Distributed Port-Hamiltonian Systems Modelling and Control

Summer School on

Port-Hamiltonian modelling and passivity-based control of physical systems. Theory and applications

CeUB, Bertinoro, July 6-8, 2017

Overview

- This lecture is devoted to two main topics
  - *Modelling* of distributed parameter systems within the port-Hamiltonian framework
  - *Energy-based control* of a class of distributed port-Hamiltonian systems with one dimensional spatial domain

Very good…

Now, let's start with some modelling!!

Modelling origins

- This part presents the formulation of *distributed parameter systems in terms of port-Hamiltonian system*
- For different examples of physical systems defined on one-dimensional spatial domains, the *Dirac structure* and the *port-Hamiltonian formulation* arise from the description of distributed parameter systems as *systems of conservation laws*
- Systems of *two conservation laws* describe two physical domains in *reversible interaction*
  - They may be formulated as port-Hamiltonian systems defined on a canonical Dirac structure called *canonical Stokes-Dirac structure*

Modelling origins

- *Boundary port-Hamiltonian systems* that extend the port-Hamiltonian formulation from lumped parameter systems
- Dynamic models of distributed parameter systems are defined by considering *time and space as independent parameters* on which the physical quantities are defined
  - Vibrating strings or plates;
  - Transmission lines or electromagnetic fields;
  - Mass and heat transfer in tubular reactors or fuel cells
- Distributed parameter systems are now formulated in terms of *systems of conservation laws*

Conservation law and irreversible thermodynamics

- *Heat diffusion in 1-dimensional medium* (e.g., a rod with cylindrical symmetry)
  - The medium is undeformable (i.e. its deformations are neglected);
  - Only the thermal domain and its dynamics are considered
- Write a *conservation law* of the conserved quantity i.e., the density of internal energy, denoted by *u(t,z):*

extensive thermodynamic variable

$$\frac{\partial u}{\partial t} = -\frac{\partial}{\partial z} J_Q, \qquad z \in Z = [a,\, b] \subset \mathbb{R}$$

spatial domain

flux variable

- *The heat flux J_Q arises from the thermodynamic non-equilibrium* and is defined by some phenomenological law, for instance defined according to Fourier's law by:

thermodynamic driving force F(z,t)

$$J_Q(t,z) = -\lambda(T,z)\frac{\partial}{\partial z}T(t,z)$$

heat conduction coeff.

Conservation law and irreversible thermodynamics

- The *thermodynamic driving force F(z,t)* characterises the non-equilibrium condition
  - Conservation and phenomenological laws should be completed by a relation between the driving force *F* and the conserved quantity *u*;
  - This relation is given by the *thermodynamical properties of the medium* which is characterized by some *thermodynamical potential*
- The thermodynamical properties are given by Gibbs relation:

$$\mathrm{d}u = T\mathrm{d}s$$

No exchange of matter and the *volume* of the medium is *constant*

  - *s is the entropy of the medium* (extensive variable);
  - Due to the irreversibility of thermodynamic processes, *T* is strictly positive, in such a way that *one may choose equivalently the internal energy or the entropy as thermodynamical potential*
- ***Who is the thermodynamic potential?***

Conservation law and irreversible thermodynamics

- **Choice #1:** the internal energy *$u = u(s)$*. Then, Gibbs relation defines the temperature as intensive variable conjugated to the entropy:

$$T = \frac{\mathrm{d}u}{\mathrm{d}s}(s)$$

- This leads to the following *entropy balance equation* (also called conservation law with source term or *Jaumann's entropy balance*):

$$\frac{\partial s}{\partial t} = -\frac{1}{T}\frac{\partial}{\partial z}J_Q = -\frac{\partial}{\partial z}J_S + \sigma$$

irreversible entropy creation

with

$$J_S = \frac{1}{T}J_Q = -\lambda(T,z)\frac{1}{T}\frac{\partial}{\partial z}T(t,z) \qquad \sigma = -\frac{1}{T}\frac{\partial T}{\partial z}J_S = \frac{\lambda(T)}{T^2}\left(\frac{\partial T}{\partial z}\right)^2 \geq 0$$

flux of entropy

- The flux of entropy may be written as a function *characterising the (irreversible) phenomenon of heat conduction* and in terms of the generating force:

$$J_S(t,z) = \frac{\lambda(t,z)}{T}F(t,z), \qquad F(t,z) = -\frac{\partial}{\partial z}\frac{\mathrm{d}u}{\mathrm{d}s}(s)$$

It depends on the *differential* of the internal *energy function*

Conservation law and irreversible thermodynamics

- **Choice #2:** chose the entropy $s = s(u)$. Then, Gibbs relation defines the inverse of the temperature as the intensive variable conjugated to the internal energy:

$$\frac{1}{T} = \frac{\mathrm{d}s}{\mathrm{d}u}(u)$$

$$\frac{\partial u}{\partial t} = -\frac{\partial}{\partial z} J_Q$$

- The heat flux is a function of the driving force

$$J_Q = \lambda(T, z) T^2 F' \qquad F'\left(\frac{1}{T}\right) = \frac{\partial}{\partial z}\frac{\mathrm{d}s}{\mathrm{d}u}(u)$$

Usual expression of heat conduction. The start is the calorimetric property (assume a constant volume):

$$\mathrm{d}u = c_V(T)\mathrm{d}T$$

By direct substitution in the conservation law, one obtains the heat equation:

$$c_V(T)\frac{\partial T}{\partial t} = \frac{\partial}{\partial z}\left(\lambda(T, z)\frac{\partial}{\partial z} T\right)$$

Note that this does not retain the structure of a conservation law

The thermodynamic axioms (near equilibrium) define dynamical systems as *conservation laws*, completed with the definition of the *flux variable* by *an irreversible phenomenological law*, expressing the flux variable as a function of the generating force which is the *spatial derivative of the differential* of some thermodynamical potential characterising the *thermodynamic (or equilibrium) properties of the system*

Reversible physical systems of 2 coupled conservation laws

- In order to define *a port-Hamiltonian formulation for infinite dimensional systems,* we shall apply the thermodynamic analysis to *reversible physical systems*
- Electromagnetic or elasto-dynamic systems are considered as two physical domain *coupled by a reversible inter-domain coupling*
  - In bond graph terms this coupling is represented by a **gyrator;**
  - For distributed parameter systems, one may define some analogous canonical inter-domain coupling which however correspond to *an extension of a symplectic gyrator*
- The thermodynamic perspective for reversible physical systems makes appear a *canonical Dirac structure* associated with some canonical inter-domain coupling

Reversible physical systems of 2 coupled conservation laws

- *Ideal lossless transmission line:*

$$\frac{\partial Q}{\partial t} = -\frac{\partial I}{\partial z}$$
$$\frac{\partial \phi}{\partial t} = -\frac{\partial V}{\partial z}$$

$$\mathcal{H}(Q,\phi) = \frac{1}{2}\left[\frac{Q^2(t,z)}{C(z)} + \frac{\phi^2(t,z)}{L(z)}\right]$$

- The current and voltage are the *flux variables* of the conservation laws

$$\begin{pmatrix} I \\ V \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} V \\ I \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \frac{\partial \mathcal{H}}{\partial Q} \\ \frac{\partial \mathcal{H}}{\partial \phi} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \frac{Q}{C} \\ \frac{\phi}{L} \end{pmatrix}$$

- The flux variable of the electrical domain is identical with the co-energy variable of the magnetic domain and vice versa
  - This is the *canonical inter-domain coupling* via a symplectic gyrator
  - This relation is the pendant of the *phenomenological law* for irreversible systems, for reversible systems
  - It expresses a coupling by a anti-diagonal matrix which has *no parameters*
  - The electro-magnetic energy “is” the thermodynamical potential

Reversible physical systems of 2 coupled conservation laws

Definition. Consider a functional H defined by

$$H[x] = \int_a^b \mathcal{H}\left(z, x, x^{(1)}, \ldots, x^{(n)}\right) \mathrm{d}z$$

for any smooth real function $x(z)$, $z \in Z$. The *variational derivative of the functional H* is denoted by

$$\frac{\delta H}{\delta x}$$

and is the only function that satisfy for every $\varepsilon \in \mathcal{R}$ and smooth real function $\delta x(z)$, $z \in Z$, such that their derivatives satisfy $\delta x^{(i)}(a) = \delta x^{(i)}(b) = 0$, $i = 0,\ldots,n$:

$$H[x + \epsilon\delta x] = H[x] + \epsilon \int_a^b \frac{\delta H}{\delta x} \delta x \mathrm{d}z + O(\epsilon^2)$$

When H does depend *only* on the function x and not its derivatives, then

$$\frac{\delta H}{\delta x} = \frac{\mathrm{d}\mathcal{H}}{\mathrm{d}x}$$

$$\mathcal{H}(Q, \phi) = \frac{1}{2}\left[\frac{Q^2(t,z)}{C(z)} + \frac{\phi^2(t,z)}{L(z)}\right]$$

Transmission line eq.

Reversible physical systems of 2 coupled conservation laws

- For the transmission line, *the co-energy variables are then interpreted as the variational derivatives of the total electromagnetic energy*

$$\alpha = \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = \begin{pmatrix} Q \\ \phi \end{pmatrix} \qquad \frac{\partial \alpha}{\partial t} = \begin{pmatrix} 0 & -\frac{\partial}{\partial z} \\ -\frac{\partial}{\partial z} & 0 \end{pmatrix} \begin{pmatrix} \frac{\delta H}{\delta \alpha_1} \\ \frac{\delta H}{\delta \alpha_2} \end{pmatrix}$$

- This system is an *infinite-dimensional Hamiltonian system* defined with respect to the matrix differential operator:

$$\mathcal{J} = \begin{pmatrix} 0 & -\frac{\partial}{\partial z} \\ -\frac{\partial}{\partial z} & 0 \end{pmatrix}$$

and generated by the Hamiltonian function *H*

- One has to check that the matrix differential operator satisfies
  - *Skew-symmetry;*
  - *Jacobi identities*

- Consider two vectors of smooth functions

$$e = \begin{pmatrix} e_1 \\ e_2 \end{pmatrix}, \quad e' = \begin{pmatrix} e'_1 \\ e'_2 \end{pmatrix} \qquad e(a) = e(b) = e'(a) = e'(b) = 0$$

Reversible physical systems of 2 coupled conservation laws

- *Skew-symmetry:*

$$\int_a^b \left(e^{\mathrm{T}}\mathcal{J}e' + e'^{\mathrm{T}}\mathcal{J}e\right)\mathrm{d}z = \int_a^b \left[e_1\left(-\frac{\partial}{\partial z}e_2'\right) + e_2\left(-\frac{\partial}{\partial z}e_1'\right) + \right.$$
$$\left. + e_1'\left(-\frac{\partial}{\partial z}e_2\right) + e_2'\left(-\frac{\partial}{\partial z}e_1\right)\right]\mathrm{d}z$$
$$= -\left[e_1 e_2' + e_2 e_1'\right]_a^b = 0$$

- *Jacobi identities:* immediate since $\mathcal{J}$ is a constant differential operator

- As in the finite-dimensional case, the Hamiltonian structure results in an additional conservation law, namely the *conservation of energy:*

$$\frac{\mathrm{d}}{\mathrm{d}t}H = \int_a^b \begin{pmatrix}\frac{\delta H}{\delta\alpha_1} & \frac{\delta H}{\delta\alpha_2}\end{pmatrix}\frac{\partial\alpha}{\partial t}\mathrm{d}z = \int_a^b \begin{pmatrix}\frac{\delta H}{\delta\alpha_1} & \frac{\delta H}{\delta\alpha_2}\end{pmatrix}\mathcal{J}\begin{pmatrix}\frac{\delta H}{\delta\alpha_1}\\ \frac{\delta H}{\delta\alpha_2}\end{pmatrix}\mathrm{d}z = 0$$

For the skew-symmetry of $\mathcal{J}$ there might be *no energy exchange at the boundaries.* This is a very particular situation, not suitable neither from a modular point of view as for control. This is the major motivation for introducing *port variables* and extend the Hamiltonian formulation to a *boundary port Hamiltonian system*

Reversible physical systems of 2 coupled conservation laws

- For functions that are *not zero at the boundary,* the differential operator is no more skew-symmetric and boundary terms appear
  - In terms of physical modelling, the *energy is not conserved*

$$\frac{\mathrm{d}H}{\mathrm{d}t} = \frac{\delta H}{\delta \alpha_1}(t,a)\frac{\delta H}{\delta \alpha_2}(t,a) - \frac{\delta H}{\delta \alpha_1}(t,b)\frac{\delta H}{\delta \alpha_2}(t,b) = V(t,a)I(t,a) - V(t,b)I(t,b)$$

- This suggests to introduce *the restriction of the co-energy variables to the boundary of the spatial domain* as external variables:

$$\begin{pmatrix} f_\partial \\ e_\partial \end{pmatrix} = \begin{pmatrix} \frac{\delta H}{\delta \alpha_1} \\ \frac{\delta H}{\delta \alpha_2} \end{pmatrix}\Bigg|_{a,b} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \frac{\delta H}{\delta \alpha_1} \\ \frac{\delta H}{\delta \alpha_2} \end{pmatrix}\Bigg|_{a,b}$$

where

$$\begin{pmatrix} e_1 \\ e_2 \end{pmatrix}\Bigg|_{a,b} = \begin{pmatrix} e_1(a) & e_1(b) & e_2(a) & e_2(b) \end{pmatrix}^{\mathrm{T}}$$

We show that this is the starting point for the definition of a *port Hamiltonian system* defined with respect to a Dirac structure called *Stokes-Dirac structure*

Reversible physical systems of 2 coupled conservation laws

$Z = [a,\, b] \quad \partial Z = \{a,\, b\}$

- *Space of flow variables:*

$$\mathcal{F} = \left\{ f = \begin{pmatrix} f_1 \\ f_2 \\ f_\partial \end{pmatrix} \in C^\infty([a,\, b]) \times C^\infty([a,\, b]) \times \mathbb{R}^{\{a,b\}} \right\}$$

- *Space of effort variables:*

$$\mathcal{E} = \left\{ e = \begin{pmatrix} e_1 \\ e_2 \\ e_\partial \end{pmatrix} \in C^\infty([a,\, b]) \times C^\infty([a,\, b]) \times \mathbb{R}^{\{a,b\}} \right\}$$

- *Non-degenerated bi-linear product or pairing:*

$$\left\langle \begin{pmatrix} e_1 \\ e_2 \\ e_\partial \end{pmatrix}, \begin{pmatrix} f_1 \\ f_2 \\ f_\partial \end{pmatrix} \right\rangle = \int_a^b (e_1 f_1 + e_2 f_2)\, \mathrm{d}z + e_\partial(b) f_\partial(b) - e_\partial(a) f_\partial(a)$$

Reversible physical systems of 2 coupled conservation laws

Proposition. The linear subset $\mathcal{D} \subset \mathcal{F} \times \mathcal{E}$ defined by:

$$\mathcal{D} = \left\{ \left(\begin{pmatrix} f_1 \\ f_2 \\ f_\partial \end{pmatrix}, \begin{pmatrix} e_1 \\ e_2 \\ e_\partial \end{pmatrix} \right) \in \mathcal{F} \times \mathcal{E} \;\middle|\; \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \begin{pmatrix} 0 & -\frac{\partial}{\partial z} \\ -\frac{\partial}{\partial z} & 0 \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \end{pmatrix} \right.$$

$$\left. \text{and } \begin{pmatrix} f_\partial \\ e_\partial \end{pmatrix} (a,b) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \end{pmatrix} (a,b) \right\}$$

is a Dirac structure with respect to the symmetric pairing:

$$\ll \begin{pmatrix} f \\ e \end{pmatrix}, \begin{pmatrix} f' \\ e' \end{pmatrix} \gg = \langle e, f' \rangle + \langle e', f \rangle \qquad \begin{pmatrix} f \\ e \end{pmatrix}, \begin{pmatrix} f' \\ e' \end{pmatrix} \in \mathcal{F} \times \mathcal{E}$$

The construction of the port variables and the Dirac structure associated with the differential operator

$$\frac{\partial}{\partial z}$$

has been extended to *higher order linear skew-symmetric operators*

A port Hamiltonian system with respect to this Dirac structure follows *similarly* to the definition for *finite dimensional systems*

Reversible physical systems of 2 coupled conservation laws

Definition. A *boundary port Hamiltonian system* with state variables

$$\alpha(t) = \begin{pmatrix} \alpha_1(t) \\ \alpha_2(t) \end{pmatrix} \in C^\infty([a,b]) \times C^\infty([a,b])$$

and *port variables*

$$\begin{pmatrix} f_\partial(t) \\ e_\partial(t) \end{pmatrix} \in \mathbb{R}^{\{a,b\}} \times \mathbb{R}^{\{a,b\}}$$

generated by the *Hamiltonian functional*

$$H[\alpha] = \int_a^b \mathcal{H}(z,\alpha)\mathrm{d}z$$

with respect to the previous *Dirac structure* is defined by

$$\left(\begin{pmatrix} \frac{\partial \alpha_1}{\partial t} \\ \frac{\partial \alpha_2}{\partial t} \\ f_\partial \end{pmatrix}, \begin{pmatrix} \frac{\delta H}{\delta \alpha_1} \\ \frac{\delta H}{\delta \alpha_2} \\ e_\partial \end{pmatrix} \right) \in \mathcal{D}$$

