modifying the essence of our contribution, considerably simplify the notation are made. First, since we consider the case where disturbances enter in the $(n-m)$ non-actuated coordinates, the internal model principle indicates that it is necessary to add $(n-m)$ integrators. To ensure solvability of the problem it is reasonable to assume that the number of control actions is sufficiently large. This leads to the following assumption

$$m \ge n - m. \tag{3}$$

If less integrators are added this restriction can be relaxed—but then the notation gets very cumbersome. See [9] for further details.

The second simplification that we introduce concerns the matrix $g(x)$. Dragging this matrix through the calculations significantly complicates the notation, therefore it will be assumed in the following that, after redefinition of the inputs and the states, the input matrix takes the form

$$g(x) = \begin{bmatrix} I_m \\ 0 \end{bmatrix}, \tag{4}$$

where I_m is the $m \times m$ identity matrix.

For notational convenience, we partition the state and disturbance vectors as $x = \mathrm{col}(x_1, x_2)$, $d = \mathrm{col}(d_1, d_2)$, where $d_1, x_1 \in \mathbb{R}^m$ and $d_2, x_2 \in \mathbb{R}^{n-m}$. Similarly, the matrix $F(x)$ is block partitioned as

$$F(x) = \begin{bmatrix} F_{11}(x) & F_{12}(x) \\ F_{21}(x) & F_{22}(x) \end{bmatrix},$$

with $F_{11}(x) \in \mathbb{R}^{m\times m}$ and $F_{22}(x) \in \mathbb{R}^{(n-m)\times(n-m)}$. With this notation the passive output is $y = \nabla_1 H(x)$. For future reference we also define a second output to be regulated as the $(n-m)$-dimensional vector

$$r = \nabla_2 H(x). \tag{5}$$

2.3. Some remarks about equilibria

In the absence of disturbances the desired assignable equilibrium $x^\star \in \mathbb{R}^n$ is an isolated minimizer of $H(x)$, that is,

$$x^\star = \arg\min H(x),$$

ensuring that $H(x)$ is positive definite. In view of (2), when $u = 0$ and $d = 0$, we have that

$$\dot{H} \le -\alpha|y|^2 \le 0,$$

and $x^\star$ is a stable equilibrium of the unperturbed open-loop system with Lyapunov function $H(x)$. Furthermore, invoking standard LaSalle arguments it is possible to prove that $\lim_{t\to\infty} y(t) = 0$ and, if y is a detectable output, that $x^\star$ is asymptotically stable. See, for instance, [8,3].

To simplify the presentation, in the sequel we identify the set of minimizers of $H(x)$ with

$$\mathcal{M} := \{x \in \mathbb{R}^n \mid \nabla H(x) = 0, \nabla^2 H(x) > 0\}. \tag{6}$$

Since the second order (Hessian positivity) condition is sufficient, but not necessary, for $x^\star$ to be a minimizer of $H(x)$, the set $\mathcal{M}$ is a subset of the minimizer set, hence the consideration is taken with a slight loss of generality.

In the perturbed case, the set of assignable equilibria of (1) and (4) is given by

$$\mathcal{E} := \{x \in \mathbb{R}^n \mid F_{21}(x)\nabla_1 H(x) + F_{22}(x)\nabla_2 H(x) = -d_2\}. \tag{7}$$

It is clear that, if the disturbances are matched, *i.e.*, $d_2 = 0$,

$$\mathcal{M} \subseteq \mathcal{E}.$$

That is, all energy minimizers are assignable equilibria and it is desirable to preserve in closed-loop the open-loop equilibria. On the other hand, in the face of unmatched disturbances, that is, when $d_2 \ne 0$,

$$\mathcal{M} \cap \mathcal{E} = \emptyset. \tag{8}$$

In other words, it is not possible to assign as equilibrium a minimizer of the energy function. As will become clear below, this situation complicates the task of rejection of unmatched disturbances.

Remark 1. A problem with the equilibria, similar to the one described above, appears when the desired value for the output to be regulated is different from zero, which is discussed in point 4 of Section 3.2.

3. Robust IC of the passive output

In this section the output regulation and disturbance rejection properties of IC of the passive output of a pH system are revisited.

[1] This control action is also known in the literature as L_gV control [8,3].

Although both properties are widely referred in the literature, to highlight the differences with our main result, a detailed analysis and some comments and extensions are given below.

