Formal Methods for the Control of Large-scale Networked Nonlinear Systems with Logic Specifications

Basilica di Santa Maria di Collemaggio, L'Aquila (Italy), 1287

Lecture L10:

Symbolic models for time-delay systems

Speaker: Giordano Pola

What's new?

In this lecture symbolic models for time-varying time-delay systems

Tools:

- countable approximations of functional spaces for approximating infinite dimensional state space
- incremental input-delay-to state stability for relating solutions of time-delay systems with different delay realizations

Lecture based on:

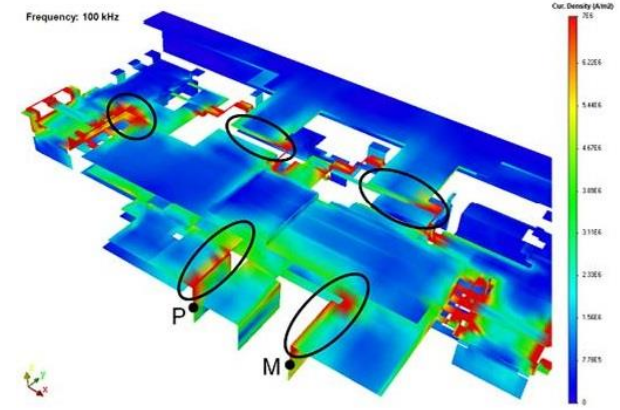
[Pola et al., IJRNC15] Pola, G., Pepe, P. Di Benedetto, M.D., Symbolic Models for Time–Varying Time–Delay Systems via Alternating Approximate Bisimulation, International Journal of Robust and Nonlinear Control, 25:2328–2347, September 2015

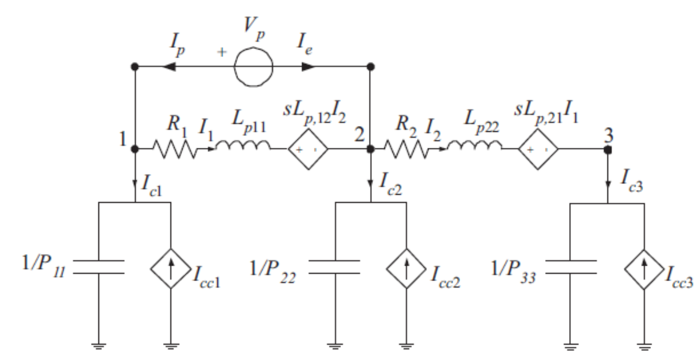
[Pola et al., SCL10] Pola, G., Pepe, P., Di Benedetto, M.D., Tabuada, P., Symbolic models for nonlinear time-delay systems using approximate bisimulation, Systems & Control Letters 59(6): 365-373, June 2010

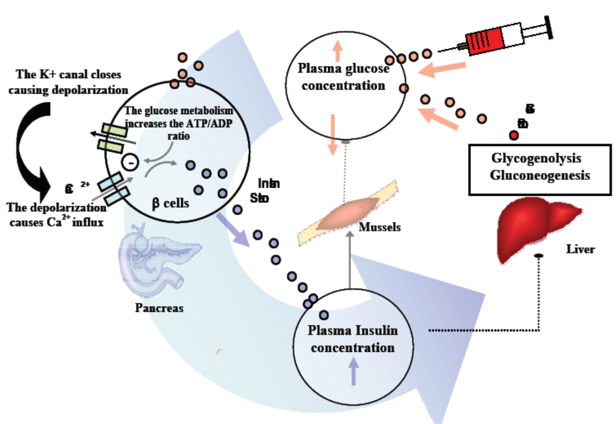
Time-varying time-delay systems

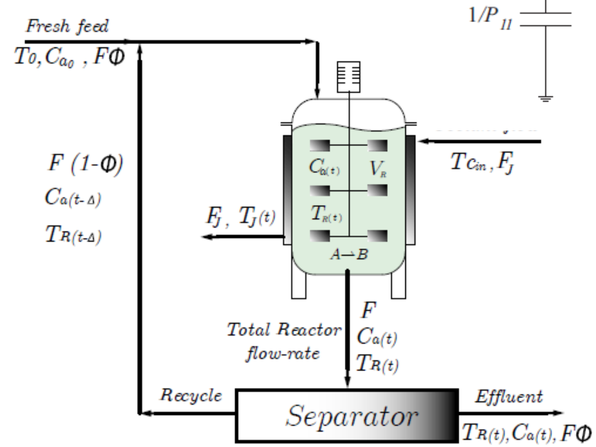
Some applications of interest :