Reversible physical systems of 2 coupled conservation laws

- Going back to the *transmission line equation...*

$$\frac{\partial Q}{\partial t} = -\frac{\partial I}{\partial z}$$
$$\frac{\partial \phi}{\partial t} = -\frac{\partial V}{\partial z}$$

conservation laws

$$\begin{pmatrix} I \\ V \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} V \\ I \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \frac{\partial \mathcal{H}}{\partial Q} \\ \frac{\partial \mathcal{H}}{\partial \phi} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \frac{Q}{C} \\ \frac{\phi}{L} \end{pmatrix}$$

co-energy variables

$$\mathcal{H}(Q,\phi) = \frac{1}{2}\left[\frac{Q^2(t,z)}{C(z)} + \frac{\phi^2(t,z)}{L(z)}\right]$$

electro-magnetic energy

$$\begin{pmatrix} f_\partial \\ e_\partial \end{pmatrix} = \begin{pmatrix} \frac{\delta H}{\delta \alpha_1} \\ \frac{\delta H}{\delta \alpha_2} \end{pmatrix}\Bigg|_{a,b} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \frac{\delta H}{\delta \alpha_1} \\ \frac{\delta H}{\delta \alpha_2} \end{pmatrix}\Bigg|_{a,b}$$

port variables

The *pairing* consists in two integral terms corresponding to the *electric and the magnetic power in the spatial domain Z* plus two terms corresponding to the *electromagnetic power at both boundary points* of the domain

Reversible physical systems of 2 coupled conservation laws

displacement
$u(t,z)$

- *Vibrating string* as a port Hamiltonian system
- *State variables:*

$$\alpha(t,z) = \begin{pmatrix} \epsilon(t,z) \\ p(t,z) \end{pmatrix} \qquad \epsilon = \frac{\partial}{\partial z} u(t,z), \; p = \mu v(t,z) = \mu \frac{\partial}{\partial t} u(t,z)$$

strain momentum

- *Total energy,* sum of kinetic and elastic energies:

$$H_0(\alpha) = \int_a^b \frac{1}{2}\left(T\alpha_1^2 + \frac{1}{\mu}\alpha_2^2\right) \mathrm{d}z$$

$$\beta_1 = \frac{\delta H_0}{\delta \alpha_1} = T\epsilon \qquad \beta_2 = \frac{\delta H_0}{\delta \alpha_2} = \frac{p}{\mu}$$

stress velocity

The energy functional depends *only on the state variables* and ***not*** on their spatial derivatives

- The model is expressed by:

$$\frac{\partial \alpha}{\partial t} = \begin{pmatrix} 0 & \frac{\partial}{\partial z} \\ \frac{\partial}{\partial z} & 0 \end{pmatrix} \frac{\delta H_0}{\delta \alpha}$$

system of two conservation laws

Reversible physical systems of 2 coupled conservation laws

- By looking at the model:

$$\frac{\partial \alpha}{\partial t} = \begin{pmatrix} 0 & \frac{\partial}{\partial z} \\ \frac{\partial}{\partial z} & 0 \end{pmatrix} \frac{\delta H_0}{\delta \alpha}$$

  - The first equation is purely *kinematic;*
  - The second equation is *Newton's law;*
  - The fluxes are expressed as a function of the generating forces by:

$$\beta = \begin{pmatrix} -\frac{\delta H_0}{\delta \epsilon} \\ -\frac{\delta H_0}{\delta p} \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \frac{\delta H_0}{\delta \alpha_1} \\ \frac{\delta H_0}{\delta \alpha_2} \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} T\epsilon \\ \frac{p}{\mu} \end{pmatrix}$$

  - *Port boundary variables:*

$$\begin{pmatrix} f_\partial \\ e_\partial \end{pmatrix} = \begin{pmatrix} \beta_2 \\ \beta_1 \end{pmatrix} \Bigg|_{a,b}$$

> By using as state variables the variables on which the *elementary conservation laws* applies and such that the *energy is a function of them* and **not** their derivatives, one may express the dynamics as a port Hamiltonian system defined on a *Dirac structure* which is *canonical,* i.e. it express simultaneously, the conservation laws, the inter-domain coupling and the interaction through the boundary of the system, independently of its energy properties

Reversible physical systems of 2 coupled conservation laws

- Usually, the model of the vibrating string is expressed in terms of the geometric state variables the *displacement u* and the *velocity v*
- *Total energy:*

$$H(x) = U(u) + K(v) \qquad U(u) = \int_a^b \frac{1}{2}\left(\frac{\partial u}{\partial z}\right)^2 \mathrm{d}z, \;\; K(v) = \int_a^b \frac{1}{2}\mu v^2 \mathrm{d}z$$

strain $\epsilon = \dfrac{\partial u}{\partial z}$ **velocity** $v = \dfrac{\partial u}{\partial t}$

- *System dynamics:*

$$\frac{\partial}{\partial t}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{\mu} \\ -\frac{1}{\mu} & 0 \end{pmatrix}\begin{pmatrix} \frac{\delta H}{\delta u} \\ \frac{\delta H}{\delta v} \end{pmatrix}$$

$$\frac{\delta H}{\delta u} = -\frac{\partial}{\partial z}\left(T\frac{\partial u}{\partial z}\right)$$

$$\frac{\delta H}{\delta v} = \mu v$$

- *This is an Hamiltonian system!* However:
  * The skew-symmetric operator depends on some parameter, and this goes against the idea of *compositional modelling* (energy and inter-domain coupling are now coupled with the definition of this matrix)
  * The Hamiltonian system is **not** expressed as a system of conservation laws

Reversible physical systems of 2 coupled conservation laws

- Even if the model look simpler, there is a drawback for the case when there is some *energy flow through the boundary* of the spatial domain
  - The *variational derivative* has to be completed by a boundary term as the Hamiltonian functional depends on the spatial derivative of the state:

$$U(u+\epsilon\eta) = U(u) - \epsilon\int_a^b \frac{\partial}{\partial z}\left(T\frac{\partial u}{\partial z}\right)\eta \mathrm{d}z + \epsilon\left[\eta\left(T\frac{\partial u}{\partial z}\right)\right]_a^b + O(\epsilon^2)$$

$$\frac{\mathrm{d}H}{\mathrm{d}t} = \int_a^d \left(\frac{\delta H}{\delta u}\frac{\partial u}{\partial t} + \frac{\delta H}{\delta v}\frac{\partial v}{\partial t}\right)\mathrm{d}z + \left[\frac{\partial u}{\partial t}\left(T\frac{\partial u}{\partial z}\right)\right]_a^b = \left[v\left(T\frac{\partial u}{\partial z}\right)\right]_a^b$$

- One may introduce the following two *boundary variables:*

$$\begin{pmatrix} f_\partial \\ e_\partial \end{pmatrix} = \begin{pmatrix} v \\ T\frac{\partial u}{\partial z} \end{pmatrix}\Bigg|_{a,b}$$

The *power continuity* properties are ***not*** visible!!

Non trivial relation with the variational derivatives and *depending on the energy parameters*

Bored????

So, it's time for talking about control!!!

Initial remarks

- This second part of the lecture deals with the *energy-based control of distributed port-Hamiltonian systems*
  - Initially, this task has been accomplished via the *energy-Casimir method*
    - Boundary control
    - Application to particular systems (e.g., transmission line, flexible structures, fluid dynamics)
    - Simple stability
    - No analysis of solutions for the open and the closed-loop systems
    - Linearity
  - There has been the need for a connection with *functional analysis, e.g. semigroup theory* for linear, distributed port-Hamiltonian systems
    - Precise formalization of the problem
    - Results on the existence of solutions, definition of boundary control systems and analysis of interconnected systems
    - Stability analysis, at least in the linear case

Initial remarks

- *The initial work has been then reformulated (and rediscovered)* in this well-structured framework
  - We now know how to prove existence of solutions, and to verify the stability properties in closed-loop
  - Complete characterisation of the energy-Casimir method in the linear case, and for one-dimensional spatial domain
- The stability proofs can take advantage from other tools, the *La Salle's invariance principle* in particular
  - *There some evil hypotheses to check*

  - *Physical interpretation* of the approach, and of the results
  - *Nonlinearities* and *state-dependent (boundary) control actions*
  - Move the focus to the *trajectories,* rather than on structural properties only
- Energy-based control beyond the *dissipation obstacle*
  - *"Geometric"* interpretation of the results
- Simple nonlinear systems

Energy-based control

- Let us consider a finite-dimensional port-Hamiltonian system:

$$\begin{cases} \dot{x}(t) = \left[J(x(t)) - R(x(t))\right] \dfrac{\partial H}{\partial x}(x(t)) + G(x(t))u(t) \\ y(t) = G^{\mathrm{T}}(x(t)) \dfrac{\partial H}{\partial x}(x(t)) \end{cases}$$

energy balancing

$$\frac{\mathrm{d}H}{\mathrm{d}t}(x(t)) = -\left(\frac{\partial H}{\partial x}(x(t))\right)^{\mathrm{T}} R(x(t)) \frac{\partial H}{\partial x}(x(t)) + y^{\mathrm{T}}(t)u(t)$$

- Standard approach is to rely on *"energy considerations"* to obtain and prove asymptotic stability of equilibria
  - *Damping injection*
  - *Energy-shaping*
- Standard assumption is *H bounded from below*

Damping injection

- Suppose that *H has an isolated minimum at a desired equilibrium*

$$\frac{\partial H}{\partial x}(x^\star) = 0 \qquad\qquad \frac{\partial^2 H}{\partial x^2}(x^\star) > 0$$

- The idea is to *dissipate energy* until the minimum is reached
  - Asymptotic stability if there is “enough dissipation”
  - Zero-state detectability
  - La Salle’s Invariance principle
- The control action is

$$u(t) = -K_D y(t), \qquad K_D = K_D^{\mathrm{T}} \geq 0$$

$$\frac{\mathrm{d}H}{\mathrm{d}t}(x(t)) = -\left(\frac{\partial H}{\partial x}(x(t))\right)^{\mathrm{T}} \big(R(x(t)) + K_D\big)\frac{\partial H}{\partial x}(x(t)) \leq 0$$

Energy-balancing control

- In general, it is necessary to *shape the open-loop Hamiltonian* to introduce a minimum at the desired equilibrium
- From the energy-balance relation we have

$$H(x(t)) - H(x(0)) = \int_0^t y^{\mathrm{T}}(\tau)u(\tau)\,\mathrm{d}\tau - d(t)$$

- The standard formulation of passivity-based control requires to *determine a control action*

$$u(t) = \beta(x(t)) + u'(t)$$

such that the *closed-loop dynamics* satisfies:

$$H_d(x(t)) - H_d(x(0)) = \int_0^t {y'}^{\mathrm{T}}(\tau)u'(\tau)\,\mathrm{d}\tau - d_d(t)$$

new energy balancing

- *H_d* is a desired energy function, while *d_d* replaces the natural dissipation
  - Energy-shaping *plus* damping injection

Energy-balancing control

- A large class of dynamical systems can be stabilized by requiring that the *supplied energy is a function of the state of the plant*

$$-\int_0^t y^{\mathrm{T}}(\tau)\beta(x(\tau))\,\mathrm{d}\tau = H_a(x(t)) + \kappa$$

- We require that along *all* system trajectories

$$-y^{\mathrm{T}}(t)\beta(x(t)) = \frac{\partial^{\mathrm{T}} H_a}{\partial x}(x(t))\dot{x}(t)$$

- The *"desired" closed-loop Hamiltonian* is then

$$H_d(x(t)) = H(x(t)) + H_a(x(t))$$

- The previous PDE provides the *class of H_a* and the *control actions,* while stability analysis follows from the energy-balance relation
  - *u' can be used to add damping*

Energy-balancing control

- The methodology can be applied to generic nonlinear systems

$$\begin{cases} \dot{x} = f(x) + g(x)u \\ y = h(x) \end{cases}$$

- From *KYP lemma,* passivity is equivalent to the *existence of a function H(x)* such that

$$\left(\frac{\partial H}{\partial x}(x)\right)^{\mathrm{T}} f(x) \leq 0 \qquad\qquad h(x) = g^{\mathrm{T}}(x)\frac{\partial H}{\partial x}(x)$$

- *Matching equation:*

supplied power

$$\left(\frac{\partial H_a}{\partial x}(x)\right)^{\mathrm{T}} \left[f(x) + g(x)\beta(x)\right] = -h^{\mathrm{T}}(x)\beta(x)$$

- At the equilibrium:

$$f(x^\star) + g(x^\star)\beta(x^\star) = 0 \quad \Rightarrow \quad h^{\mathrm{T}}(x^\star)\beta(x^\star) = 0$$

dissipation obstacle

Energy-Casimir method

- Let us consider a port-Hamiltonian controller

$$\begin{cases} \dot{\xi}(t) = \left[J_C(\xi(t)) - R_C(\xi(t))\right] \dfrac{\partial H_C}{\partial \xi}(\xi(t)) + G_C(\xi(t))u_C(t) \\ y_C(t) = G_C^{\mathrm{T}}(\xi(t)) \dfrac{\partial H_C}{\partial \xi}(\xi(t)) \end{cases}$$

$$\begin{cases} \dot{x}(t) = \left[J(x(t)) - R(x(t))\right] \dfrac{\partial H}{\partial x}(x(t)) + G(x(t))u(t) \\ y(t) = G^{\mathrm{T}}(x(t)) \dfrac{\partial H}{\partial x}(x(t)) \end{cases}$$

- Closed-loop Hamiltonian:

$$H(x(t)) + H_C(\xi(t))$$

- How can we select H_C to properly *shape the closed-loop Hamiltonian in the "x-direction"?*
  - We need to introduce the concept of *Casimir function*
  - We look for, or we generate, a set of functions that are *constant independently from the Hamiltonians,* i.e. for all the possible achievable trajectories in closed-loop

Energy-Casimir method

- We look for *Casimir functions* in the form

$$C(x,\xi) = \xi - \Psi(x)$$

- In this way, for the *closed-loop system* we have

$$H(x) + H_C(\xi) \equiv H(x) + H_C(\Psi(x) + \kappa)$$

- The advantage is that
  - There is no need to explicitly deal with the *trajectories* of the system
  - Stability can be analysed by looking at the *energy-balance* only
  - Complete *characterisation of all the energy-balancing controllers*

$$\left(\frac{\partial \Psi}{\partial x}(x)\right)^{\mathrm{T}} J(x) \frac{\partial \Psi}{\partial x} = J_C(\xi) \qquad R(x)\frac{\partial \Psi}{\partial x}(x) = 0$$

$$R_C(\xi) = 0 \qquad \left(\frac{\partial \Psi}{\partial x}(x)\right)^{\mathrm{T}} J(x) = G_C(\xi) G^{\mathrm{T}}(x)$$

The solution is determined by the *Dirac structure* and independent from the *resistive relation*

Control with state-modulated source

- The idea is to compute a state feed-back action

$$u(t) = \beta(x(t)) + u'(t)$$

  so that *the open-loop system is mapped into a new one,* but with a desired Hamiltonian

$$H_d(x(t)) = H(x(t)) + H_a(x(t))$$

$$\dot{x}(t) = \left[J(x(t)) - R(x(t))\right]\frac{\partial H_d}{\partial x}(x(t)) + G(x(t))u'(t)$$

- A direct computation leads to

$$G(x)\beta(x) = \left[J(x) - R(x)\right]\frac{\partial H_a}{\partial x}(x) \qquad \textit{matching condition}$$

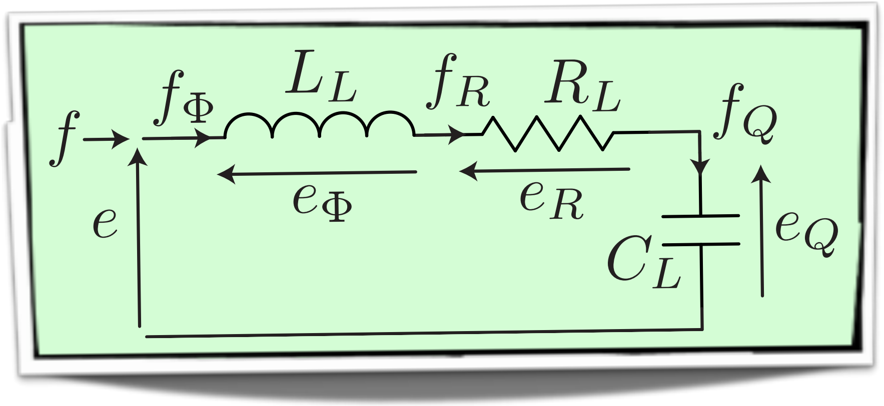
- A further generalization leads to the *IDA-PBC control technique,* where we shape
  * Hamiltonian
  * Interconnection and resistive structure