3.1. Robustness to matched disturbances

Proposition 1. *Consider the perturbed pH system*

$$\dot{x} = F(x)\nabla H(x) + \begin{bmatrix} I_m \\ 0 \end{bmatrix}(u + d_1)$$
$$y = \nabla_1 H(x) \tag{9}$$

with an equilibrium $x^\star \in \mathcal{M}$, and $d_1 \in \mathbb{R}^m$ a constant disturbance, in closed-loop with the IC

$$\dot{\eta} = K_i y$$
$$u = -\eta, \tag{10}$$

where $K_i \in \mathbb{R}^{m\times m}$ is an arbitrary symmetric positive definite matrix.

(i) *(Stability of the equilibrium). The equilibrium $(x^\star, d_1)$ is stable.*
(ii) *(Output regulation). There exists a (closed) ball, centered at $(x^\star, d_1)$ such that for all initial states $(x(0), \eta(0)) \in \mathbb{R}^n \times \mathbb{R}^m$ inside the ball the trajectories are bounded and*

$$\lim_{t\to\infty} y(t) = 0.$$

(iii) *(Asymptotic stability). If, moreover, y is a detectable output for the closed-loop system (9) and (10), the equilibrium is asymptotically stable.*

The properties (i)–(iii) are global if $H(x)$ is globally positive definite and radially unbounded.

Proof 1. Define the Lyapunov function candidate

$$W(x,\eta) := H(x) + \frac{1}{2}(\eta - d_1)^\top K_i^{-1}(\eta - d_1). \tag{11}$$

The closed-loop system (9) and (10) may be written in the pH form

$$\begin{bmatrix} \dot{x} \\ \dot{\eta} \end{bmatrix} = \begin{bmatrix} F(x) & \begin{bmatrix} -K_i \\ 0 \end{bmatrix} \\ \begin{bmatrix} K_i & 0 \end{bmatrix} & 0 \end{bmatrix} \nabla W(x,\eta). \tag{12}$$

Clearly, in view of (2) and (4), we get $\dot{W} \leq -\alpha|y|^2$. The proof is completed invoking standard Lyapunov and LaSalle arguments [10,3]. □

Remark 2. It is clear from (9) that, to ensure $x^\star \in \mathcal{M}$ remains an equilibrium of the closed-loop system, the desired value for u, and consequently for $-\eta$, is $-d_1$. This aspect is also reflected in (11). The fact that in IC the disturbances fix the equilibrium value of their state, will also be exploited in the case of unmatched disturbances, allowing us to concentrate our attention on the x components of the equilibrium set.

3.2. Discussion and extensions

1. Proposition 1 is a global result that holds for arbitrary positive values of the integral gain. The fact that PBC yields high-performance, easily tunable, simple designs (like PI control) explains its wide-spread popularity in applications.

2. Proposition 1 applies *verbatim* for a general input matrix $g(x)$. In this case, the closed–loop is the pH system

$$\begin{bmatrix} \dot{x} \\ \dot{\eta} \end{bmatrix} = \begin{bmatrix} F(x) & \begin{bmatrix} -g(x)K_i \\ 0 \end{bmatrix} \\ \begin{bmatrix} K_i g^\top(x) & 0 \end{bmatrix} & 0 \end{bmatrix} \nabla W(x,\eta).$$

On the other hand, the presence of $g(x)$ in the subsequent material considerably complicates the notation. Hence, our assumption (4).

3. Looking at the linearization of the closed-loop system (9) and (10), it is possible to show that, if $x^\star \in \mathcal{M}$ and the (2, 2) block of the matrix $F(x)$, evaluated at $x^\star$ is full rank, $x^\star$ is an exponentially stable equilibrium. The rank condition holds if and only if the triple

$$\left(F^\star, \begin{bmatrix} -K_i \\ 0 \end{bmatrix}, \begin{bmatrix} K_i & 0 \end{bmatrix}\right)$$

has no transmission zeros at the origin. This assumption is standard for integral control of nonlinear systems. See, *e.g.*, Section 12.3 of [10].

4. If the desired value for the output y is different from zero, say $y_d \in \mathbb{R}^m$, it is common in practice to use a PI controller

$$\dot{\vartheta} = \nabla_1 H(x) - y_d$$
$$u = -K_p[\nabla_1 H(x) - y_d] - K_i\vartheta,$$

where the proportional term, with $K_p \geq 0$, replaces the previous damping injection. Local stability of this scheme can be established looking at its linearization. It is not clear to the authors under which conditions is it possible to establish a global result—like the one obtained in Proposition 1. A particular case when this is so is when the matrix $F(x)$ is constant. Then, following the analysis of [11], it is possible to show that the shifted Hamiltonian qualifies as a global Lyapunov function.