- Electric engineering
  partial element equivalent circuits
- Chemical engineering
  continuous stirred tank reactor with recycle
- Systems biology
  human glucose insulin system
- …









Time-varying time-delay systems

We consider the following nonlinear time-varying time-delay system:

$$\begin{cases} \dot{x}(t) = f(x(t), x(t-\Delta(t)), u(t-r)) \\ x(t) = \xi_0(t), t \in [-\Delta_{max}, 0] \end{cases} \qquad (*)$$

where:

- $x(t) \in \mathbb{R}^n$ and $x_t \in \chi = C^0([-\Delta_{max}, 0], \mathbb{R}^n)$ is the state at time t
- $\xi_0 \in \chi$ is the initial condition
- $u(t) \in \mathbb{R}^m$ is the control input at time $t \in [-r, +\infty[$ and r is the constant control input delay
- $\Delta: \mathbb{R}_0^+ \to [\Delta_{min}, \Delta_{max}]$ is the unknown time-varying state delay and $\Delta_{min}, \Delta_{max} \in \mathbb{R}_0^+$
- $f: \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is the vector field

In the sequel we refer for simplicity, to the time-varying time-delay system above as a time-delay system

... unknown time-varying delay in the state
and known constant delay in the control !

Time-delay systems

A0: Technical assumptions:

1. $\mathrm{f}(0,0,0) = 0$ and locally Lipschitz
2. $|\mathrm{u}(\mathrm{t})| \leq \mathrm{B_U}$ for all $\mathrm{t} \in [-\mathrm{r}, +\infty[$ for some known $\mathrm{B_U} > 0$
3. Delay bounds $\Delta_{min}, \Delta_{max} \in \mathbb{R}_0^+$ are known
4. Initial condition $\xi_0 \in C^1([-\Delta_{max}, 0], \mathbb{R}^n)$ bounded and with bounded derivative over $[-\Delta_{max}, 0]$
5. Input functions u belonging to the functional space $\boldsymbol{U}$ of all measurable control inputs u: $[-\mathrm{r}, +\infty[\to \mathbf{B}_{\mathrm{B_U}}(0)$
6. Delay realizations Δ belonging to the functional space $\boldsymbol{D}$ of all continuously differentiable functions Δ: $\mathbb{R}_0^+ \to [\Delta_{min}, \Delta_{max}]$ with derivative bounded by $d_{min} \in [0,1[$ known

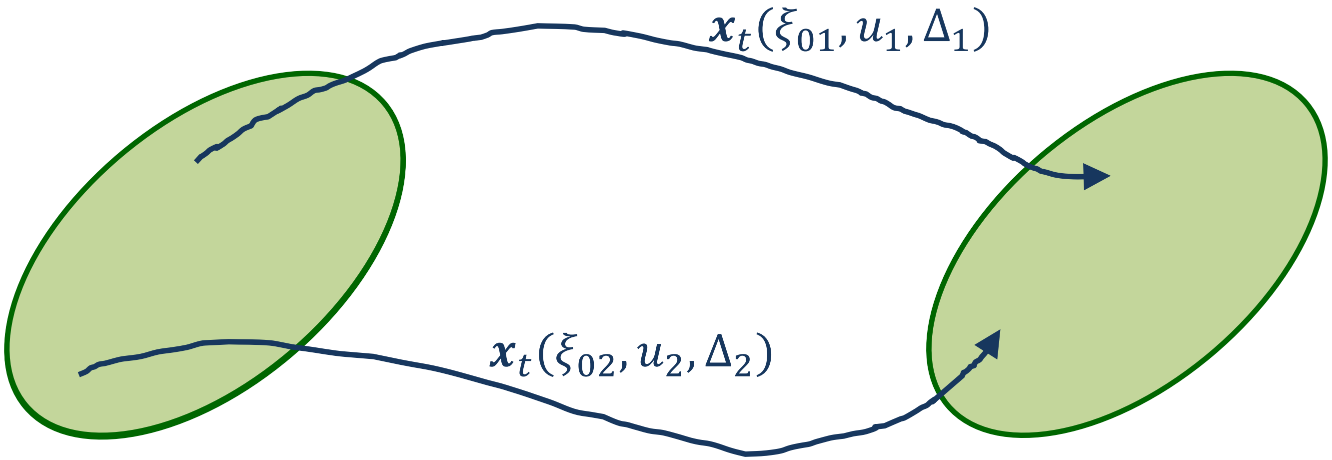
In the sequel we will denote:

- time-delay system satisfying Assumption A0 by the tuple

$$\Sigma = (\mathbb{R}^n, \chi, \xi_0, U, \boldsymbol{U}, \boldsymbol{D}, f)$$

- the solution of (*), starting from ξ_0, with control input u and time-delay Δ, in $\mathbb{R}^n$ and χ, by $\boldsymbol{x}(t, \xi_0, u, \Delta)$ and $\boldsymbol{x}_t(\xi_0, u, \Delta)$, respectively

Some questions …



May I leverage previous results and definitions?

(Q1) Since Δ can be thought of a disturbance, may I use results of L8?

No! Here, the state is a function and not a vector!

Countable approximation of infinite dimensional systems for approximating the state space !

(Q2) May I use stability notions introduced in L9?

No! Here, I need a notion of incremental stability wrt different delay realizations!

Incremental input-delay-to state stability !

Countable approximation of functional spaces

Definition

Given a functional space $\boldsymbol{Y} \subseteq C^0([a,b],\mathbb{R})$, a map

$$A_{\boldsymbol{Y}}: \mathbb{R}^+ \to 2^{C^0([a,b],\mathbb{R})}$$

is a countable approximation of $\boldsymbol{Y}$ if for any desired accuracy $\lambda > 0$

- $A_{\boldsymbol{Y}}(\lambda)$ is a countable set
- For any $y \in \boldsymbol{Y}$ there exists $z \in A_{\boldsymbol{Y}}(\lambda)$ such that $\|y - z\|_\infty \le \lambda$
- For any $z \in A_{\boldsymbol{Y}}(\lambda)$ there exists $y \in \boldsymbol{Y}$ such that $\|y - z\|_\infty \le \lambda$

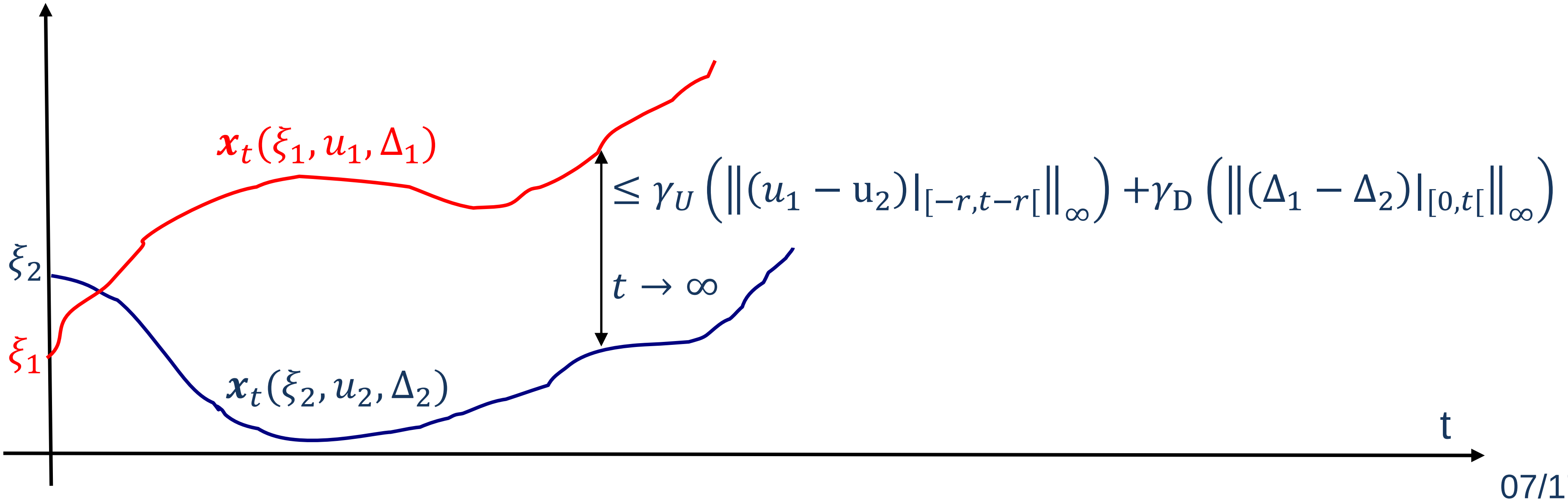
Map $A_{\boldsymbol{Y}}$ is said a finite approximation of $\boldsymbol{Y}$ if functional space $A_{\boldsymbol{Y}}(\lambda)$ is finite for any accuracy $\lambda > 0$