Example: “series” RLC circuit

- Port-Hamiltonian model:

$$\dot{x}_Q = \frac{\partial H}{\partial x_\Phi}$$
$$\dot{x}_\Phi = -\frac{\partial H}{\partial x_Q} - R\frac{\partial H}{\partial x_\Phi} + u$$

$$H(x_Q, x_\Phi) = \frac{1}{2}\left(\frac{x_Q^2}{C} + \frac{x_\Phi^2}{L}\right)$$



- Equilibrium configuration

$$(x_Q^\star, x_\Phi^\star) = (Cv^\star, 0)$$

- If the controller is an *integrator,* Casimir function are given by

$$\dot{\xi} = u_C$$
$$y_C = \frac{\partial H_C}{\partial \xi}(\xi)$$

$$C(x_Q, x_\Phi, \xi) = \xi - x_Q$$

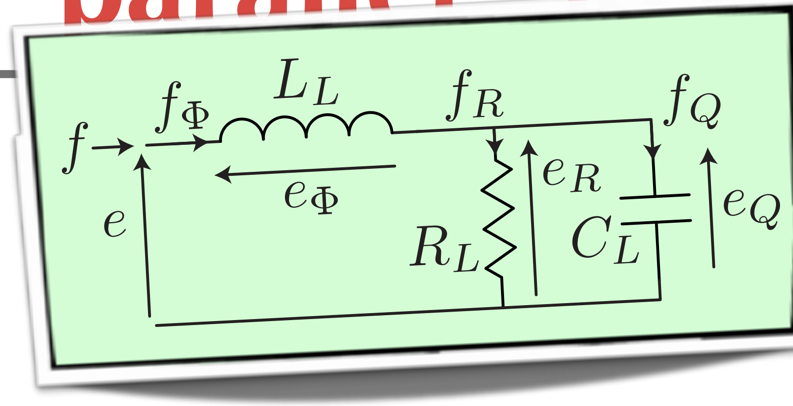
- *Asymptotic stability* with

$$H_C(\xi) = \frac{1}{2C_a}\xi^2 - e^\star\left(1 + \frac{C}{C_a}\right)\xi + \kappa$$

Same result with **energy-balancing**

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}t}(H + H_C) = -R\left(\frac{x_\Phi}{L}\right)^2 \leq 0}$$

Example: “parallel” RLC circuit



- Port-Hamiltonian model

$$\dot{x}_Q = -\frac{1}{R}\frac{\partial H}{\partial x_Q} + \frac{\partial H}{\partial x_\Phi}$$

$$\dot{x}_\Phi = -\frac{\partial H}{\partial x_Q} + u$$

$$H(x_Q, x_\Phi) = \frac{1}{2}\left(\frac{x_Q^2}{C} + \frac{x_\Phi^2}{L}\right)$$

- Desired equilibrium configuration

$$\left(\frac{x_Q^\star}{C}, \frac{x_\Phi^\star}{L}\right) = \left(e^\star, \frac{e^\star}{R}\right)$$

- From the matching equation we have that

$$-\frac{1}{R}\frac{\partial H_a}{\partial x_Q}(x_Q, x_\Phi) + \frac{\partial H_a}{\partial x_\Phi}(x_Q, x_\Phi) = 0$$

$$-\frac{\partial H_a}{\partial x_Q}(x_Q, x_\Phi) = \beta(x_Q, x_\Phi)$$

$$G(x)\beta(x) = \left[J(x) - R(x)\right]\frac{\partial H_a}{\partial x}(x)$$

$$H_a(x_Q, x_\Phi) = F(\underbrace{x_\Phi + Rx_Q}_{=\xi})$$

$$H_a(\xi) = \frac{K}{2}\left(\xi - \xi^\star\right)^2 - Re^\star\xi$$

Some stability tools

- For *linear, distributed, port-Hamiltonian systems,* it is convenient to rely on the *semi-group theory* for proving
  - Existence of solution, also in case of boundary control action
  - (Exponential) stability in closed-loop
- On the other hand, the strict connection between port-Hamiltonian systems and *"physical intuition,"* suggests that it could be convenient to rely on physical considerations for
  - The *synthesis* of the control law
  - (Asymptotic) stability *analysis*
- Here, we give some of tools for
  - Determining when the Hamiltonian has an *isolated minimum*
  - Studying the *steady state* behaviour of the state trajectories, i.e. an extension of *La Salle's invariance principle*

Some stability tools

- The idea behind the stability of distributed parameter systems remains the *same* of the finite dimensional case, but the *positive definiteness of the second differential of the Hamiltonian* is ***not,*** in general, sufficient to guarantee asymptotic stability
  - When dealing with distributed parameter systems, *it is necessary to specify the norm* associated with the stability argument
- **Definition.** Denote by x^* an equilibrium configuration. Then, x^* is said to be *stable in the sense of Lyapunov with respect to the norm* $\|\ \|$ if, for every $\varepsilon > 0$ there exists $\delta_\varepsilon > 0$ such that

$$\|x(0) - x^\star\| < \delta_\epsilon \quad \Rightarrow \quad \|x(t) - x^\star\| < \epsilon$$

- We refer to *Arnold's first and second stability theorems* for linear and nonlinear infinite dimensional systems
  - Here, we speak about *Arnold's first nonlinear stability theorem*
  - *A constructive procedure is now illustrated*

Some stability tools

- Denote by H a candidate Lyapunov function, i.e. the Hamiltonian
- Show that the equilibrium is an *extremum of the Hamiltonian,* i.e.

$$\nabla H(x^\star) = 0$$

check that b.c. are compatible

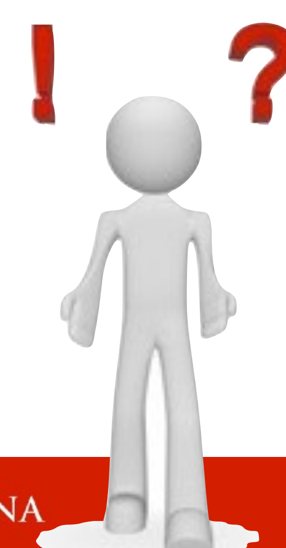
- Introduce the nonlinear functional

$$\mathcal{N}(\Delta x) = H(x^\star + \Delta x) - H(x^\star) \qquad \mathcal{N}(\Delta x) \approx \frac{1}{2}\Delta x^{\mathrm{T}} \nabla^2 H(x^\star) \Delta x$$

- Verify if the functional satisfies the following *convexity condition with respect to a suitable norm,* in order to assure its positive definiteness:

$$C_1 \left\| \Delta x \right\|^2 \leq \mathcal{N}(\Delta x) \leq C_2 \left\| \Delta x \right\|^\alpha \qquad \alpha, C_1, C_2 > 0$$

- If it is the case, the configuration is *stable* in the sense of the previous definition...
- ...but can we do something for proving *asymptotic stability??*



Some stability tools

- To prove asymptotic stability, La Salle's invariance principle could be "too much," so it is better to rely on other methods, e.g. the *energy-multipliers*
- Just take $\alpha = 2$, and assume that $x^* = 0$ with $H(x^*) = 0$

$$C_1 \|x\|^2 \leq \mathcal{N}(x) \leq C_2 \|x\|^2$$

- Suppose that there exists a function ρ such that

$$|\rho(x)| \leq C_\rho \|x\|^2$$

for some $C_\rho > 0$, and a constant $\varepsilon > 0$, supposed "small," such that

$$V(x) = H(x) + \varepsilon\rho(x)$$

satisfies

$$\frac{\mathrm{d}V}{\mathrm{d}t}(x(t)) \leq -C_\varepsilon \|x(t)\|^2$$

for some $C_\varepsilon > 0$, then *x^* is an asymptotically stable equilibrium*

Some stability tools

- In fact, we have that

$$(C_1 - \varepsilon C_\rho)\,\|x\|^2 \le V(x) \le (C_2 + \varepsilon C_\rho)\,\|x\|^2$$

$$\frac{\mathrm{d}V}{\mathrm{d}t}(x(t)) \le -\frac{C_\varepsilon}{C_2 + \varepsilon C_\rho} V(x(t))$$

$$V(x(t)) \to 0$$

$$\|x(t)\| \to 0$$

$$\frac{\mathrm{d}V}{\mathrm{d}t}(x(t)) \le -C_\varepsilon \|x(t)\|^2$$

- Approaches similar to this can be often found in literature
  - Z. H. Luo, B. Z. Guo, O. Morgul, Stability and stabilization of infinite dimensional systems with applications, Springer–Verlag, London, 1999
- Another way is to rely on *La Salle's arguments*

Some stability tools

- *La Salle's invariance principle in brief:* if in a domain about the equilibrium we can find a Lyapunov function whose derivative along system trajectories is negative semidefinite, and no trajectory can stay identically at point where it is zero except at the equilibrium, then this configuration is asymptotically stable
- For a distributed parameter system, with state space *X*, consider the following operator

$$\Phi(t) : X \to X, \quad x(t) = \Phi(t)x(0)$$

  - Φ(t) is a family of bounded and continuous operators which is called *C_0-semi-group on X*
  - The operator Φ gives the *solutions of the associated PDE* once initial and boundary conditions are specified
- Denote the *set of all orbits passing through x* by

$$\gamma(x) = \bigcup_{t \geq 0} \Phi(t)x$$

Some stability tools

- Then, define the (possibly empty) *ω-limit set of x* as

$$\omega(x) = \left\{ \bar{x} \in X \,|\, \bar{x} = \lim_{n\to\infty} \Phi(t_n)x, \text{ with } t_n \to \infty \text{ as } n \to \infty \right\}$$

  - *ω(x) is always positively invariant,* i.e.

$$\Phi(t)\omega(x) \subseteq \omega(x)$$

  - and closed

A set V is pre-compact (or, relatively compact) if ***its closure is compact***

- **Theorem.** If $x \in X$ and *γ(x) is pre-compact,* then **ω**(x) is nonempty, compact, and connected. Moreover,

$$\lim_{t\to\infty} d\left(\Phi(t)x, \omega(x)\right) = 0, \qquad d(\bar{x}, \Omega) = \inf_{\omega\in\Omega} \|\bar{x} - \omega\|$$

distance

- **La Salle's invariance principle.** Denote by H a continuous Lyapunov function, and by $\mathcal{B}$ the largest invariant subset of

$$\left\{ x \in X \,|\, \dot{H}(x) = 0 \right\}, \text{ i.e. } \Phi(t)\mathcal{B} = \mathcal{B}, \, \forall t \geq 0$$

If γ(x) is pre-compact, then

$$\lim_{t\to\infty} d\left(\Phi(t)x, \mathcal{B}\right) = 0$$

A class of distributed port-Hamiltonian systems

- Linear PDE, with one dimensional domain

$$\frac{\partial x}{\partial t}(t,z) = P_1 \frac{\partial}{\partial z}\big(\mathcal{L}(z)x(t,z)\big) + (P_0 - G_0)\mathcal{L}(z)x(t,z)$$

$X = L_2(a,b;\mathbb{R}^n)$

$H(x(t)) = \frac{1}{2}\|x(t)\|^2_{\mathcal{L}}$

- *Boundary port:*

$$\begin{pmatrix} f_\partial \\ e_\partial \end{pmatrix} = \underbrace{\frac{1}{\sqrt{2}}\begin{pmatrix} P_1 & -P_1 \\ I & I \end{pmatrix}}_{=R} \begin{pmatrix} (\mathcal{L}x)(b) \\ (\mathcal{L}x)(a) \end{pmatrix}$$

$$\dot{H}(x(t)) \leq e_\partial^{\mathrm{T}}(t) f_\partial(t)$$

$W\Sigma W^{\mathrm{T}} = \tilde{W}\Sigma\tilde{W}^{\mathrm{T}} = 0$

$W\Sigma\tilde{W}^{\mathrm{T}} = I$

- *Inputs and outputs:*

$$u(t) = W\begin{pmatrix} f_\partial(t) \\ e_\partial(t) \end{pmatrix} \qquad y(t) = \tilde{W}\begin{pmatrix} f_\partial(t) \\ e_\partial(t) \end{pmatrix}$$

$$\frac{\mathrm{d}H}{\mathrm{d}t}(x(t)) \leq y^{\mathrm{T}}(t)u(t)$$

- Important properties:
  - This system with input *u* is a *boundary control system*
  - Existence of a *contraction semigroup on X* when *u(t) = 0*

$$\bar{\mathcal{J}}x = P_1 \frac{\partial}{\partial z}\mathcal{L}x + (P_0 - G_0)\mathcal{L}x$$

$$D(\bar{\mathcal{J}}) = \left\{\mathcal{L}x \in H^1(a,b;\mathbb{R}^n) \mid u(t) = 0\right\}$$

A class of distributed port-Hamiltonian systems

- For this class, let us study what happens when a *finite dimensional, linear port-Hamiltonian system* is interconnected to the boundary

$$\begin{cases} \dot{x}_C = (J_C - R_C)\, Q_C x_C + (G_C - P_C)\, u_C \\ y_C = (G_C + P_C)^{\mathrm{T}} Q_C x_C + (M_C + S_C)\, u_C \end{cases}$$

$$\begin{pmatrix} R_C & P_C \\ P_C^{\mathrm{T}} & S_C \end{pmatrix} \geq 0$$

$$Q_C = Q_C^{\mathrm{T}} > 0$$

$$(A_C,\, B_C,\, C_C,\, D_C)$$

$$A_C = (J_C - R_C)\, Q_C \qquad B_C = G_C - P_C$$
$$C_C = (G_C + P_C)^{\mathrm{T}} Q_C \qquad D_C = M_C + S_C$$

- *Boundary interconnection:*

$$\begin{pmatrix} u \\ y \end{pmatrix} = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix} \begin{pmatrix} u_C \\ y_C \end{pmatrix} + \begin{pmatrix} u' \\ 0 \end{pmatrix}$$

$$H_{cl}(t) = \frac{1}{2} \|x(t)\|^2_{\mathcal{L}} + \underbrace{\frac{1}{2} x_C^{\mathrm{T}}(t) Q_C x_C(t)}_{=: H_C(t)}$$

- In compact form:

$$u = \mathcal{B} x$$

$$\dot{\xi} = \mathcal{J}_{cl} \xi$$
$$u' = (\mathcal{B} + D_C \mathcal{C} \;\; C_C)\, \xi =: \mathcal{B}' \xi$$

$$\xi = \begin{pmatrix} x \\ x_C \end{pmatrix} \in \Xi := \begin{pmatrix} X \\ \mathbb{R}^{n_C} \end{pmatrix}$$

$$\mathcal{J}_{cl} \xi := \begin{pmatrix} \mathcal{J} & 0 \\ B_C \mathcal{C} & A_C \end{pmatrix} \begin{pmatrix} x \\ x_C \end{pmatrix}$$

$$D(\mathcal{J}_{cl}) = D(\mathcal{J}) \oplus \mathbb{R}^{n_C}$$

The extended system is a *boundary control system*

IMPORTANT

(Boundary) Energy-Casimir control

- The control by *energy shaping* relies on the choice of the controller Hamiltonian to *shape the closed-loop energy to introduce a minimum* in the desired equilibrium configuration
  - Such minimum is reached if “sufficient” dissipation has been introduced
  - A common way is to relate the state variable of the controller to the state variable of the plant by means of *a set of Casimir functions*
- **Definition.** Consider the (autonomous) port-Hamiltonian system introduced before. A *Casimir function* is a function C that is constant along the solutions for every possible choice of $\mathcal{L}$ and H_C

$$
\begin{aligned}
C(x(t), x_C(t)) &= \Gamma^{\mathrm{T}} x_C(t) + \langle \Psi \mid x(t) \rangle \\
&= \Gamma^{\mathrm{T}} x_C(t) + \int_a^b \Psi^{\mathrm{T}}(z) x(t,z) \mathrm{d}z
\end{aligned}
$$

We establish a *constant algebraic relation* between state of the plant and of the controller

(Boundary) Energy-Casimir control

- The *characterisation* of the possible Casimir function is given by

$$P_1 \frac{\partial}{\partial z}\Psi(z) + (P_0 + G_0)\Psi(z) = 0$$

$R_C = P_C = M_C = S_C = 0$

constraint $\quad J_C \Gamma + G_C \tilde{W} R \begin{pmatrix} \Psi(b) \\ \Psi(a) \end{pmatrix} = 0$

control action $\quad G_C^{\mathrm{T}} \Gamma + W R \begin{pmatrix} \Psi(b) \\ \Psi(a) \end{pmatrix} = 0$

$$P_1 \frac{\partial}{\partial z}\Psi(z) + P_0 \Phi(z) = 0 \qquad\qquad G_0 \Psi(z) = 0$$

dissipation obstacle
(again…)