5. Another difficulty that arises when $y_d \neq 0$ is that a necessary condition to achieve output regulation is the existence of $x^\star \in \mathbb{R}^n$ verifying

$$x^\star \in \mathcal{E} \cap \{x \in \mathbb{R}^n \mid \nabla_1 H(x) = y_d\}.$$

That is, an assignable equilibrium such that the output function, evaluated at this equilibrium, takes the desired value. If $y_d \neq 0$, it is clear that $x^\star \notin \mathcal{M}$. This, unfortunately, makes the expression of the linearized system rather complicated and it does not seem to be possible to easily complete the analysis with an assumption like the rank condition of point 3 above.

4. A feedback equivalence problem

As shown in the proof of Proposition 1 the key property to prove that IC of the passive output rejects matched disturbances is the preservation of the pH structure, moreover, with a separable energy function, see (11) and (12).[2] A key contribution of the paper is the proof that, under some conditions, it is possible to retain these properties in the unmatched disturbance case. More precisely, it is proposed to add a new dynamic extension and a change of coordinates, without modifying the functional relations in the matrix $F(x)$ nor the energy function $H(x)$.[3] Preserving the energy function avoids the need to solve a partial differential equation, while keeping the same interconnection and damping matrix, simplifies the nonlinear algebraic equations. This motivates the following definition of feedback equivalence.

Definition 1. The perturbed system

$$\dot{x} = F(x)\nabla H(x) + \begin{bmatrix} I_m \\ 0 \end{bmatrix} u + \begin{bmatrix} 0 \\ d_2 \end{bmatrix} \tag{13}$$

[2] This property is a consequence of the well-known fact that power-preserving interconnections of pH systems—through power-port variables—preserve the pH structure with energy the sum of the energies of the pH systems. See [2] for a detailed study of this fundamental property.

[3] See Remark 4 for a clarification of this point.

is said to be feedback equivalent to a matched disturbance integral controlled system—for short, MDICS equivalent—if there exists two mappings

$$\hat{u}, \psi : \mathbb{R}^m \times \mathbb{R}^{n-m} \times \mathbb{R}^{n-m} \to \mathbb{R}^m,$$

with

$$\mathrm{rank}\{\nabla_1 \psi(x_1, x_2, \zeta)\} = m, \tag{14}$$

such that the system in closed-loop with the "integral" control

$$\begin{aligned}\dot{\zeta} &= K_i[\nabla_2 H(\psi(x_1, x_2, \zeta), x_2)]\\ u &= \hat{u}(x_1, x_2, \zeta),\end{aligned} \tag{15}$$

expressed in the coordinates,

$$\begin{aligned}z_1 &= \psi(x_1, x_2, \zeta)\\ z_2 &= x_2\\ z_3 &= \zeta,\end{aligned} \tag{16}$$

takes the pH-form

$$\dot{z} = \begin{bmatrix} F(z_1, z_2) & \begin{bmatrix} 0 \\ -K_i \end{bmatrix} \\ \begin{bmatrix} 0 & K_i \end{bmatrix} & 0 \end{bmatrix} \nabla U(z), \tag{17}$$

where

$$U(z) := H(z_1, z_2) + \frac{1}{2}(z_3 - d_2)^\top K_i^{-1}(z_3 - d_2). \tag{18}$$

It is said to be robustly MDICS equivalent if the mappings $\psi(x_1, x_2, \zeta)$ and $\hat{u}(x_1, x_2, \zeta)$ can be computed without knowledge of d_2.[4]

MDICS equivalence guarantees that the transformed closed-loop system takes the desired form (17). (Compare with (12).) The rank condition (14) ensures that (16) is a diffeomorphism that maps the set of equilibria of the (x, ζ)-system into the equilibria of the z-system. This is, of course, necessary to be able to infer stability of one system from stability of the other one.

At this point we make the important observation that choosing the desired value for z_3 to be equal to d_2 is necessary to be able to solve the robust MDICS equivalence problem. Indeed, since in the change of coordinates (16) we fixed $z_2 = x_2$, and these are unactuated coordinates, it is necessary that d_2, which appears in $\dot{x}_2$, appears also in $\dot{z}_2$. This fact will become evident in the next section, when we give the solution to the MDICS equivalence problem. Remark that, since $z_3 = \zeta$, the equilibrium value for ζ is also d_2.