Incremental input-delay-to-state stability

Definition

A time-delay system Σ is incrementally Input-Delay-to-State Stable (δ-IDSS) if it is forward complete and there exist a KL function β and K functions γ_U and γ_D such that that for any initial conditions $\xi_1, \xi_2 \in \chi$, for any inputs $u_1, u_2 \in \boldsymbol{U}$ and any time-delay realizations $\Delta_1, \Delta_2 \in \boldsymbol{D}$, the corresponding solutions $\boldsymbol{x}_t(\xi_1, u_1, \Delta_1)$ and $\boldsymbol{x}_t(\xi_2, u_2, \Delta_2)$ exist for any time $t \geq 0$ and satisfy

$$\|\boldsymbol{x}_t(\xi_1, u_1, \Delta_1) - \boldsymbol{x}_t(\xi_2, u_2, \Delta_2)\|_\infty \leq$$
$$\beta(\|\xi_1 - \xi_2\|_\infty, t) + \gamma_U\left(\left\|(u_1 - \mathrm{u}_2)|_{[-r,t-r[}\right\|_\infty\right) + \gamma_{\mathrm{D}}\left(\left\|(\Delta_1 - \Delta_2)|_{[0,t[}\right\|_\infty\right)$$



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A δ-IDSS Lyapunov-Krasovskii functional can be properly defined by which

Theorem

Time delay system Σ is δ-IDSS if it admits a δ-IDSS Lyapunov-Krasovskii functional

To recap …

We have provided

- Countable approximations of functional spaces to deal with infinite dimensionality of the state space and with the delay realization
- The notion of δ-IDSS for comparing trajectories with different delay realizations

One question:

Which approximation scheme do we need to consider?

1. Exact bisimulation
2. Approximate bisimulation
3. Approximate and alternating bisimulation

Time-discretization of time-delay systems …

Given the time-delay system Σ and a sampling time $\tau > 0$ consider the following metric transition system

$$T_\tau(\Sigma) = (X_\tau, X_{0\tau}, U_\tau, \longrightarrow_\tau, X_{m\tau}, Y_\tau, H_\tau)$$

where:

- $X_\tau = X_{m\tau} = \chi$
- $X_{0\tau} = \{\xi_0\}$
- $U_\tau = \boldsymbol{U}_\tau \times \boldsymbol{D}_\tau$
- $\xi \xrightarrow{(u,\Delta)}_\tau \xi'$, if $\boldsymbol{x}_\tau(\xi, u, \Delta) = \xi'$
- $Y_\tau = \chi$
- $H_\tau(\xi) = \xi$, for all $\xi \in \chi$

and

- $\boldsymbol{U}_\tau$ is the collection of constant functions u: $[-r, -r+\tau[\rightarrow U$ in $\boldsymbol{U}$ for which $\boldsymbol{x}_t(\xi, u, \Delta)$ is defined for any $\Delta \in \boldsymbol{D}_\tau$
- $\boldsymbol{D}_\tau$ is the collection of functions Δ: $[0, \tau[\rightarrow [\Delta_{min}, \Delta_{max}]$ in $\boldsymbol{D}$ for which $\boldsymbol{x}_t(\xi, u, \Delta)$ is defined for any $u \in \boldsymbol{U}_\tau$