- A "practical" choice is to have

$$x_{C,i}(t) = \langle \Psi_i \mid x(t) \rangle = \int_a^b \Psi_i^{\mathrm{T}}(z) x(t,z) \mathrm{d}z + \kappa_i$$

These results are valid also when the plant is described by a *PDE plus ODE*

(Boundary) Energy-Casimir control

- A simple proof (with $G_0 = 0$)

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} C &= \Gamma^{\mathrm{T}} \dot{x}_C + \int_a^b \Psi^{\mathrm{T}} \frac{\partial x}{\partial t} \mathrm{d}z \\
&= \Gamma^{\mathrm{T}} \left(J_C \frac{\partial H_C}{\partial x_C} + G_C u_C \right) + \\
&\quad + \int_a^b \Psi^{\mathrm{T}} \left(P_1 \frac{\partial}{\partial z} (\mathcal{L} x) + P_0 \mathcal{L} x \right) \mathrm{d}z
\end{aligned}
$$

$$
\begin{pmatrix} f_\partial \\ e_\partial \end{pmatrix} = \Sigma \left(W^{\mathrm{T}} \gamma_1 + \tilde{W}^{\mathrm{T}} \gamma_2 \right)
$$

$$
u_C = y = \tilde{W} \begin{pmatrix} f_\partial \\ e_\partial \end{pmatrix} = \gamma_1
$$

$$
G_C^{\mathrm{T}} \frac{\partial H_C}{\partial x_C} = y_C = -u = -W \begin{pmatrix} f_\partial \\ e_\partial \end{pmatrix} = -\gamma_2
$$

$$
\begin{pmatrix} f_\partial \\ e_\partial \end{pmatrix} = \Sigma \left(W^{\mathrm{T}} \gamma_1 - \tilde{W}^{\mathrm{T}} G_C^{\mathrm{T}} \frac{\partial H_C}{\partial x_C} \right)
$$

$$
\begin{aligned}
\Psi^{\mathrm{T}} \left(P_1 \frac{\partial}{\partial z} (\mathcal{L} x) + P_0 \mathcal{L} x \right) &= \\
&= \frac{\partial}{\partial z} \left((\mathcal{L} x)^{\mathrm{T}} P_1 \Psi \right) - (\mathcal{L} x)^{\mathrm{T}} \left(P_1 \frac{\partial}{\partial z} \Psi + P_0 \Psi \right)
\end{aligned}
$$

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} C &= - \frac{\partial^{\mathrm{T}} H_C}{\partial x_C} J_C \Gamma + \gamma_1^{\mathrm{T}} G_C^{\mathrm{T}} \Gamma \\
&\quad - \int_a^b (\mathcal{L} x)^{\mathrm{T}} \left(P_1 \frac{\partial}{\partial z} \Psi + P_0 \Psi \right) \mathrm{d}z \\
&\quad + \begin{pmatrix} (\mathcal{L} x)(b) \\ (\mathcal{L} x)(a) \end{pmatrix}^{\mathrm{T}} \begin{pmatrix} P_1 & 0 \\ 0 & -P_1 \end{pmatrix} \begin{pmatrix} \Psi(b) \\ \Psi(a) \end{pmatrix}
\end{aligned}
$$

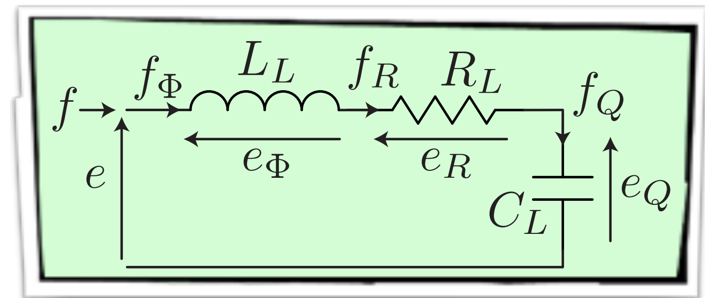
$$
= \left(\gamma_1^{\mathrm{T}} W - \frac{\partial^{\mathrm{T}} H_C}{\partial x_C} G_C \tilde{W} \right) R \begin{pmatrix} \Psi(b) \\ \Psi(a) \end{pmatrix}
$$

Example: transmission line + sRLC

- The *distributed port-Hamiltonian* formulation of the line is

$$\begin{cases} \dfrac{\partial x_q}{\partial t}(t,z) = -\dfrac{\partial}{\partial z}\dfrac{\delta H_{\mathrm{TL}}}{\delta x_\phi}(t,z) \\ \dfrac{\partial x_\phi}{\partial t}(t,z) = -\dfrac{\partial}{\partial z}\dfrac{\delta H_{\mathrm{TL}}}{\delta x_q}(t,z) \end{cases}$$

$$H_{\mathrm{TL}}(x_q, x_\phi) = \frac{1}{2}\int_0^\ell \left(\frac{x_q^2}{C} + \frac{x_\phi^2}{L}\right) \mathrm{d}z$$



- *RLC load* in series configuration:

$$\dot{x}_Q = \frac{\partial H_L}{\partial x_\Phi}$$

$$\dot{x}_\Phi = -\frac{\partial H_L}{\partial x_Q} - R_L\frac{\partial H_L}{\partial x_\Phi} + V$$

$$I = \frac{\partial H_L}{\partial x_\Phi}$$

$$H_{\mathrm{L}}(x_Q, x_\Phi) = \frac{1}{2}\left(\frac{x_Q^2}{C_L} + \frac{x_\Phi^2}{L_L}\right)$$

Desired equilibrium

$$\left(\frac{x_q^\star}{C}, \frac{x_\phi^\star}{L}, \frac{x_Q^\star}{C_L}, \frac{x_\Phi^\star}{L_L}\right) = (e^\star, 0, e^\star, 0)\,, \quad e^\star \in \mathbb{R}$$

- Boundary interconnection:

$$(I_0, V_0) = \left(\frac{\delta\mathcal{H}}{\delta x_\phi}(0),\ \frac{\delta\mathcal{H}}{\delta x_q}(0)\right)$$

$$(I_\ell, V_\ell) = \left(-\frac{\delta\mathcal{H}}{\delta x_\phi}(\ell),\ \frac{\delta\mathcal{H}}{\delta x_q}(\ell)\right)$$

$$(I_\ell, V_\ell) = (-I, V)$$

Example: transmission line + sRLC

- Let us fix the *port-Hamiltonian controller* as follows

$$\dot{\xi}(t) = u_C(t) = I_0(t)$$

$$y_C(t) = \frac{\partial H_C}{\partial \xi}(\xi(t)) = -V_0(t)$$

- It is possible to prove that *one Casimir function* is present

$$C(\xi, x_Q, \cancel{x_\Phi}, x_q, \cancel{x_\phi}) = \xi - x_Q - \int_0^\ell x_q \, \mathrm{d}z$$

$$H_C(\xi) \equiv H_C\left(x_Q + \int_0^\ell x_q \, \mathrm{d}z\right)$$

- *Stability* can be easily achieved by selecting

$$H_C(\xi) = \frac{K_C}{2}\left(\xi - \xi^\star\right)^2 - e^\star \left[1 + K_C\left(C + C_L\right)\right]\xi$$

control action

$$V_0(t) = -K_C\left(\xi(t) - \xi^\star\right) + \left[1 + K_C\left(C + C_L\right)\right]e^\star$$

Physical interpretation???

Example: transmission line + sRLC

$$u = y_C + u'$$

- *Asymptotic stability* follows from La Salle's arguments
  - We *add damping* through the further input *u′*

$$u'(t) = -K_D y'(t), \qquad K_D > 0$$

$$H_d = H_{\mathrm{L}} + H_{\mathrm{TL}} + H_C$$

  - Coordinate change (move to 0 the equilibrium)

$$\tilde{x}_q = x_q - x_q^\star \qquad \tilde{x}_\phi = x_\phi - \cancel{x_\phi^\star}$$

$$\tilde{x}_Q = x_Q - x_Q^\star \qquad \tilde{x}_\Phi = x_\Phi - \cancel{x_\Phi^\star}$$

$$\tilde{\xi} = \tilde{x}_Q + \int_0^\ell \tilde{x}_q \, \mathrm{d}z$$

$$u'(t) = \frac{\delta \tilde{H}_d}{\delta \tilde{x}_q}(t,0) = \frac{\tilde{x}_q(t,0)}{C} + K_C \tilde{\xi}(t)$$

$$y'(t) = \frac{\delta \tilde{H}_d}{\delta \tilde{x}_\phi}(t,0) = \frac{\tilde{x}_\phi(t,0)}{L}$$

$$\frac{\mathrm{d}\tilde{H}_d}{\mathrm{d}t} = -R_L \left(\frac{\tilde{x}_\Phi}{L_L}\right)^2 - K_D \left(\frac{\tilde{x}_\phi(0)}{L}\right)^2$$

energy balancing

In steady state we have that *x_Q, x_Φ and ξ are constant,* with *H_d constant,* which implies that the *energy stored in the line is constant,* and the line as constant applied voltages and currents

The only *invariant solution* is with constant voltage and current in the line

Boundary d-pH systems and damping injection

- What happens when *boundary control by damping injection* is applied to a boundary control system in port-Hamiltonian form?
  - See e.g. the previous situation with transmission line and sRLC
  - What about the *existence of solutions,* i.e. associated C_0-semigroup?
- Given the class of linear, distributed port-Hamiltonian systems, a contraction C_0-semigroup is generated when

$$\bar{u}(t) = \bar{W} \begin{pmatrix} f_\partial(t) \\ e_\partial(t) \end{pmatrix} = 0, \qquad \bar{W}\Sigma\bar{W}^{\mathrm{T}} \geq 0$$

- If the system is in *impedance form,* we have that

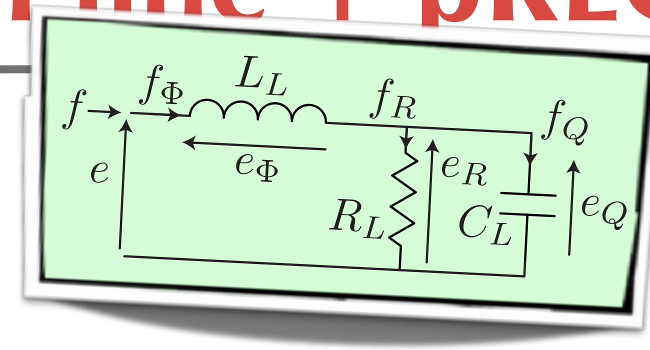
$$W\Sigma W^{\mathrm{T}} = \tilde{W}\Sigma\tilde{W}^{\mathrm{T}} = 0$$
$$W\Sigma\tilde{W}^{\mathrm{T}} = I$$

$$u(t) = W \begin{pmatrix} f_\partial(t) \\ e_\partial(t) \end{pmatrix} \qquad y(t) = \tilde{W} \begin{pmatrix} f_\partial(t) \\ e_\partial(t) \end{pmatrix}$$

- Let's introduce damping

$$0 = \bar{u} = u + K_D y = \underbrace{(W + K_D\tilde{W})}_{=\bar{W}} \begin{pmatrix} f_\partial \\ e_\partial \end{pmatrix} \qquad \bar{W}\Sigma\bar{W}^{\mathrm{T}} > 0$$

Example: transmission line + pRLC



- *RLC load* in parallel configuration:

$$\dot{x}_Q = -\frac{1}{R_L}\frac{\partial H_L}{\partial x_Q} + \frac{\partial H_L}{\partial x_\Phi}$$

$$\dot{x}_\Phi = -\frac{\partial H_L}{\partial x_Q} + V$$

$$I = \frac{\partial H_L}{\partial x_\Phi}$$

Desired equilibrium

$$\frac{x_Q^\star}{C_L} = \frac{x_q^{i,\star}}{C_i} = e^\star \qquad \frac{x_\Phi^\star}{L_L} = \frac{x_\phi^{i,\star}}{L_i} = \frac{e^\star}{R_L}$$

- It is easy to check that there are *no Casimir functions* in closed-loop
  - This is coherent with the fact that the *supplied power at the equilibrium* must be in general different from 0
  - *The energy-Casimir method fails,* as all the possible energy-balancing controller

(Boundary) Energy-shaping control

- Let us consider the next PDE, with the "usual" choice of *u* and *y*

$$\frac{\partial x}{\partial t}(t,z) = P_1 \frac{\partial}{\partial z}\left(\mathcal{L}(z)x(t,z)\right) + (P_0 - \cancel{G_0})\mathcal{L}(z)x(t,z)$$

- Is the classical *energy-balance control* applicable?

$$u(t) = \beta(x(t)) + u'(t) \qquad -\int_0^t y^{\mathrm{T}}(\tau)\beta(x(\tau))\,\mathrm{d}\tau = H_a(x(t)) + \kappa$$

$$H_d = H + H_a$$

$$-\beta^{\mathrm{T}}(x(t))y(t) = \int_a^b \left(\frac{\delta H_a}{\delta x}(x(t,z))\right)^{\mathrm{T}} \frac{\partial x}{\partial t}(t,z)\,\mathrm{d}z$$

$$-\beta^{\mathrm{T}}\tilde{W}R\begin{pmatrix}(\mathcal{L}x)(b)\\(\mathcal{L}x)(a)\end{pmatrix} = \int_a^b \left(\frac{\delta H_a}{\delta x}\right)^{\mathrm{T}}\left(P_1\frac{\partial}{\partial z}(\mathcal{L}x) + P_0(\mathcal{L}x)\right)\mathrm{d}z$$

$$\cancel{\begin{pmatrix}(\mathcal{L}x)(b)\\(\mathcal{L}x)(a)\end{pmatrix}^{\mathrm{T}}}\left[R^{\mathrm{T}}\tilde{W}^{\mathrm{T}}\beta + \begin{pmatrix}P_1 & 0\\0 & -P_1\end{pmatrix}\begin{pmatrix}\frac{\delta H_a}{\delta x}(b)\\\frac{\delta H_a}{\delta x}(a)\end{pmatrix}\right] = \int_a^b \cancel{(\mathcal{L}x)^{\mathrm{T}}}\left[P_1\frac{\partial}{\partial z}\left(\frac{\delta H_a}{\delta x}\right) + P_0\left(\frac{\delta H_a}{\delta x}\right)\right]\mathrm{d}z$$

(Boundary) Energy-shaping control

- The energy-balance controller exists if

$$P_1 \frac{\partial}{\partial z}\left(\frac{\delta H_a}{\delta x}\right) + P_0\left(\frac{\delta H_a}{\delta x}\right) = 0$$

$$\begin{pmatrix} P_1 & 0 \\ 0 & -P_1 \end{pmatrix} = R^{\mathrm{T}}\Sigma R$$

$$R^{\mathrm{T}}\tilde{W}^{\mathrm{T}}\beta + \begin{pmatrix} P_1 & 0 \\ 0 & -P_1 \end{pmatrix}\begin{pmatrix} \frac{\delta H_a}{\delta x}(b) \\ \frac{\delta H_a}{\delta x}(a) \end{pmatrix} = 0$$

$$\beta = -WR\begin{pmatrix} \frac{\delta H_a}{\delta x}(b) \\ \frac{\delta H_a}{\delta x}(a) \end{pmatrix}$$

control action

$$0 = \tilde{W}R\begin{pmatrix} \frac{\delta H_a}{\delta x}(b) \\ \frac{\delta H_a}{\delta x}(a) \end{pmatrix}$$

boundary condition

- Now it is necessary to analyse *what happens in closed-loop*
  - Existence of solution
  - Energy-shaping
  - Stability analysis
- Note that *the boundary action is state dependent*