As explained in Section 2.3 the equilibrium sets of (13), (15) and (17) are not just different, but they are actually disjoint, see (8). Indeed, while the (x components of the) former are in the set

$$\mathcal{E}_{cl} := \mathcal{E} \cap \{x \in \mathbb{R}^n \mid [\nabla_2 H](\psi(x_1, x_2, d_2), x_2) = 0\}, \tag{19}$$

the (z_1, z_2) components of the latter are in $\mathcal{M}$. In spite of that, the fact that (16) is a diffeomorphism ensures that the implication

$$[(x_1, x_2) \in \mathcal{E}_{cl} \Rightarrow (\psi(x_1, x_2, d_2), x_2) \in \mathcal{M}], \tag{20}$$

is true, which will be essential for future developments.

Remark 3. The proposed control (15) is, in general, not an integral action because of the possible dependence of $\psi(x_1, x_2, \zeta)$ with respect to ζ. We have decided to keep the name because in the z coordinates it is, indeed, an integral action of the form

$$\dot{z}_3 = K_i \nabla_2 H(z_1, z_2). \tag{21}$$

[4] See Remark 6 for a precise explanation.

Remark 4. It is important to underscore that in the feedback equivalence problem considered here the matrix $F(z_1, z_2)$ and energy function $H(z_1, z_2)$ are just the evaluations of the original functions of the x system in the z coordinates, without applying the (inverse) change of coordinates.[5] That is, $H(x_1, x_2) \neq H(z_1, z_2) \circ \psi(\chi)$, but simply $H(z_1, z_2) = H(x_1, x_2)|_{x_1=z_1, x_2=z_2}$. This, rather arbitrary, choice is done to be able to translate MDICS equivalence into an algebraic problem.

5. Conditions for MDICS equivalence

In this section we present two propositions that identify conditions for MDICS equivalence. The first one is global and identifies the matching conditions that the mapping $\psi(x_1, x_2, \zeta)$ has to satisfy. The second one gives a necessary and a sufficient condition for existence of a local result in terms of controllability and a rank condition of the linearized systems, respectively. To simplify the notation we introduce the $2n - m$ state vector

$$\chi := \mathrm{col}(x_1, x_2, \zeta).$$

5.1. Global MDICS equivalence

Proposition 2. *The perturbed pH system (13) satisfying condition (3) is MDICS equivalent if the mapping $\psi(\chi)$ verifies (14) and the following algebraic equation:*

(DyM) *(Dynamics matching)*

$$\begin{aligned}\zeta = &-F_{21}(x)\nabla_1 H(x) - F_{22}(x)\nabla_2 H(x)\\ &+ F_{21}(\psi(\chi), x_2)[\nabla_1 H(\psi(\chi), x_2)]\\ &+ F_{22}(\psi(\chi), x_2)[\nabla_2 H(\psi(\chi), x_2)].\end{aligned} \tag{22}$$

Moreover, the control signal $\hat{u}(\chi)$ is independent of d_2 if $\psi(\chi)$ verifies

(DiM) *(Disturbance matching)*

$$\nabla_2 \psi(\chi) d_2 = 0. \tag{23}$$

Proof 2. We will prove that, under the condition (22), there exists $\hat{u}(\chi)$ such that the closed-loop system (13), (15) takes, in the z-coordinates, the pH form (17). Furthermore, if (23) holds, the mapping $\hat{u}(\chi)$ is independent of d_2. For, computing $\dot{\psi}$ and setting it equal to $\dot{z}_1$, as defined in (17), yields

$$\begin{aligned}\dot{\psi} &= \nabla\psi(\chi)\dot{\chi}\\ &= \nabla_1\psi(\chi)[F_{11}(x)\nabla_1 H(x) + F_{12}(x)\nabla_2 H(x) + \hat{u}(\chi)]\\ &\quad + \nabla_2\psi(\chi)[F_{21}(x)\nabla_1 H(x) + F_{22}(x)\nabla_2 H(x) + d_2]\\ &\quad + \nabla_3\psi(\chi)[\nabla_2 H(\psi(\chi), x_2)] \equiv \dot{z}_1\\ &= [F_{11}(z)\nabla_1 H(z) + F_{12}(z)\nabla_2 H(z)]|_{z_1=\psi(\chi), z_2=x_2}.\end{aligned} \tag{24}$$

Since $\nabla_1\psi(\chi)$ is full rank, (24) has a unique solution that defines the mapping $\hat{u}(\chi)$. Notice that the disturbance enters through the term $\nabla_2\psi(\chi)d_2$, which cancels if $\psi(\chi)$ satisfies (23).