Symbolic models

Given

- a sampling time $\tau > 0$
- a state space quantization $\lambda_X > 0$
- an input space quantization $\lambda_U > 0$
- a delay space quantization $\lambda_D > 0$

define the following metric transition system

$$T_q(\Sigma) = (X_q, X_{0q}, U_q, \longrightarrow_q, X_{mq}, Y_q, H_q)$$

where $q = (\tau, \lambda_X, \lambda_U, \lambda_D)$ and

- $X_q = X_{mq} = A_\chi(\lambda_X)$
- $X_{0q} = \{\xi_{0q}\}$ such that $\xi_{0q} \in A_\chi(\lambda_X)$ and $\|\xi_0 - \xi_{0q}\|_\infty \leq \lambda_X$
- $U_q = A_{\boldsymbol{U}_\tau}(\lambda_U) \times A_{\boldsymbol{D}_\tau}(\lambda_D)$
- $\xi \xrightarrow{(u,\Delta)}_q \xi'$, if $\|\xi' - \boldsymbol{x}_\tau(\xi, u, \Delta)\|_\infty \leq \beta(\lambda_X, \tau) + \gamma_D(\lambda_D) + \lambda_X$
- $Y_q = \chi$
- $H_q(\xi) = \xi$, for all $\xi \in X_q$

and A_χ, $A_{\boldsymbol{U}_\tau}$ and $A_{\boldsymbol{D}_\tau}$ are countable approximations of $Reach_\tau(\Sigma)$, $\boldsymbol{U}_\tau$ and $\boldsymbol{D}_\tau$

Existence of symbolic models

Theorem

Consider a time-delay system Σ and a desired accuracy $\mu > 0$.

Suppose:

- Σ is δ-IDSS and choose $\tau > 0$ s.t. $\beta(\mu, \tau) < \mu$.
- Existence of countable approximations A_X and A_D of $Reach_\tau(\Sigma)$ and $\boldsymbol{D}_\tau$.

Then, for any $\lambda_X > 0$, $\lambda_U > 0$ and $\lambda_D > 0$ satisfying

$$\beta(\mu, \tau) + \gamma_U(\lambda_U) + \gamma_D(\lambda_D) + \max\{\beta(\lambda_X, \tau), \gamma_D(\lambda_D)\} + \lambda_X \leq \mu$$

transition systems $T_\tau(\Sigma)$ and $T_q(\Sigma)$ are alternatingly approximately bisimilar with accuracy μ

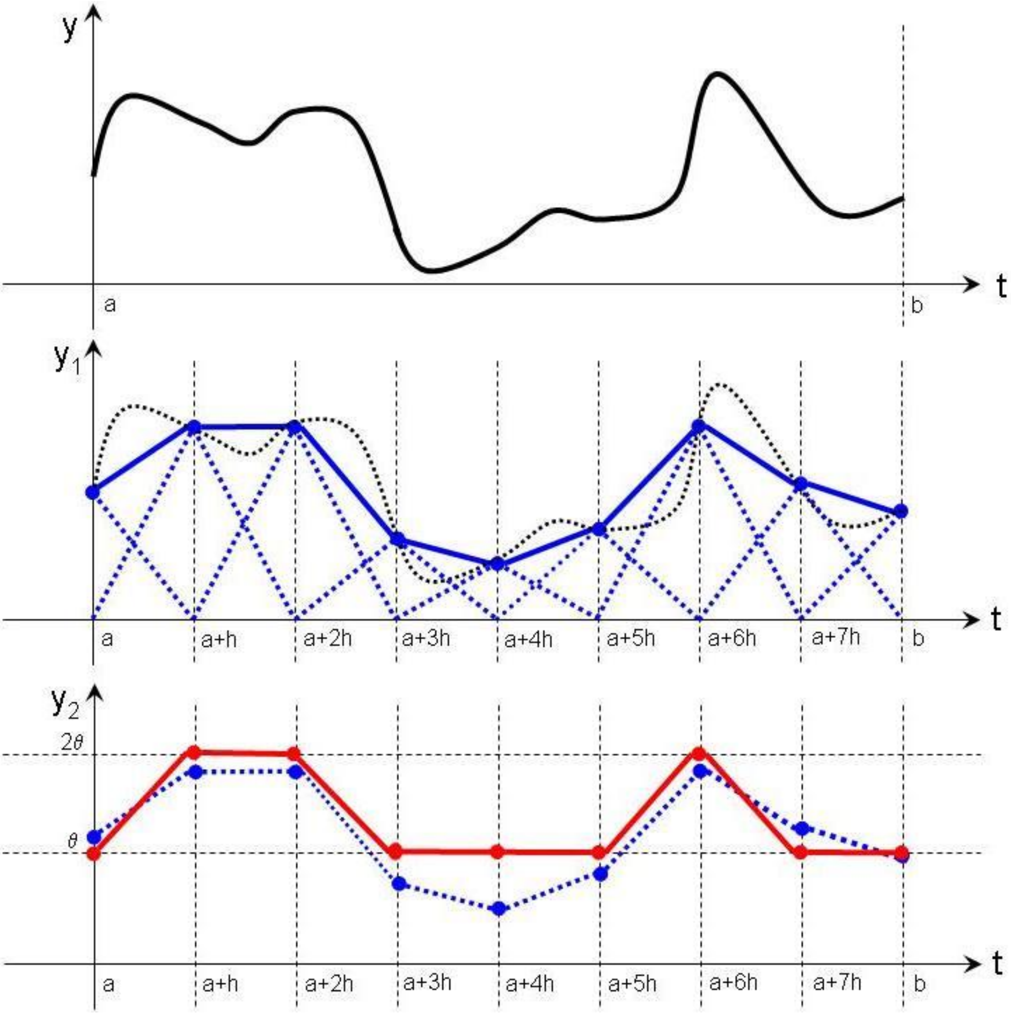
Remark: The «completeness property»