Energy-shaping vs. energy-Casimir

- The idea is to see under which conditions the *energy-balance* control law can be obtained from the *energy-Casimir* method
  - All the results about existence of solutions, stability, and so on can be used

$$P_1 \frac{\partial}{\partial z}\Psi(z) + P_0 \Psi(z) = 0$$

$$J_C = 0 \qquad G_C = I$$

$$J_C \Gamma + G_C \tilde{W} R \begin{pmatrix} \Psi(b) \\ \Psi(a) \end{pmatrix} = 0$$

$$G_C^{\mathrm{T}} \Gamma + W R \begin{pmatrix} \Psi(b) \\ \Psi(a) \end{pmatrix} = 0$$

$$C(x(t), x_C(t)) = \Gamma^{\mathrm{T}} x_C(t) + \langle \Psi \mid x(t) \rangle = \Gamma^{\mathrm{T}} x_C(t) + \int_a^b \Psi^{\mathrm{T}}(z) x(t,z) \mathrm{d}z$$

$$H_C(x_C(t)) \equiv H_C\left(\int_a^b \hat{\Psi}^{\mathrm{T}}(z) x(t,z) \mathrm{d}z \right)$$

$$y_C = \frac{\partial H_C}{\partial x_C} = -WR \begin{pmatrix} \frac{\delta H_C}{\delta x}(b) \\ \frac{\delta H_C}{\delta x}(a) \end{pmatrix}$$

control action

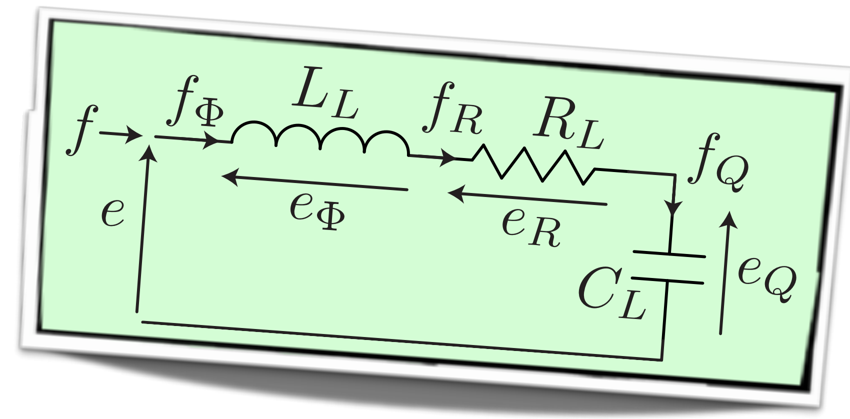
The relation is much clearer if analysed from a *"geometric" point of view,* as in finite dimensions

$$P_1 \frac{\partial}{\partial z}\left(\frac{\delta H_C}{\delta x}\right) + P_0 \left(\frac{\delta H_C}{\delta x}\right) = 0 \quad + \quad \tilde{W} R \begin{pmatrix} \frac{\delta H_C}{\delta x}(b) \\ \frac{\delta H_C}{\delta x}(a) \end{pmatrix} = 0$$

admissible energy-shaping

Example: transmission line + sRLC (2)

- The stabilisation can be alternatively performed by relying directly on *energy-balancing* considerations



  - The class of functions H_a is given by

$$H_a(\xi), \quad \xi = x_Q + \int_0^\ell x_q \,\mathrm{d}z$$

  - The control action is then given by

$$\beta(x_q, x_Q) = -\frac{\partial H_a}{\partial \xi}(\xi(x_q, x_Q))$$

- Clearly, the result is the same, provided that $\boldsymbol{H_C = H_a}$

$$H_a(\xi) = \frac{K_C}{2}(\xi - \xi^\star)^2 - e^\star[1 + K_C(C_L + C)]\xi$$

State-modulated boundary control

- Energy-Casimir method fails because we look for H_a (or H_C) functions that are *independent from the Hamiltonian and from the dissipative structure*
  - If you restrict the dynamics of the closed-loop system on the invariant, in finite dimensions it is immediate to see that the feedback action maps the open-loop system into a "desired" one

$$\dot{x} = (J - R)\frac{\partial H}{\partial x} + Gu = (J - R)\frac{\partial H}{\partial x} - GG_C^{\mathrm{T}}\frac{\partial H_C}{\partial x_C}$$

$$R\frac{\partial \Psi}{\partial x} = 0$$

$$\frac{\partial^{\mathrm{T}} \Psi}{\partial x} J = G_C G^{\mathrm{T}}$$

$$\dot{x} = (J - R)\frac{\partial}{\partial x}(H + H_C)$$

  - Is it the same in the *infinite dimensional* scenario????

- Why don't we try to do the same for a *specific system dynamics,* as in the *control with state-modulated source,* here applied to the *boundary* of the domain????

State-modulated boundary control

- *Open-loop system* (H is not necessarily quadratic)

$$\frac{\partial x}{\partial t} = P_1 \frac{\partial}{\partial z}\frac{\delta H}{\delta x} + (P_0 - G_0)\frac{\delta H}{\delta x} \qquad u = WR\begin{pmatrix} \frac{\delta H}{\delta x}(b) \\ \frac{\delta H}{\delta x}(a) \end{pmatrix}$$

- *Target (or desired) dynamics*

$$\frac{\partial x}{\partial t} = P_1 \frac{\partial}{\partial z}\frac{\delta H_d}{\delta x} + (P_0 - G_0)\frac{\delta H_d}{\delta x} \qquad u' = WR\begin{pmatrix} \frac{\delta H_d}{\delta x}(b) \\ \frac{\delta H_d}{\delta x}(a) \end{pmatrix}$$

- *Matching conditions*

$$u = \beta(x) + u' \qquad H_d = H + H_a$$

same state evolution!!

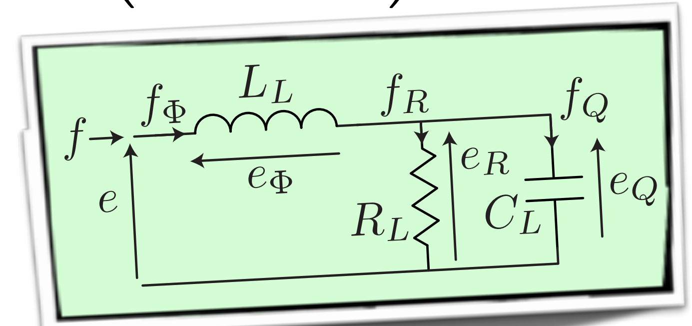
$$P_1 \frac{\partial}{\partial z}\frac{\delta H_a}{\delta x} + (P_0 - G_0)\frac{\delta H_a}{\delta x} = 0 \qquad \beta + WR\begin{pmatrix} \frac{\delta H_a}{\delta x}(b) \\ \frac{\delta H_a}{\delta x}(a) \end{pmatrix} = 0$$

An *energy-balancing controller* satisfies these conditions, but the inverse implication is not true

Example: transmission line + pRLC

- The *distributed port-Hamiltonian* formulation of the line is

$$\begin{cases} \dfrac{\partial x_q}{\partial t}(t,z) = -\dfrac{\partial}{\partial z}\dfrac{\delta H_{\mathrm{TL}}}{\delta x_\phi}(t,z) \\[2ex] \dfrac{\partial x_\phi}{\partial t}(t,z) = -\dfrac{\partial}{\partial z}\dfrac{\delta H_{\mathrm{TL}}}{\delta x_q}(t,z) \end{cases} \quad + \quad H_{\mathrm{TL}}(x_q, x_\phi) = \frac{1}{2}\int_0^\ell \left(\frac{x_q^2}{C} + \frac{x_\phi^2}{L} \right) \mathrm{d}z$$



- *RLC load* in parallel configuration:

$$\dot{x}_Q = -\frac{1}{R_L}\frac{\partial H_L}{\partial x_Q} + \frac{\partial H_L}{\partial x_\Phi} \qquad H_{\mathrm{L}}(x_Q, x_\Phi) = \frac{1}{2}\left(\frac{x_Q^2}{C_L} + \frac{x_\Phi^2}{L_L} \right)$$

$$\dot{x}_\Phi = -\frac{\partial H_L}{\partial x_Q} + V \quad +$$

$$I = \frac{\partial H_L}{\partial x_\Phi}$$

Desired equilibrium

$$\left(\frac{x_q^\star}{C}, \frac{x_\phi^\star}{L}, \frac{x_Q^\star}{C_L}, \frac{x_\Phi^\star}{L_L} \right) = \left(e^\star, \frac{e^\star}{R_L}, e^\star, \frac{e^\star}{R_L} \right)$$

- Boundary interconnection

$$(I_0, V_0) = \left(\frac{\delta \mathcal{H}}{\delta x_\phi}(0), \frac{\delta \mathcal{H}}{\delta x_q}(0) \right)$$

$$(I_\ell, V_\ell) = \left(-\frac{\delta \mathcal{H}}{\delta x_\phi}(\ell), \frac{\delta \mathcal{H}}{\delta x_q}(\ell) \right) \quad + \quad (I_\ell, V_\ell) = (-I, V)$$

Example: transmission line + pRLC

- Complete model of the plant:

$$\begin{pmatrix} \frac{\partial x_q}{\partial t} \\ \frac{\partial x_\phi}{\partial t} \\ \dot{x}_Q \\ \dot{x}_\Phi \end{pmatrix} = \left\{ \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \frac{\partial}{\partial z} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{R_L} & 1 \\ \cdot\,|_\ell & 0 & -1 & 0 \end{pmatrix} \right\} \begin{pmatrix} \frac{\delta \bar{H}}{\delta x_q} \\ \frac{\delta \bar{H}}{\delta x_\phi} \\ \frac{\partial \bar{H}}{\partial x_Q} \\ \frac{\partial \bar{H}}{\partial x_\Phi} \end{pmatrix}$$

$$0 = -\frac{\delta \bar{H}}{\delta x_\phi}(\ell) + \frac{\partial \bar{H}}{\partial x_\Phi}$$

$$u = \frac{\delta \bar{H}}{\delta x_q}(0)$$ ***boundary control action***

$$\boxed{\bar{H} = \frac{1}{2}\left(\frac{x_Q^2}{C_L} + \frac{x_\Phi^2}{L_L}\right) + \frac{1}{2}\int_0^\ell \left(\frac{x_q^2}{C} + \frac{x_\phi^2}{L}\right) \mathrm{d}z}$$

- A similar expression is valid for the desired dynamics, but with a different Hamiltonian
  - *Open-loop and target systems have the same "interconnection structure," and resistive relation*
  - Only the Hamiltonian is changed thanks to the control action

Example: transmission line + pRLC

- We obtain that

$$0 = \frac{\partial}{\partial z}\frac{\delta H_a}{\delta x_\phi}$$

$$0 = \frac{\partial}{\partial z}\frac{\delta H_a}{\delta x_q}$$

$$0 = -\frac{1}{R_L}\frac{\partial H_a}{\partial x_Q} + \frac{\partial H_a}{\partial x_\Phi}$$

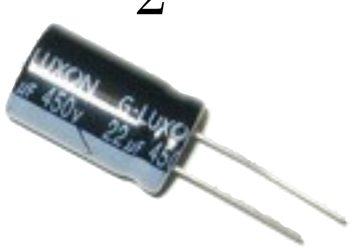
$$0 = \frac{\delta H_a}{\delta x_q}(\ell) - \frac{\partial H_a}{\partial x_Q}$$

$$0 = -\frac{\delta H_a}{\delta x_\phi}(\ell) + \frac{\partial H_a}{\partial x_\Phi}$$

$$\beta = -\frac{\delta H_a}{\delta x_q}(0)$$ ***boundary control action***

$$H_a(\xi), \quad \xi = x_\Phi + R_L x_Q + \int_0^\ell (x_\phi + R_L x_q)\,\mathrm{d}z$$

$$H_a(\xi) = \frac{1}{2}K_L(\xi - \xi^\star)^2 - \frac{e^\star}{R_L}\xi, \quad K_L > 0$$



$$\beta(x_q, x_\phi, x_Q, x_\Phi) = -R_L\frac{\partial H_a}{\partial \xi}(\xi)$$

Example: transmission line + pRLC

- Stability follows from *La Salle's arguments,* once further dissipation is introduced in *z = 0*

$$u'(t) = -K_D y'(t), \qquad K_D > 0$$

$$u'(t) = \frac{\delta \tilde{H}_d}{\delta \tilde{x}_q}(t,0) = \frac{\tilde{x}_q(t,0)}{C} + K_L R_L \tilde{\xi}(t)$$
$$y'(t) = \frac{\delta \tilde{H}_d}{\delta \tilde{x}_\phi}(t,0) = \frac{\tilde{x}_\phi(t,0)}{L} + K_L \tilde{\xi}(t)$$

- The starting point is the *energy-balance relation:*

$$\frac{\mathrm{d}\tilde{H}_d}{\mathrm{d}t} = -\frac{1}{R_L}\left(\frac{\tilde{x}_Q}{C_L} + K R_L \tilde{\xi}\right)^2 - K_D\left(\frac{\tilde{x}_\phi(0)}{L} + K_L\tilde{\xi}\right)^2$$

$$\tilde{x}_q = x_q - x_q^\star \qquad \tilde{x}_\phi = x_\phi - x_\phi^\star$$
$$\tilde{x}_Q = x_Q - x_Q^\star \qquad \tilde{x}_\Phi = x_\Phi - x_\Phi^\star$$

$$\frac{\tilde{x}_Q}{C_L} = -K R_L \tilde{\xi} = \frac{\tilde{x}_q(0)}{C} = R_L \frac{\tilde{x}_\phi(0)}{L}$$ ***steady state***

$$\frac{\mathrm{d}}{\mathrm{d}t}$$

$$\frac{\tilde{x}_Q}{C_L} = R_L \frac{\tilde{x}_\Phi}{L_L} \qquad \frac{\tilde{x}_Q}{C_L} = \frac{\tilde{x}_q(\ell)}{C} = R_L \frac{\tilde{x}_\phi(\ell)}{L} = -K R_L \tilde{\xi} = \text{const.}$$

In steady state we have that *x_Q, x_Φ and ξ are constant,* with *H_d constant,* which implies that the *energy stored in the line is constant,* and the line as constant applied voltages and currents

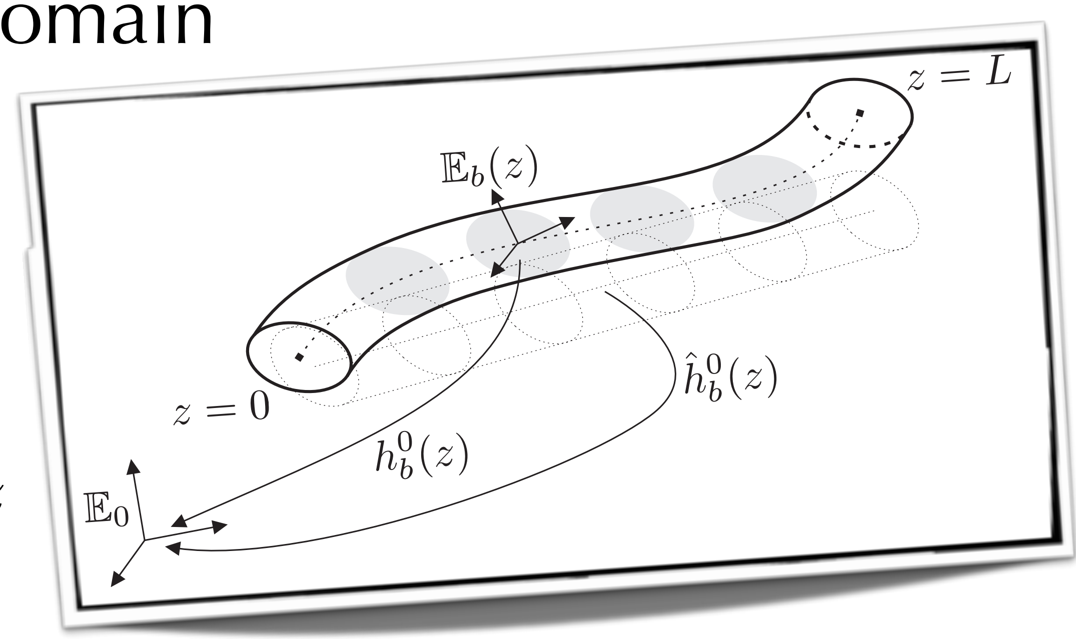
The only *invariant solution* is with constant voltage and current in the line

"Almost linear" d-pH systems

- The motivating problem has been the *stability analysis of a nonlinear flexible link,* with one-dimensional spatial domain

$$\begin{cases} \dfrac{\partial q}{\partial t} = \dfrac{\partial}{\partial z}\dfrac{\delta H}{\delta p} + \mathrm{ad}_{(q+\hat{n})}\dfrac{\delta H}{\delta p} \\ \dfrac{\partial p}{\partial t} = \dfrac{\partial}{\partial z}\dfrac{\delta H}{\delta q} - \mathrm{ad}^*_{(q+\hat{n})}\dfrac{\delta H}{\delta q} + p \wedge \dfrac{\delta H}{\delta p} \end{cases}$$

$$+ \quad H(q,p) = \frac{1}{2}\int_Z \Big(\langle p \mid p \rangle_Y + \langle q \mid q \rangle_{C^{-1}} \Big) \mathrm{d}z$$



- "Natural" boundary ports

$$(T_0(t), W_0(t)) = \left(\frac{\delta H}{\delta p}(t,0), -\frac{\delta H}{\delta q}(t,0) \right)$$

$$(T_L(t), W_L(t)) = \left(\frac{\delta H}{\delta p}(t,L), \frac{\delta H}{\delta q}(t,L) \right)$$

$$\Rightarrow \quad \frac{\mathrm{d}H}{\mathrm{d}t}(t) = \langle W_L(t), T_L(t) \rangle + \langle W_0(t), T_0(t) \rangle$$