Proceeding now with $\dot{x}_2$, and setting it equal to $\dot{z}_2$, leads to

$$\begin{aligned}\dot{x}_2 &= F_{21}(x)\nabla_1 H(x) + F_{22}(x)\nabla_2 H(x) + d_2 \equiv \dot{z}_2\\ &= [F_{21}(z)\nabla_1 H(z) + F_{22}(z)\nabla_2 H(z)\\ &\quad - (z_3 - d_2)]|_{z_1=\psi(\chi), z_2=x_2, z_3=\zeta},\end{aligned}$$

which is the matching equation (22). It is important to note that the disturbance d_2, that enters through $\dot{x}_2$, is canceled with the term $\dot{z}_2$, which also contains this signal.

Finally, the third coordinate $\dot{z}_3$ is equal to $\dot{\zeta}$ by construction. □

[5] To avoid cluttering the notation the same symbols, $H(\cdot)$ and $F(\cdot)$, have been used for both functions.

5.2. Local MDICS equivalence

To streamline the presentation of the next result define the linearization of the pH system (13) at the points $x^\star \in \mathcal{E}_{cl}$ and $\bar{x} \in \mathcal{M}$ as

$$A := \nabla(F(x)\nabla H(x))|_{x=x^\star}, \qquad E := (F(x)\nabla^2 H(x))|_{x=\bar{x}}. \tag{25}$$

These $n \times n$ matrices are block partitioned as

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix},$$

with $A_{11} \in \mathbb{R}^{m\times m}$ and $A_{22} \in \mathbb{R}^{(n-m)\times(n-m)}$, with a similar partition for E. Notice that, since $\nabla H(\bar{x}) = 0$, the linearization at a point in the minimizer set takes a simpler form.

Proposition 3. *Consider the perturbed pH system (13) satisfying condition (3) and two points: $x^\star \in \mathcal{E}_{cl}$ and $\bar{x} \in \mathcal{M}$.*

(NC) *A necessary condition for MDICS equivalence is that the linearizations of the pH system at the points $x^\star$ and $\bar{x}$ are controllable. That is, the pairs*

$$\left(A, \begin{bmatrix} I_m \\ 0 \end{bmatrix}\right), \qquad \left(E, \begin{bmatrix} I_m \\ 0 \end{bmatrix}\right)$$

are controllable pairs.

(SC) *A sufficient condition for MDICS equivalence is that the $(2, 1)$ blocks of the matrices A and E defined in (25) are full rank. That is,*

$$\mathrm{rank}\{A_{21}\} = \mathrm{rank}\{E_{21}\} = n - m. \tag{26}$$

Moreover, the system is robustly MDICS equivalent if

$$A_{22} = E_{22} \tag{27}$$

$$A_{21}x_1^\star = -d_2. \tag{28}$$

Proof 3. Since we are interested in local solutions we will solve the MDICS equivalence problem for the linearization of the systems (13), (15) and (17)—around their corresponding equilibrium points. In particular, we are interested in their unactuated dynamics, x_2 and z_2, for which we get[6]

$$\dot{x}_2 = A_{21}(x_1 - x_1^\star) + A_{22}(x_2 - x_2^\star),$$

and

$$\dot{z}_2 = E_{21}(z_1 - \bar{x}_1) + E_{22}(z_2 - \bar{x}_2) - z_3 + d_2.$$

The linearization of the mapping $\psi(\chi)$ at $(x^\star, d_2)$ yields

$$\psi(\chi) = \psi^\star + T_1(x_1 - x_1^\star) + T_2(x_2 - x_2^\star) + T_3(\zeta - d_2), \tag{29}$$

with the constant matrices $T_i := \nabla_i\psi^\star$, $i = 1, 2, 3$. Setting $\dot{x}_2$ equal to $\dot{z}_2$—evaluated at (16)—yields the dynamics matching equation

$$\begin{aligned} &A_{21}(x_1 - x_1^\star) + A_{22}(x_2 - x_2^\star) \\ &\quad\equiv E_{21}[T_1(x_1 - x_1^\star) + T_2(x_2 - x_2^\star) + T_3(\zeta - d_2)] \\ &\qquad + E_{22}(x_2 - x_2^\star) - \zeta + d_2, \end{aligned} \tag{30}$$

where the identities $x_2^\star = \bar{x}_2$ and $\psi^\star = \bar{x}_1$, that stem from (20), are used. Eq. (30) has a solution if and only if the matrices T_i satisfy

$$A_{21} = E_{21}T_1, \qquad A_{22} = E_{21}T_2 + E_{22}, \qquad E_{21}T_3 = I_{n-m}. \tag{31}$$

Remark that, in view of (3), the matrices A_{21} and E_{21} are not tall, being either square or fat.