If a control strategy exists for enforcing some specification on $T_q(\Sigma)$ then a control strategy exists for enforcing the same specification on $T_\tau(\Sigma)$ up to a given accuracy, and vice versa

… how to compute countable/finite approximations of $Reach_\tau(\Sigma)$ and $\boldsymbol{D}_\tau$?

Computation of countable approximations

How to Approximate $\boldsymbol{Y} \subseteq C^0([a,b],\mathbb{R})$? … Spline Analysis



Approximating error:

$$\Lambda(N,\Theta,M) = h^2\frac{M}{8} + (N+2)\Theta$$

where:

- N number of samples
- $h = \frac{b-a}{N+1} + 1$ time quantization
- Θ space quantization
- M bound on $\|.\|_\infty$ of 2nd derivative

Given λ and M find Θ and N such that:

$$\Lambda(N,\Theta,M) \leq \lambda$$

Approximate $\boldsymbol{Y} \subseteq C^0([a,b],\mathbb{R})$ by $\Psi_{\lambda,M}(\boldsymbol{Y})$

$$\Psi_{\lambda,M}(y) = \sum_{i\in[0,N+1]} \tilde{y}_i s_i(t)$$
$$|\tilde{y}_i - y(a+ih)| \leq \Theta$$

Example

Given

$$\Sigma: \begin{cases} \dot{x}_1(t) = -8\, x_1(t) + \tanh(x_2(t-\Delta(t))) \\ \dot{x}_2(t) = -9x_2(t) + \sin\Big(x_1\big(t-\Delta(t)\big)\Big) + \cos\big(x_2(t)\big)\, u(t-r) \end{cases}$$

with $\Delta_{min} = 10^{-3}$, $\Delta_{max} = 10^{-2}$, $r = 2$ and $d_{min} = 0.2$

Find a control strategy enforcing the following specification robustly wrt to time-delay realizations and with accuracy $\mu = 0.12$

Synchronization specification:
Starting from the origin, remain in the positive orthant for all times, reach the set $X_1 = [0.01, \infty[\times [0.01, \infty[$ in no more than $4s$, stay in the set X_1 for at least $4s$, reach the set $X_2 = [0.02, 0.16] \times [0.02, 0.16]$ and finally remain in X_2 for at least $12s$

Remarks This specification is

- Relevant in multi-agent systems with shared resourses
- Difficult to enforce by using known techniques in time-delay systems