- *The system has a nonlinearity in the algebraic skew-symmetric term*
  - *Existence of solutions* with algebraic boundary control
  - *Stability analysis*

“Almost linear” d-pH systems

- We study the following class of distributed port-Hamiltonian systems

$$\frac{\partial x}{\partial t}(t,z) = P_1 \frac{\partial}{\partial z}\big(\mathcal{L}(z)x(t,z)\big) + P_0(x,z)\mathcal{L}(z)x(t,z)$$

$$X = C^1(a,b;\mathbb{R}^n)$$

$$H(x(t,z)) = \frac{1}{2}\int_a^b x^{\mathrm{T}}(t,z)\mathcal{L}(z)x(t,z)\,\mathrm{d}z$$

- We use the same parametrisation for the definition of *u* and *y*
  - *Boundary port:*

$$\begin{pmatrix} f_\partial \\ e_\partial \end{pmatrix} = \underbrace{\frac{1}{\sqrt{2}}\begin{pmatrix} P_1 & -P_1 \\ I & I \end{pmatrix}}_{=R}\begin{pmatrix} (\mathcal{L}x)(b) \\ (\mathcal{L}x)(a) \end{pmatrix} \qquad \dot{H}(x(t)) = e_\partial^{\mathrm{T}}(t) f_\partial(t)$$

$$W\Sigma W^{\mathrm{T}} = \tilde{W}\Sigma\tilde{W}^{\mathrm{T}} = 0$$
$$W\Sigma\tilde{W}^{\mathrm{T}} = I$$

  - *Boundary input and output:*

$$u(t) = W\begin{pmatrix} f_\partial(t) \\ e_\partial(t) \end{pmatrix} \qquad y(t) = \tilde{W}\begin{pmatrix} f_\partial(t) \\ e_\partial(t) \end{pmatrix} \qquad \frac{\mathrm{d}H}{\mathrm{d}t}(x(t)) = y^{\mathrm{T}}(t)u(t)$$

- It is possible to “reveal” the *power traveling along the spatial domain,* in both the direction

"Almost linear" d-pH systems

Proposition. There always exists a coordinate change that puts the system in the following form:

terms of class C[1]

$$\frac{\partial \xi}{\partial t}(t,z) = \Lambda(z)\frac{\partial \xi}{\partial z}(t,z) + M(\xi,z)\xi(t,z)$$

diagonal

$$\Lambda(z) = \begin{pmatrix} \Lambda_-(z) & 0 \\ 0 & -\Lambda_+(z) \end{pmatrix}$$

$$\sqrt{\mathcal{L}}(z) P_1 \sqrt{\mathcal{L}}(z) = \Phi^{\mathrm{T}}(z)\Lambda(z)\Phi(z)$$

$$\xi(t,z) = \Phi(z)\sqrt{\mathcal{L}}(z)x(t,z)$$

Energy and energy balance:

scattering decomposition

$$\big(s_{+,a}(t), s_{-,a}(t)\big) = \big(\xi_+(t,a), \xi_-(t,a)\big)$$
$$\big(s_{+,b}(t), s_{-,b}(t)\big) = \big(\xi_+(t,b), \xi_-(t,b)\big)$$

$$\bar{H}(t) = \frac{1}{2}\int_a^b \xi^{\mathrm{T}}(t,z)\xi(t,z)\,\mathrm{d}z$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\bar{H} = \frac{1}{2}\Big[\xi_-^{\mathrm{T}}(b)\Lambda_-(b)\xi_-(b) - \xi_+^{\mathrm{T}}(b)\Lambda_+(b)\xi_+(b)\Big] + \\ + \frac{1}{2}\Big[\xi_+^{\mathrm{T}}(a)\Lambda_+(a)\xi_+(a) - \xi_-^{\mathrm{T}}(a)\Lambda_-(a)\xi_-(a)\Big]$$

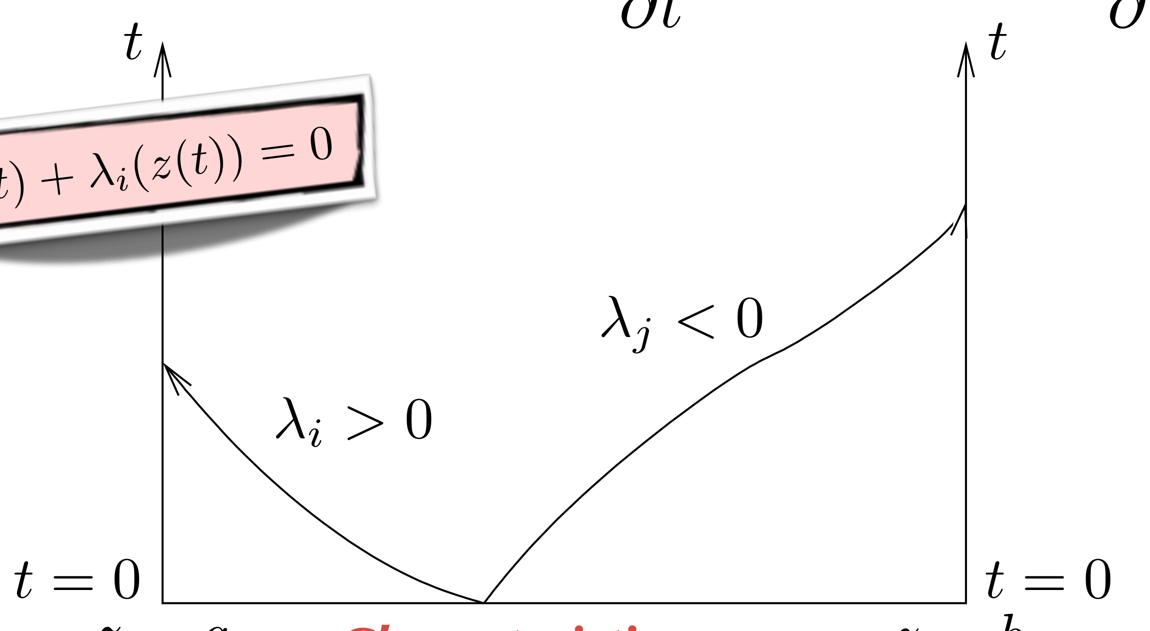
“Almost linear” d-pH systems

- In the context of *quasi-linear hyperbolic PDEs,* we have the dynamics in terms of the *Riemann invariants* or coordinates

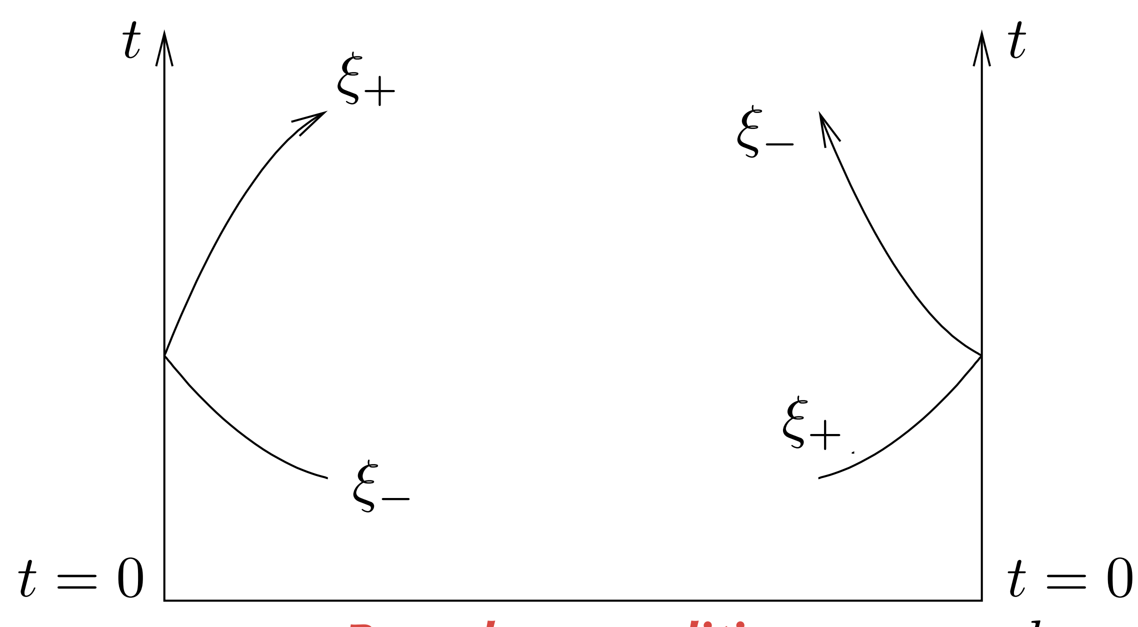
$\boxed{M(\xi, z) = 0}$

$$\frac{\partial \xi_i}{\partial t}(t,z) = \lambda_i(z)\frac{\partial \xi_i}{\partial z}(t,z), \quad i = 1, \ldots, n$$

$\boxed{\dot z(t) + \lambda_i(z(t)) = 0}$



Characteristic curves



Boundary conditions

- If P_0 and $\mathcal{L}$ do not depend on z, we have

$$\frac{\partial}{\partial t}\begin{pmatrix}\xi_-(t,z)\\ \xi_+(t,z)\end{pmatrix} = \begin{pmatrix}\Lambda_- & 0\\ 0 & -\Lambda_+\end{pmatrix}\frac{\partial}{\partial z}\begin{pmatrix}\xi_-(t,z)\\ \xi_+(t,z)\end{pmatrix} + \bar P_0(\xi_-(t,z), \xi_+(t,z))\begin{pmatrix}\xi_-(t,z)\\ \xi_+(t,z)\end{pmatrix}$$

port-Hamiltonian system

$$u_\xi(t) = \begin{pmatrix} s_{-,b}(t)\\ s_{+,a}(t)\end{pmatrix}$$

$$y_\xi(t) = \begin{pmatrix} s_{-,a}(t)\\ s_{+,b}(t)\end{pmatrix}$$

scattering variables

$\xi(t,z) = \Phi(z)\sqrt{\mathcal{L}}(z)x(t,z)$

“Almost linear” d-pH systems

- Inputs and outputs, and *algebraic boundary control*

$$\begin{pmatrix} u_\xi \\ y_\xi \end{pmatrix} = \begin{pmatrix} I & 0 & 0 & 0 \\ 0 & 0 & 0 & I \\ 0 & 0 & I & 0 \\ 0 & I & 0 & 0 \end{pmatrix} \times$$

$$\times \begin{pmatrix} \left(\Phi\sqrt{\mathcal{L}^{-1}}\right)(b) & 0 \\ 0 & \left(\Phi\sqrt{\mathcal{L}^{-1}}\right)(a) \end{pmatrix} \begin{pmatrix} (\mathcal{L}x)(b) \\ (\mathcal{L}x)(a) \end{pmatrix}$$

$$u_\xi(t) = g(y_\xi(t))$$

control action

- Some definitions…
  - *Norm on $\mathcal{R}^n$:*

$$|\xi| := \max(|\xi_i|,\ i = 1, \ldots, n)$$

  - *Norms for C^0 and C^1 functions:*

$$|f_0|_{C^0(a,b)} = \max_{z \in [a,\, b]} |f_0(z)|$$

$$|f_1|_{C^1(a,b)} = |f_1|_{C^0(a,b)} + |f_1'|_{C^0(a,b)}$$

  - *Spectral radius of a matrix A:*

$$\rho(A) = \max_i(|\lambda_i|)$$

“Almost linear” d-pH systems

- **Theorem *(Prieur et al. [2008]).*** Let us consider the “almost linear” port-Hamiltonian system, with the boundary control introduced before. Given $\varepsilon_0 > 0$ and $M > 0$, if

$$\rho\left(\text{abs}\left(\nabla g(0)\right)\right) < 1$$

and

$$\left|\nabla\left(M(\xi)\xi\right)\right|_{\xi=0} \leq M$$

then there exists $0 < \varepsilon_1 < \varepsilon_0$, $\mu > 0$ and $C > 0$ such that, for all continuously differentiable initial conditions $\xi^{\#} \in B_C(\varepsilon_1)$, there exists an unique solution for the PDE, and the solution satisfies

$$|\xi(\cdot,t)|_{C^1(0,L)} \leq Ce^{-\mu t}|\xi^\sharp|_{C_1(0,L)}, \quad \forall t \geq 0$$

- Note that *asymptotic stability* is obtained e.g. if

$$u_\xi = \begin{pmatrix} 0 & k_b \\ k_a & 0 \end{pmatrix} y_\xi \quad + \quad |k_a k_b| < 1$$

This is equivalent to have *full boundary dissipation* at least on one side of the domain

Stabilization of a nonlinear flexible beam

- Let's apply the previous result to the nonlinear beam equation

$$x(t,z) = \begin{pmatrix} q(t,z) \\ p(t,z) \end{pmatrix} \quad \mathcal{L} = \begin{pmatrix} C^{-1} & 0 \\ 0 & I_\rho^{-1} \end{pmatrix} \quad P_1 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$$

$$(\mathcal{L}x)(t,z) = \begin{pmatrix} C^{-1}q(t,z) \\ I_\rho^{-1}p(t,z) \end{pmatrix} = \begin{pmatrix} W(t,z) \\ T(t,z) \end{pmatrix}$$

co-energy variables

- For the *boundary input and output,* we proceed as before

$$f_\partial = \frac{1}{\sqrt{2}} \begin{pmatrix} T_L - T_0 \\ W_L + W_0 \end{pmatrix} \quad e_\partial = \frac{1}{\sqrt{2}} \begin{pmatrix} W_L - W_0 \\ T_L + T_0 \end{pmatrix}$$

$$\hat{W} = \frac{\sqrt{2}}{2} \begin{pmatrix} 0 & I & 0 & -I \\ 0 & I & 0 & I \end{pmatrix} \quad \tilde{W} = \frac{\sqrt{2}}{2} \begin{pmatrix} -I & 0 & I & 0 \\ I & 0 & I & 0 \end{pmatrix}$$

$$u = \begin{pmatrix} W_0 \\ W_L \end{pmatrix} \qquad y = \begin{pmatrix} T_0 \\ T_L \end{pmatrix}$$

system in impedance form

Stabilization of a nonlinear flexible beam

- Scattering decomposition:

$$\sqrt{C^{-1}}\sqrt{I_\rho^{-1}} = \Psi^{\mathrm{T}}\Gamma\Psi \qquad \Lambda = \begin{pmatrix}\Gamma & 0\\ 0 & -\Gamma\end{pmatrix}$$

- After the coordinate change...

$$\xi_-(t,z) = \frac{1}{\sqrt{2}}\Psi\left(\sqrt{C^{-1}}q(t,z) + \sqrt{I_\rho^{-1}}p(t,z)\right) \qquad \xi_-(t,z) = \frac{1}{\sqrt{2}}\Psi\left(\sqrt{C}W(t,z) + \sqrt{I_\rho}T(t,z)\right)$$

$$\xi_+(t,z) = \frac{1}{\sqrt{2}}\Psi\left(\sqrt{C^{-1}}q(t,z) - \sqrt{I_\rho^{-1}}p(t,z)\right) \qquad \xi_+(t,z) = \frac{1}{\sqrt{2}}\Psi\left(\sqrt{C}W(t,z) - \sqrt{I_\rho}T(t,z)\right)$$

- ...and on the boundary

$$(s_{+,0}, s_{-,0}) = \frac{-\Psi}{\sqrt{2}}\left(\sqrt{C}W_0 + \sqrt{I_\rho}T_0, \sqrt{C}W_0 - \sqrt{I_\rho}T_0\right)$$

$$(s_{+,L}, s_{-,L}) = \frac{\Psi}{\sqrt{2}}\left(\sqrt{C}W_L - \sqrt{I_\rho}T_L, \sqrt{C}W_L + \sqrt{I_\rho}T_L\right)$$

- In case of *free-end in z = L,* and *control action in z = 0,* we have

$$u = \begin{pmatrix}W_0\\ W_L\end{pmatrix} = -\begin{pmatrix}K_D & 0\\ 0 & 0\end{pmatrix}\underbrace{\begin{pmatrix}T_0\\ T_L\end{pmatrix}}_{=y} \qquad \begin{aligned} s_{+,0} &= \left(\sqrt{C}K_D - \sqrt{I_\rho}\right)\left(\sqrt{C}K_D + \sqrt{I_\rho}\right)^{-1} s_{-,0}\\ s_{-,L} &= -s_{+,L}\end{aligned}$$

$$\boxed{K_D > 0}$$

Dirac structures & control synthesis

- A Dirac structure describes *internal power flows* and the *power exchange* between the system and the environment
- Denote by $\mathcal{F} \times \mathcal{E}$ the space of *power variables,* and by $\langle e, f \rangle$ the *power* associated to the port $(f, e) \in \mathcal{F} \times \mathcal{E}$
- **Definition.** A (constant) Dirac structure on $\mathcal{F}$ is a *linear subspace* $\mathcal{D} \subset \mathcal{F} \times \mathcal{E}$ such that

$$\dim \mathcal{D} = \dim \mathcal{F} \qquad \qquad \langle e, f \rangle = 0, \ \forall (f, e) \in \mathcal{D}$$