We now proceed to prove (SC). Assume $\mathrm{rank}\{A_{21}\} = \mathrm{rank}\{E_{21}\} = n-m$. Then, $E_{21}E_{21}^\top$ is invertible and, defining the pseudo-inverse, $E_{21}^\dagger := E_{21}^\top(E_{21}E_{21}^\top)^{-1}$, propose

$$T_1 = E_{21}^\dagger A_{21}, \qquad T_2 = E_{21}^\dagger(A_{22} - E_{22}), \qquad T_3 = E_{21}^\dagger \tag{32}$$

as solutions of (31). Notice that T_1 is the product of full-rank matrices, hence is full rank, and the condition (14) is satisfied.

To prove (NC) assume a solution of (31) exists. Then,

$$\mathrm{rank}\{E_{21}T_3\} = \mathrm{rank}\{I_{n-m}\} = n - m.$$

Since $\mathrm{rank}\{AB\} \le \min\{\mathrm{rank}\{A\}, \mathrm{rank}\{B\}\}$, the identity above implies that $\mathrm{rank}\{E_{21}\} = n - m$. Now, from the Popov–Belevitch–Hautus test we have that the linearized system (E, g) is controllable if and only if, for all $v \in \mathbb{C}^{n-m}$, the following implication is true

$$\left(v^\top E_{21} = 0, E_{22}^\top v = \lambda v, \lambda \in \mathbb{C} \Rightarrow v = 0\right). \tag{33}$$

The rank condition ensures then that the system (E, g) is controllable. It only remains to prove that (A, g) is also controllable. Towards this end, note that $A_{21} = E_{21}T_1$. The rank condition on T_1, (14), imposes that A_{21} is full rank that, once again, implies controllability of (A, g).

The claim of robust MDICS equivalence follows noting that, on one hand, (27) and (31) imply $T_2 = 0$, hence ensuring (23). On the other hand, replacing (28) and (32) in (29), yields the resulting mapping

$$\psi(\chi) = \bar{x}_1 + E_{21}^\dagger(A_{21}x_1 + \zeta), \tag{34}$$

which is, obviously, independent of d_2. □

Unfortunately, there is a gap between the necessary and the sufficient conditions of Proposition 3. Indeed, controllability of the linearized systems is necessary, but not sufficient, for MDICS equivalence. The gap stems from the fact that, without further qualifications on E_{22}, the implication (33) does not ensure that $\mathrm{rank}\{E_{21}\} = n-m$. On the other hand, it is obvious that (26) implies controllability.

Proposition 3 establishes that, if (26)–(28) hold, the system is locally robustly MDICS equivalent—in a neighborhood of $(x_1^\star, x_2^\star, d_2)$—to the linear mapping (34). Of course, there might be other, possibly nonlinear, admissible mappings valid in a large region of the state space. It is shown in Section 7, that this is the case for linear systems and nonlinear mechanical systems.

Remark 5. Condition (27) imposes restrictions on the dependence of $F(x)$ and $H(x)$ with respect to the unactuated coordinate x_2. Condition (28), on the other hand, is related to the form of the assignable equilibrium set $\mathcal{E}$. Recalling that the matrices A and E are linearizations of the same vector field at two different points, it is clear that both sets $\mathcal{E}_{cl}$ and $\mathcal{M}$ play a role in these assumptions. Interestingly, even though these assumptions are now technical, they are satisfied in the examples of Section 7, as well as in the motor example of [7].

Remark 6. In Definition 1 the feedback equivalence was said to be robust—for obvious reasons—if the mappings $\psi(\chi)$ and $\hat{u}(\chi)$ can be computed without knowledge of the disturbance d_2. As seen from the proof of Proposition 2, $\hat{u}(\chi)$ may, indeed, depend on d_2. However, from the dynamics matching equation (22) that defines $\psi(\chi)$, it is not clear why would it depend on d_2. The reason is that, as shown in Proposition 3, when looking for a local solution around the equilibria, these depend on d_2. See (29) and (30).