Example

Control strategy designed

$$(0,0) \xrightarrow{186} (4,30) \xrightarrow{-396} (3,22) \xrightarrow{248} (4,31) \xrightarrow{-562} (3,20) \xrightarrow{-546} (3,24) \xrightarrow{-484} (2,20) \xrightarrow{388} (3,21)$$
$$(4,33) \xrightarrow{234} (4,31) \xrightarrow{-220} (3,25) \xrightarrow{542} (4,35) \xrightarrow{-560} (3,19) \xrightarrow{-74} (3,27) \xrightarrow{-142} (3,26)$$

where

$$(n_1, n_2) \xrightarrow{u} (n'_1, n'_2)$$

stands for

$$(n_1\vartheta_X, n_2\vartheta_X) \xrightarrow{u\vartheta_U} (n'_1\vartheta_X, n'_2\vartheta_X)$$

with $\vartheta_X = 0.04$ and $\vartheta_U = 0.0005$

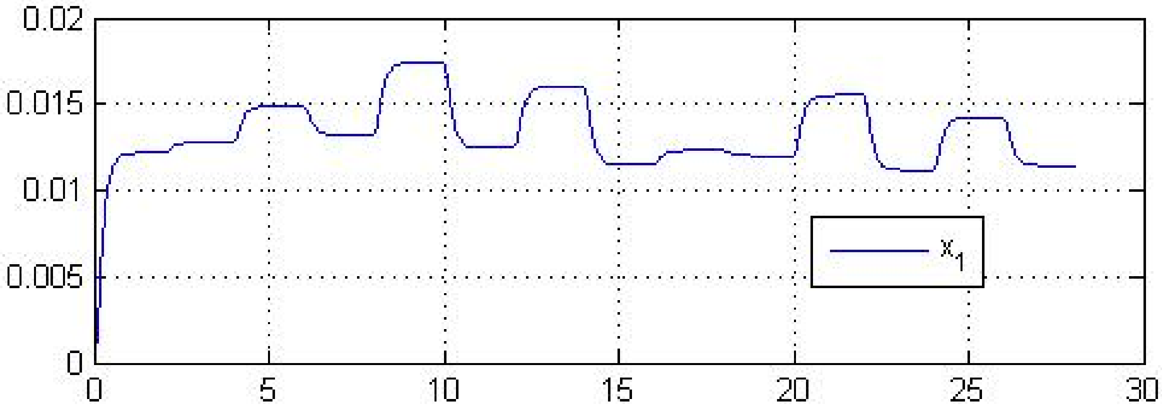
Time of computation

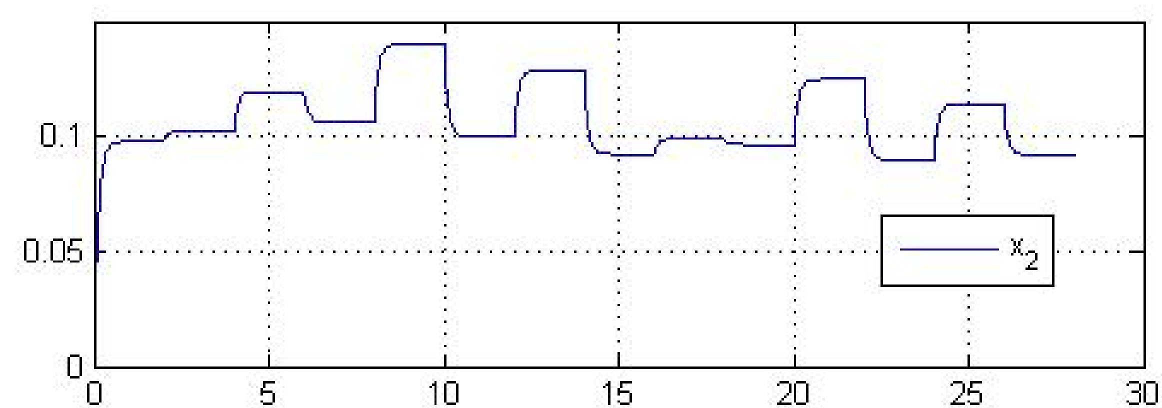
7692s on an Intel Core 2 Duo T5500 at 1.66 GHz

Example

Simulation results

for delay realization $\Delta(t) = \frac{\Delta_{max}+\Delta_{min}}{2} + \frac{\Delta_{max}-\Delta_{min}}{2}\sin(0.01t)$





... specification met !

Conclusions

- Incremental Input-Delay-to-State Stability (δ-IDSS)
- Existence of symbolic models approximating nonlinear control systems with unknown and time-varying delays
- Construction of symbolic models through spline analysis
- Example with synchronization specification