- Coordinate representations:

image

$$\mathcal{D} = \left\{ (f, e) \in \mathcal{F} \times \mathcal{E} \mid f = E^{\mathrm{T}} \lambda, \ e = F^{\mathrm{T}} \lambda, \ \lambda \in \mathbb{R}^n \right\}$$

$$EF^{\mathrm{T}} + FE^{\mathrm{T}} = 0$$
$$\operatorname{rank} (F \mid E) = n$$

kernel

$$\mathcal{D} = \left\{ (f, e) \in \mathcal{F} \times \mathcal{E} \mid Ff + Ee = 0 \right\}$$

$$f = -\dot{x} \quad e = \frac{\partial H}{\partial x}$$

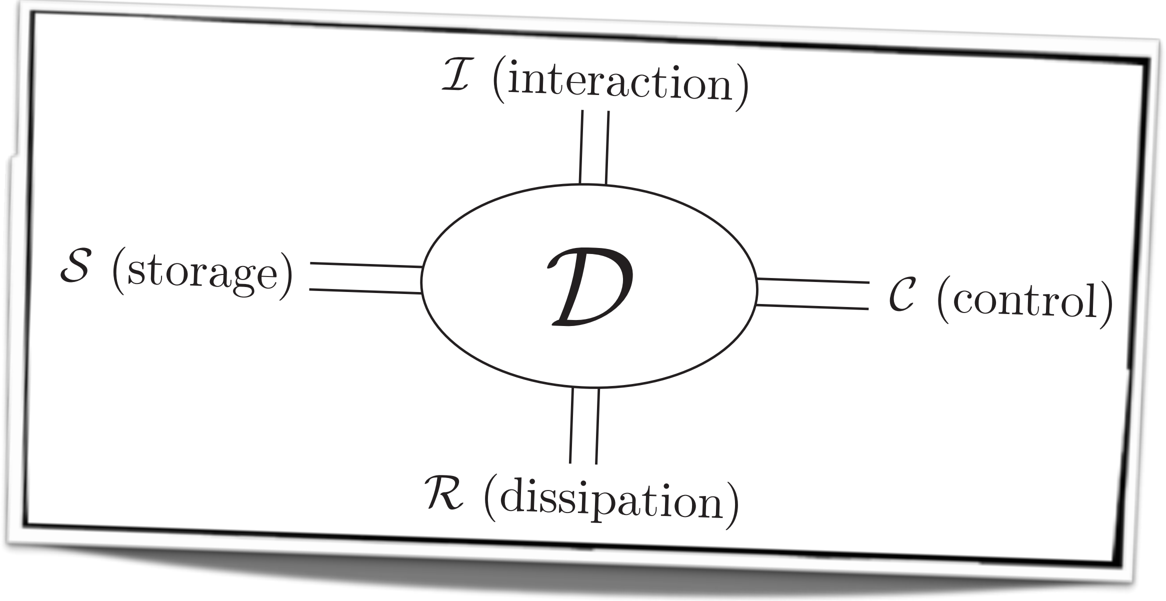
$$-F\dot{x} + E \frac{\partial H}{\partial x} = 0, \quad x(0) = x_0 \in \mathcal{X}$$

port-Hamiltonian system

Dirac structures & control synthesis

☑ In the general case:

$$\mathcal{D} = \Big\{ (f_S, f_R, f_C, f_I, e_S, e_R, e_C, e_I) \in \mathcal{F} \times \mathcal{E} \;\; \Big| \\ F_S f_S + F_R f_R + F_C f_C + F_I f_I + \\ + E_S e_S + E_R e_R + E_C e_C + E_I e_I = 0 \Big\}$$



$$R_f f_R + R_e e_R = 0$$

resistive relation

$$R_f R_e^{\mathrm{T}} = R_e R_f^{\mathrm{T}} > 0$$
$$\operatorname{rank}\,(R_f \mid R_e) = \dim \mathcal{F}_R$$

$$-F_S \dot{x} + E_S \frac{\partial H}{\partial x} + F_R f_R + E_R e_R + \\ + F_C f_C + E_C e_C + F_I f_I + E_I e_I = 0 \\ R_f f_R + R_e e_R = 0$$

$$\frac{\mathrm{d}}{\mathrm{d}t} H \leq e_C^{\mathrm{T}} f_C + e_I^{\mathrm{T}} f_I$$

This is the *most general* formulation of a port-Hamiltonian system. Usually, the *spatial discretisation* of a distributed parameter systems is in this form

Dirac structures & control synthesis

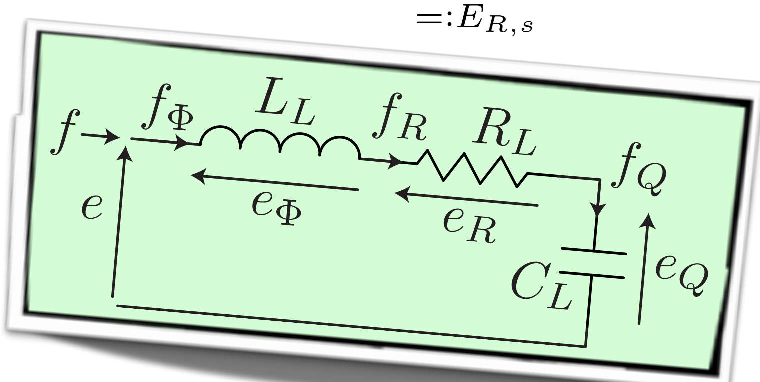
☑ *Example:* the RLC circuit

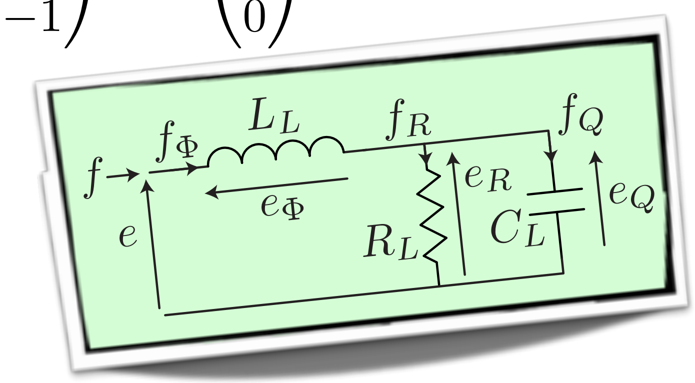
$$\underbrace{\begin{pmatrix}1&0\\0&1\\0&0\\0&0\end{pmatrix}}_{=:F_{S,s}}\begin{pmatrix}f_Q\\f_\Phi\end{pmatrix}+\underbrace{\begin{pmatrix}0&1\\-1&0\\0&1\\0&1\end{pmatrix}}_{=:E_{S,s}}\begin{pmatrix}e_Q\\e_\Phi\end{pmatrix}+\underbrace{\begin{pmatrix}0\\0\\-1\\0\end{pmatrix}}_{=:F_{R,s}}f_R+$$

$$+\underbrace{\begin{pmatrix}0\\1\\0\\0\end{pmatrix}}_{=:E_{R,s}}e_R+\begin{pmatrix}0\\0\\0\\-1\end{pmatrix}f+\begin{pmatrix}0\\1\\0\\0\end{pmatrix}e=0$$

$$\underbrace{\begin{pmatrix}1&0\\0&1\\0&0\\0&0\end{pmatrix}}_{=:F_{S,p}}\begin{pmatrix}f_Q\\f_\Phi\end{pmatrix}+\underbrace{\begin{pmatrix}0&1\\-1&0\\1&0\\0&1\end{pmatrix}}_{=:E_{S,p}}\begin{pmatrix}e_Q\\e_\Phi\end{pmatrix}+\underbrace{\begin{pmatrix}1\\0\\0\\0\end{pmatrix}}_{=:F_{R,p}}f_R+$$

$$+\underbrace{\begin{pmatrix}0\\0\\-1\\0\end{pmatrix}}_{=:E_{R,p}}e_R+\begin{pmatrix}0\\0\\0\\-1\end{pmatrix}f+\begin{pmatrix}0\\1\\0\\0\end{pmatrix}e=0$$





$$H_L(x_Q,x_\Phi)=\frac{1}{2}\left(\frac{x_Q^2}{C_L}+\frac{x_\Phi^2}{L_L}\right)$$

$$R_L f_R+e_R=0$$

Dirac structures & control synthesis

- In a similar manner for *Dirac structures on Hilbert spaces*
- Assume that the space of flows $\mathcal{F} \equiv \mathcal{E}$ is an Hilbert space, and denote by $\langle \cdot \mid \cdot \rangle$ the *inner product* on $\mathcal{F}$
  - The Cartesian space $\mathcal{F} \times \mathcal{E}$ is an Hilbert space with the inner product
    $$\langle (f_1, e_1) \mid (f_2, e_2) \rangle_{\mathcal{F} \oplus \mathcal{E}} = \langle f_1 \mid f_2 \rangle + \langle e_1 \mid e_2 \rangle$$
  - The *bond space* $\mathcal{B}$ is $\mathcal{F} \times \mathcal{E}$ with the indefinite product
    $$\ll (f_1, e_1), (f_2, e_2) \gg = \langle f_1 \mid e_2 \rangle + \langle f_2 \mid e_1 \rangle$$
  - Given a linear space $\mathcal{A} \subset \mathcal{B}$, its *orthogonal complement* is
    $$\mathcal{A}^\perp = \left\{ a' \in \mathcal{B} \mid \ll a, a' \gg = 0,\ \forall a \in \mathcal{A} \right\}$$
- **Definition.** $\mathcal{D}$ is a Dirac structure on $\mathcal{B}$ if
  $$\mathcal{D} = \mathcal{D}^\perp$$

$$FE^* + EF^* = 0$$
$$\operatorname{ran}\begin{pmatrix} F & E \end{pmatrix} = \Lambda$$

$$\mathcal{D} = \left\{ (f, e) \in \mathcal{B} \mid f = E^* \lambda,\ e = F^* \lambda,\ \forall \lambda \in \Lambda \right\}$$

image

$$\mathcal{D} = \left\{ (f, e) \in \mathcal{B} \mid Ff + Ee = 0 \right\}$$

kernel

Dirac structures & control synthesis

☑ The "usual" class of distributed port-Hamiltonian systems

$$\frac{\partial x}{\partial t}(t,z) = P_1 \frac{\partial}{\partial z}\big(\mathcal{L}(z)x(t,z)\big) + (P_0 - G_0)\mathcal{L}(z)x(t,z)$$

$$H(x(t)) = \frac{1}{2}\|x(t)\|^2_{\mathcal{L}}$$

✱ *Boundary port:*

$$\begin{pmatrix} f_\partial \\ e_\partial \end{pmatrix} = \underbrace{\frac{1}{\sqrt{2}}\begin{pmatrix} P_1 & -P_1 \\ I & I \end{pmatrix}}_{=R} \begin{pmatrix} (\mathcal{L}x)(b) \\ (\mathcal{L}x)(a) \end{pmatrix} \qquad \dot{H}(x(t)) \leq e_\partial^{\mathrm{T}}(t) f_\partial(t)$$

✱ *Inputs and outputs:*

$$W\Sigma W^{\mathrm{T}} = \tilde{W}\Sigma\tilde{W}^{\mathrm{T}} = 0 \qquad W\Sigma\tilde{W}^{\mathrm{T}} = I$$

$$u(t) = W\begin{pmatrix} f_\partial(t) \\ e_\partial(t) \end{pmatrix} \qquad y(t) = \tilde{W}\begin{pmatrix} f_\partial(t) \\ e_\partial(t) \end{pmatrix} \qquad \frac{\mathrm{d}H}{\mathrm{d}t}(x(t)) \leq y^{\mathrm{T}}(t)u(t)$$

✱ *Dirac structure:*

$$\boxed{\mathcal{F} = \mathcal{F}_S \times \mathcal{F}_R \times \mathcal{F}_C}$$

$$F_S = \begin{pmatrix} I \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad E_S = \begin{pmatrix} P_1\frac{\partial}{\partial z} + P_0 \\ -G_R^{\mathrm{T}} \\ -WR\mathcal{B} \\ -\tilde{W}R\mathcal{B} \end{pmatrix} \quad F_R = \begin{pmatrix} 0 \\ I \\ 0 \\ 0 \end{pmatrix}$$

$$E_R = \begin{pmatrix} G_R \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad F_C = \begin{pmatrix} 0 \\ 0 \\ 0 \\ I \end{pmatrix} \quad E_C = \begin{pmatrix} 0 \\ 0 \\ I \\ 0 \end{pmatrix}$$

$$\mathcal{F}_S = L_2(a,b;\mathbb{R}^n) \qquad \mathcal{F}_R = \mathbb{R}^r \qquad \mathcal{F}_C = \mathbb{R}^2$$

$$\boxed{f_S = -\frac{\partial x}{\partial t} \qquad e_S = \frac{\delta H}{\delta x} = \mathcal{L}x \qquad e_R = -\bar{G} f_R}$$

Dirac structures & control synthesis

☑ *Kernel representation:*

$$\operatorname{dom}\begin{pmatrix} F & E \end{pmatrix} = \left\{ (f,e) \in \mathcal{F} \times \mathcal{E} \mid e_S \text{ absolutely continuous, and } \frac{\partial e_S}{\partial z} \in L_2(a,b;\mathbb{R}^n) \right\}$$

☑ *Image representation:*

$$\operatorname{dom}\begin{pmatrix} F^* \\ E^* \end{pmatrix} = \left\{ \lambda \in \Lambda \mid \begin{pmatrix} \lambda_u \\ \lambda_y \end{pmatrix} = \begin{pmatrix} W \\ \tilde{W} \end{pmatrix} R\mathcal{B}\lambda_S \right\}$$

$\lambda = (\lambda_S,\, \lambda_R,\, \lambda_y,\, \lambda_u)$

$$\Lambda = L_2(a,b;\mathbb{R}^n) \times \mathbb{R}^r \times \mathbb{R}^n \times \mathbb{R}^n$$

$$E_S^* = \begin{pmatrix} -P_1 \frac{\partial}{\partial z} - P_0 & -G_R & 0 & 0 \end{pmatrix}$$

The image representation
$$\mathcal{D} = \left\{ (f,e) \in \mathcal{B} \mid f = E^*\lambda,\ e = F^*\lambda,\ \forall \lambda \in \Lambda \right\}$$
allows an easy mapping of the *effect of the boundary (inputs) on the system dynamics:* for this reason, it is the main tool used in the control synthesis

Dirac structures & control synthesis

- *Energy-balancing control* in case of *finite dimensional systems*

$$-\beta^{\mathrm{T}}(x(t))f_C(t) = \left(\frac{\partial H_a}{\partial x}(x(t))\right)^{\mathrm{T}}\dot{x}(t)$$

$(y,u) = (f_C, e_C)$

$$\left(-\frac{\partial^{\mathrm{T}} H_a}{\partial x}E_S^{\mathrm{T}} + \beta^{\mathrm{T}}E_C^{\mathrm{T}}\right)\lambda = 0$$

$$-E_S\frac{\partial H_a}{\partial x} + E_C\beta = 0$$

- A sufficient condition is that

$$\begin{pmatrix} 0 \\ 0 \\ 0 \\ -\frac{\partial H_a}{\partial x} \\ 0 \\ \beta \end{pmatrix} \in \mathrm{Im}\begin{pmatrix} E_S^{\mathrm{T}} \\ E_R^{\mathrm{T}} \\ E_C^{\mathrm{T}} \\ F_S^{\mathrm{T}} \\ F_R^{\mathrm{T}} \\ F_C^{\mathrm{T}} \end{pmatrix} \equiv \mathcal{D}$$

This is equivalent to the control by interconnection and *Casimir generation*

dissipation obstacle???