6. Robust integral control of a non-passive output

In this section the main result of the paper is presented. Namely, the design of an IC, which is robust *vis-à-vis* unmatched disturbances. More precisely, the controller preserves stability of the equilibrium and ensures regulation (to zero) of the signal (5) that, being of relative degree larger than one, is not a passive output.

[6] With an obvious abuse of notation the same symbols for the original equations and their linearizations are used.

Proposition 4. *Consider the perturbed pH system (13) satisfying condition (3). Assume there exist two points, $x^\star \in \mathcal{E}_{cl}$ and $\bar{x} \in \mathcal{M}$, that is, an assignable equilibrium and a minimizer of the energy $H(x)$, such that (26)–(28) hold, with A and E defined in (25). Under these conditions, there exist two mappings*

$$\hat{u}, \psi : \mathbb{R}^m \times \mathbb{R}^{n-m} \times \mathbb{R}^{n-m} \to \mathbb{R}^m,$$

such that the "integral" control (15) ensures the following properties.

(i) *(Stability of the equilibrium). The equilibrium $(x_1^\star, x_2^\star, d_2)$ is stable.*
(ii) *(Regulation of the passive output). There exists a (closed) ball, centered at the equilibrium, such that for all initial states $(x(0), \zeta(0)) \in \mathbb{R}^n \times \mathbb{R}^{n-m}$ inside the ball the trajectories are bounded and*

$$\lim_{t\to\infty} y(t) = 0.$$

(iii) *(Asymptotic stability). If, moreover, y is a detectable output for the closed-loop system (13) and (15), the equilibrium is asymptotically stable.*
(iv) *(Regulation of the non-passive output). Under condition (iii), there exists a (closed) ball, centered at the equilibrium, such that for all initial states $(x(0), \zeta(0)) \in \mathbb{R}^n \times \mathbb{R}^{n-m}$ inside the ball the trajectories are bounded and the output (5) satisfies*

$$\lim_{t\to\infty} r(t) = 0.$$

The properties (i)–(iv) hold globally if the function $H(x)$ is globally positive definite and proper (with respect to $\bar{x}$) and the mapping $\psi(x_1, x_2, \zeta)$ satisfies (globally) the conditions (22) and (23) of Proposition 2.

Proof 4. The proof is an immediate corollary of Propositions 2 and 3. Indeed, under the conditions of the proposition, the perturbed pH system (13) is robustly MDICS equivalent to (17). That is, (16) is a diffeomorphism that transform the closed-loop system into (17). Now, since $\bar{x} \in \mathcal{M}$, $U(z)$ is a positive definite function with respect to $(\bar{x}, d_2)$. Computing the derivative of $U(z)$ along the trajectories of (17), and using (2), yields

$$\dot{U} \le -\alpha |y|^2.$$

The proof of (i)–(iii) is completed, as the proof of Proposition 1, invoking standard Lyapunov and LaSalle arguments. Claim (iv) follows from asymptotic stability of the equilibrium and the fact that $\nabla_2 H(\bar{x}) = 0$. □

7. Examples

In this section we prove that the proposed IC ensures global asymptotic stability for linear systems and nonlinear mechanical systems.

7.1. Linear systems

Proposition 5. *Consider the linear perturbed pH system (13) satisfying condition (3), with F constant verifying*

$$x^\top (F + F^\top) x \le -\alpha |x|^2, \quad \alpha > 0,$$

for all $x \in \mathbb{R}^n$, and with[7] $H(x) = \frac{1}{2}|x|^2$. Assume rank$\{F_{21}\} = n - m$. The IC

$$\dot{\zeta} = K_i x_2$$
$$u = -F_{21}^\dagger K_i x_2 + F_{11} F_{21}^\dagger \zeta, \tag{35}$$

ensures the equilibrium $(-F_{21}^\dagger d_2, 0, d_2)$, is globally asymptotically stable with Lyapunov function

$$V(x,\zeta) := \frac{1}{2}\left(|x_1 + F_{21}^\dagger \zeta|^2 + |x_2|^2 + (\zeta - d_2)^\top K_i^{-1} (\zeta - d_2)\right).$$

Proof 5. In this case

$$\mathcal{E} = \mathcal{E}_{cl} = \{x \in \mathbb{R}^n \mid F_{21} x_1 = -d_2, x_2 = 0\}, \qquad \mathcal{M} = \{x = 0\},$$

and $F = A = E$. Hence, the conditions for robust MDICS equivalence of Proposition 3 are satisfied. The mapping (34) takes the form $\psi(\chi) = x_1 + F_{21}^\dagger \zeta$. The proof is completed computing the expression of u from (24), which yields (35). □

7.2. Mechanical systems

Proposition 6. *Consider an m-degrees of freedom, fully actuated, fully damped, perturbed mechanical system represented in pH form (13), with state $x = col(p, q)$, where $q, p \in \mathbb{R}^m$ are the generalized positions and "momenta",[8] respectively, and*

$$F = \begin{bmatrix} -K_p & -I_m \\ I_m & 0 \end{bmatrix}.$$

The energy function is given by $H(x) = \frac{1}{2} x_1^\top M^{-1} x_1 + P(x_2)$, with $M \in \mathbb{R}^{m\times m}$ the positive definite, constant inertia matrix, and $P(x_2)$ the potential energy function. Assume $\bar{x}_2 = \arg\min P(x_2)$, and it is isolated and global.

The IC

$$\dot{\zeta} = K_i \nabla P(x_2)$$
$$u = -K_p \zeta - M K_i \nabla P(x_2), \tag{36}$$

ensures the equilibrium $(-M d_2, \bar{x}_2, d_2)$ is globally asymptotically stable with Lyapunov function

$$V(x,\zeta) := \frac{1}{2}(x_1 + M\zeta)^\top M^{-1} (x_1 + M\zeta) + P(x_2) + \frac{1}{2}(\zeta - d_2)^\top K_i^{-1} (\zeta - d_2).$$

Proof 6. A global solution to the dynamics matching equation (22) is given by $\psi(\chi) = x_1 + M\zeta$, which clearly satisfies (23). Hence, the conditions for global asymptotic stability of Proposition 4 are satisfied. The proof is completed computing the expression of u above from (24). □

The disturbance considered in the example represents a bias term in the measurement of velocity that propagates into the system through the damping injection. This fact is clear writing the dynamics of the open-loop system in Euler–Lagrange form

$$M\ddot{q} + K_p(\dot{q} - d_2) + \nabla P(q) = u.$$

It is interesting to note that, after differentiation, the closed-loop system is given by

$$M\dddot{q} + K_p \ddot{q} + (I_m + M K_i) \nabla^2 P(q) \dot{q} + K_p K_i \nabla P(q) = 0.$$

Hence, the stabilization mechanism is akin to the introduction of nonlinear *gyroscopic forces* plus a suitable weighting of the potential energy term.

The result can be extended—under some assumptions—to the case of nonconstant inertia matrix. Indeed, it is easy to verify that

[7] The choices of decoupled energy function and zero equilibrium are done for simplicity and without loss of generality.

[8] Notice the non-standard definition of the state. See also the discussion after the proof for the physical meaning of the model, which explains the use of quotation marks for the momenta.

the mapping $\psi(\chi) = x_1 + M(x_2)\zeta$, is a global solution of the dynamics matching equation (22). However, additional constraints on $M(x_2)$ and/or d_2 are needed to satisfy the disturbance matching equation (23). Namely, that the i-th component of the disturbance vector is zero if $M(x_2)$ depends on the i-th element of x_2.

8. Concluding remarks and future work

Motivated by the developments of [7] a new IC that ensures regulation (to zero) of the passive output, as well as the non-passive output $\nabla_2 H(x)$, of the pH system (13)—in spite of the presence of disturbances in the non-actuated coordinates—has been proposed. Because of its simplicity and widespread popularity, we have concentrated here on basic IC solutions. An alternative approach to reject the unmatched disturbance is to use the well-known output regulation techniques as done, for instance, in [12–14], which clearly lead to more complicated state-feedback designs. See also [15].

Robustness with respect to input disturbances of the proposed IC is unclear and is currently being investigated. If the system is fully damped, it can be shown that it is input-to-state stable and, consequently, for a constant input it has a steady state [16]. However, it would be interesting to analyze the effect of adding to the new IC a standard integral action in the passive output, as done in the simulation example of [7].

Another research avenue that we are currently pursuing is to add a new degree of freedom modifying the matrix $F(x)$ in the z-dynamics. To avoid the need to solve a partial differential equation, it is desirable to keep the same energy function, however there is no particular reason to keep the same matrix—as long as the symmetric part of new one is also negative semidefinite. Towards this end, the matching equation and the control are accordingly modified, but the new algebraic equations are more complicated because of the particular way they depend on the new matrix. Finally, as pointed out in Remark 5, we have a poor understanding of the meaning of conditions (27) and (28) that, at this point, are just technically motivated.

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