- For *distributed-parameter systems,* we have

$$-\beta^{\mathrm{T}}(x(t))y(t) = \int_a^b \frac{\delta^{\mathrm{T}} H_a}{\delta x}(t,z)\frac{\partial x}{\partial t}(t,z)\mathrm{d}z$$

$$\left\langle E_C\beta - E_S\frac{\delta H_a}{\delta x} \mid \lambda \right\rangle = 0$$

$$\left(0,\, 0,\, 0,\, \tfrac{\delta H_a}{\delta x},\, 0,\, -\beta\right) \in \mathcal{D}$$

$$\begin{pmatrix} 0 \\ \frac{\delta H_a}{\delta x} \\ 0 \\ 0 \\ 0 \\ -\beta \end{pmatrix} = \begin{pmatrix} E_S^* \\ F_S^* \\ E_R^* \\ F_R^* \\ E_C^* \\ F_C^* \end{pmatrix} \lambda$$

$$\lambda \in \operatorname{dom}\begin{pmatrix} F^* \\ E^* \end{pmatrix}$$

All the results based on *energy-Casimir method* for distributed port-Hamiltonian systems are solution of this PDE

Dirac structures & control synthesis

- Let us consider at first the *finite element model* of a transmission line, which is characterized by a Dirac structure with matrices

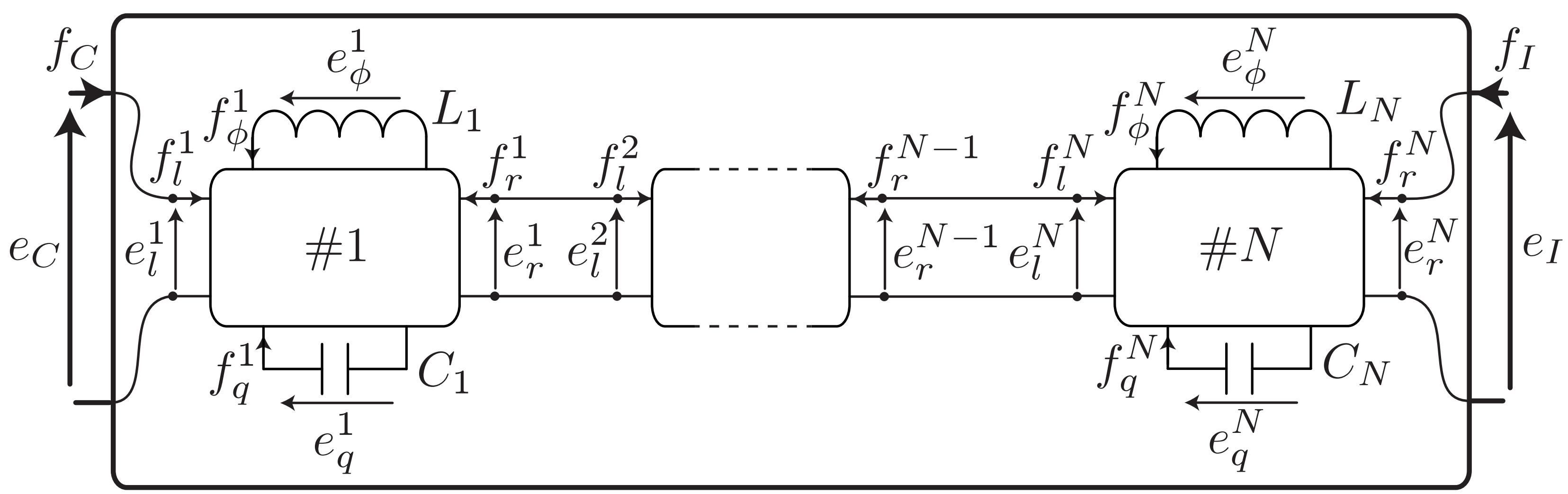
$$F_\infty = \begin{pmatrix} F_{S,\infty} & F_{C,\infty} & F_{I,\infty} \end{pmatrix} \qquad E_\infty = \begin{pmatrix} E_{S,\infty} & E_{C,\infty} & E_{I,\infty} \end{pmatrix}$$

and an Hamiltonian

$$H_\infty(x_\infty) = \frac{1}{2}\sum_{i=1}^{N}\left(\frac{{x_q^i}^2}{C_i} + \frac{{x_\phi^i}^2}{L_i}\right)$$

$$x_\infty = \begin{pmatrix} x_q^1 & x_\phi^1 & \cdots & x_q^N & x_\phi^N \end{pmatrix}^{\mathrm{T}}$$

controller



load

The plant is a finite dimensional port-Hamiltonian system with control port (f_C, e_C)

Dirac structures & control synthesis

- For the complete system we have

$$x = \begin{pmatrix} x_Q & x_\Phi & x_\infty \end{pmatrix}^{\mathrm{T}} \qquad H(x) = H_\infty(x_\infty) + H_L(x_Q, x_\Phi)$$

- Simple physical considerations lead to the *desired equilibrium:*

$$\frac{x_Q^\star}{C_L} = \frac{x_q^{i,\star}}{C_i} = e^\star \qquad \frac{x_\Phi^\star}{L_L} = \frac{x_\phi^{i,\star}}{L_i} = 0$$

- The *energy-balance controller* follows if it exists λ such that

$$\begin{aligned} -\frac{\partial H_a}{\partial x} &= {F_S}^{\mathrm{T}}\lambda \\ \beta &= {F_C}^{\mathrm{T}}\lambda \\ 0 &= E_S^{\mathrm{T}}\lambda = E_R^{\mathrm{T}}\lambda = F_R^{\mathrm{T}}\lambda = E_C^{\mathrm{T}}\lambda \end{aligned}$$

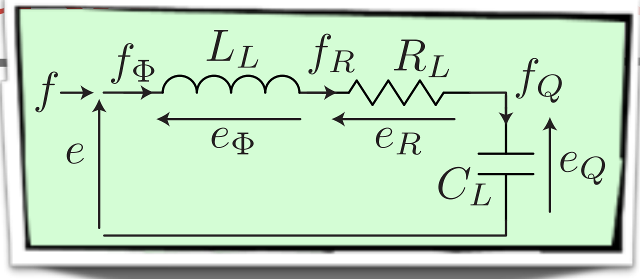
RLC load in series conf.

$$H_a(x) = H_a(\xi)\Big|_{\xi = x_Q + \sum_{i=1}^N x_q^i}$$

$$\beta(x) = -\frac{\partial H_a}{\partial \xi}\Big|_{\xi = x_Q + \sum_{i=1}^N x_q^i}$$

$$\boxed{H_a(\xi) = \frac{1}{2}\frac{\xi^2}{C_C} - e^\star\left(1 + \frac{C_L}{C_C} + \sum_{i=1}^N \frac{C_i}{C_C}\right)\xi + \kappa}$$

Dirac structures & control synthesis



- The *port-Hamiltonian* formulation of the line is

$$\begin{cases} \dfrac{\partial x_q}{\partial t}(t,z) = -\dfrac{\partial}{\partial z}\dfrac{\delta H_{\mathrm{TL}}}{\delta x_\phi}(t,z) \\ \dfrac{\partial x_\phi}{\partial t}(t,z) = -\dfrac{\partial}{\partial z}\dfrac{\delta H_{\mathrm{TL}}}{\delta x_q}(t,z) \end{cases}$$

$$H_{\mathrm{TL}}(x_q, x_\phi) = \frac{1}{2}\int_0^\ell \left(\frac{x_q^2}{C} + \frac{x_\phi^2}{L}\right) \mathrm{d}z$$

- In case of RLC load in series configuration:

$$F_S = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad E_S = \begin{pmatrix} 0 & -\partial_z & 0 & 0 \\ -\partial_z & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ \cdot|_\ell & 0 & -1 & 0 \\ 0 & \cdot|_\ell & 0 & -1 \\ 0 & 0 & 0 & 1 \\ -\cdot|_0 & 0 & 0 & 0 \\ 0 & -\cdot|_0 & 0 & 0 \end{pmatrix}$$

$$F_R = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \quad E_R = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad F_C = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \quad E_C = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

$$\mathcal{F}_S = L_2(0,\ell;\mathbb{R}^2) \times \mathbb{R}^2$$
$$\mathcal{F}_R = \mathcal{F}_C = \mathbb{R}$$

$$e_R = -R_L f_R$$

Desired equilibrium

$$\left(\frac{x_q^\star}{C}, \frac{x_\phi^\star}{L}, \frac{x_Q^\star}{C_L}, \frac{x_\Phi^\star}{L_L}\right) = (e^\star, 0, e^\star, 0)\,, \qquad e^\star \in \mathbb{R}$$

Dirac structures & control synthesis

- For the image representation of the Dirac structure we have

$$E_S^* = \begin{pmatrix} 0 & \partial_z & 0 & 0 & 0 & 0 & 0 & 0 \\ \partial_z & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 & 0 & 0 \end{pmatrix}$$

$$\Lambda = L_2(0,\ell;\mathbb{R}^2) \times \mathbb{R}^6$$

$$\lambda = (\lambda_q, \lambda_\phi, \lambda_Q, \lambda_\Phi, \lambda_{qL}, \lambda_R, \lambda_{\phi 0}, \lambda_{q0})$$

$$\operatorname{dom}\begin{pmatrix} E^* \\ F^* \end{pmatrix} = \Big\{ \lambda \in \Lambda \mid \lambda_q(0) = \lambda_{q0},\ \lambda_\phi(0) = \lambda_{\phi 0},\ \lambda_q(L) = \lambda_{qL},\ \lambda_\phi(L) = \lambda_\Phi \Big\}$$

- To have a boundary energy-balancing control we need that

$$\lambda_\phi = \lambda_\Phi = \lambda_{\phi 0} = \lambda_R = \frac{\delta H_a}{\delta x_\phi} = \frac{\partial H_a}{\partial x_\Phi} = 0$$

$$\lambda_q = \lambda_Q = \lambda_{qL} = \lambda_{q0} = \frac{\delta H_a}{\delta x_q} = \frac{\partial H_a}{\partial x_Q} = -\beta$$

$$H_a(\xi), \quad \xi = x_Q + \int_0^L x_q \, \mathrm{d}z \qquad \beta(x_q, x_Q) = -\frac{\partial H_a}{\partial \xi}(\xi)$$

$$\boxed{H_a(\xi) = \frac{1}{2}\frac{(\xi - \xi^\star)^2}{C_C} - e^\star \left(1 + \frac{C_L + C}{C_C}\right)\xi}$$

Dirac structures & control synthesis

- Finding the *EB regulator* means finding a state dependent control action able to shape the open-loop Hamiltonian, in such a way that closed loop and target dynamics have the *same behaviour at the storage, resistive and control ports*
  - ***Very strong requirement!***
  - Let us ask less: *just a matching* between open-loop *plus* controller, and target dynamics (with desired stability properties)

$$
\begin{aligned}
-\dot{x} &= E_S^{\mathrm{T}} \lambda \\
\frac{\partial H}{\partial x} &= F_S^{\mathrm{T}} \lambda \\
0 &= \left(R_f E_R^{\mathrm{T}} + R_e F_R^{\mathrm{T}}\right) \lambda \\
e_C &= F_C^{\mathrm{T}} \lambda
\end{aligned}
$$

controlled system

$$e_C = \beta(x) + e_C'$$

$$
\begin{aligned}
-\dot{x} &= E_S^{\mathrm{T}} \lambda' \\
\frac{\partial H_d}{\partial x} &= F_S^{\mathrm{T}} \lambda' \\
0 &= \left(R_f E_R^{\mathrm{T}} + R_e F_R^{\mathrm{T}}\right) \lambda' \\
e_C' &= F_C^{\mathrm{T}} \lambda'
\end{aligned}
$$

"desired" system

$$H_d(x) = H(x) + H_a(x)$$

Dirac structures & control synthesis

- Since trajectories are required to be the same

$$
\begin{aligned}
0 &= E_S^{\mathrm{T}}\left(\lambda - \lambda'\right) \\
-\frac{\partial H_a}{\partial x} &= F_S^{\mathrm{T}}\left(\lambda - \lambda'\right) \\
0 &= \left(R_f E_R^{\mathrm{T}} + R_e F_R^{\mathrm{T}}\right)\left(\lambda - \lambda'\right) \\
\beta &= F_C^{\mathrm{T}}\left(\lambda - \lambda'\right)
\end{aligned}
$$

$$
\begin{pmatrix} 0 \\ -\frac{\partial H_a}{\partial x} \\ 0 \\ \beta \end{pmatrix} \in \operatorname{Im} \begin{pmatrix} E_S^{\mathrm{T}} \\ F_S^{\mathrm{T}} \\ R_f E_R^{\mathrm{T}} + R_e F_R^{\mathrm{T}} \\ F_C^{\mathrm{T}} \end{pmatrix}
$$

- It is possible to prove that *the open-loop system is mapped into the desired closed-loop one,* for which the Hamiltonian function H_d is selected so that "nice" stability properties are satisfied
  - *Asymptotic stability follows as in case of energy-balancing regulators*

In a similar manner, this result can be obtained in the *distributed-parameter scenario*

$$
\begin{pmatrix} 0 \\ \frac{\delta H_a}{\delta x} \\ 0 \\ -\beta \end{pmatrix} = \begin{pmatrix} E_S^* \\ F_S^* \\ \bar{R} E_R^* + F_R^* \\ F_C^* \end{pmatrix} \tilde{\lambda}
$$

Dirac structures & control synthesis

- In case of pRLC with the fem of the line, the *desired equilibrium* is

$$\frac{x_Q^\star}{C_L} = \frac{x_q^{i,\star}}{C_i} = e^\star \qquad \frac{x_\Phi^\star}{L_L} = \frac{x_\phi^{i,\star}}{L_i} = \frac{e^\star}{R_L}$$

- The control synthesis requires to find λ, such that

$$\begin{aligned} -\frac{\partial H_a}{\partial x} &= {F_S}^{\mathrm{T}}\lambda \\ \beta &= {F_C}^{\mathrm{T}}\lambda \\ 0 &= E_S^{\mathrm{T}}\lambda = (R_L E_R^{\mathrm{T}} + F_R^{\mathrm{T}})\lambda \end{aligned}$$

$$\begin{aligned} H_a(x) &= H_a(\xi)\Big|_{\xi = x_\Phi + R_L x_Q + \sum_{i=1}^N \left(x_\phi^i + R_L x_q^i\right)} \\ \beta(x) &= -R_L \frac{\partial H_a}{\partial \xi}\Big|_{\xi = x_\Phi + R_L x_Q + \sum_{i=1}^N \left(x_\phi^i + R_L x_q^i\right)} \end{aligned}$$

- A possible *choice for H_a* can be the following:

$$H_a(\xi) = \frac{1}{2}\frac{(\xi - \xi^\star)^2}{L_C} - \frac{e^\star}{R_L}\xi + \kappa$$

Asymptotic stability is a consequence of the *energy dissipation inequality*

Dirac structures & control synthesis

- In the distributed parameter case, the *desired equilibrium* is

$$\left(\frac{x_q^\star}{C}, \frac{x_\phi^\star}{L}, \frac{x_Q^\star}{C_L}, \frac{x_\Phi^\star}{L_L}\right) = \left(e^\star, \frac{e^\star}{R_L}, e^\star, \frac{e^\star}{R_L}\right)$$

- The Dirac structure is similar to the previous case, with the same domains, and this only noticeable difference:

$$\Lambda = L_2(0, \ell; \mathbb{R}^2) \times \mathbb{R}^6$$

$$E_S^* = \begin{pmatrix} 0 & \partial_z & 0 & 0 & 0 & 0 & 0 & 0 \\ \partial_z & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 \end{pmatrix}$$

$$\lambda = (\lambda_q, \lambda_\phi, \lambda_Q, \lambda_\Phi, \lambda_{qL}, \lambda_R, \lambda_{\phi 0}, \lambda_{q0})$$

$$\tilde\lambda_\phi = \tilde\lambda_\Phi = -\tilde\lambda_R = \frac{\delta H_a}{\delta x_\phi} = \frac{\partial H_a}{\partial x_\Phi}$$

$$\tilde\lambda_q = \tilde\lambda_Q = \tilde\lambda_{qL} = \tilde\lambda_{q0} = -\beta = \frac{\delta H_a}{\delta x_q} = \frac{\partial H_a}{\partial x_Q}$$

$$R_L \tilde\lambda_R + \tilde\lambda_Q = 0$$

$$H_a(\xi), \quad \xi = x_\Phi + R_L x_Q + \int_0^L (x_\phi + R_L x_q)\,\mathrm{d}z$$

$$\beta(x_q, x_\phi, x_Q, x_\Phi) = -R_L \frac{\partial H_a}{\partial \xi}(\xi)$$

$$\boxed{H_a(\xi) = \frac{1}{2} K (\xi - \xi^\star)^2 - \frac{e^\star}{R_L}\xi, \quad K > 0}$$

Conclusions and open problems

- From finite-dimensional energy-based control, to the distributed parameter scenario
  - *Energy-balancing method*
  - *Energy-Casimir method*
  - *Control by state-modulated source*

- Connection between port-Hamiltonian systems and *semigroup theory*
  - Existence of solutions for the open and closed-loop systems
  - Stability analysis
- *La Salle's invariance principle* in the distributed-parameter case
  - Powerful tool, with some technical difficulties to be applied
  - Strong connection with physics, and with the finite dimensional case
- Initial results on *nonlinear distributed port-Hamiltonian systems*
  - The Riemann invariants, or scattering coordinates, as a possible framework to tackle the problem

ALMA MATER STUDIORUM
UNIVERSITÀ DI BOLOGNA

Alessandro Macchelli

Department of Electrical, Electronic, and Information Engineering (DEI)

alessandro.macchelli@unibo.it

http://www.unibo.it