Formal Methods for the Control of Large-scale Networked Nonlinear Systems with Logic Specifications

Basilica di Santa Maria di Collemaggio, L'Aquila (Italy), 1287

Lecture L11:

Symbolic models for possibly unstable nonlinear control systems

Speaker: Giordano Pola

What's new?

- In this lecture we will remove the incremental stability assumption placed on the plant

Tools:

- Incremental forward completeness, and
- Alternating approximate simulation

Lecture based on:

[Zamani et al., TAC12] Zamani, M., Pola, G., Mazo, M., Tabuada, P., Symbolic models for nonlinear control systems without stability assumptions, IEEE Transactions on Automatic Control, 57(7):1804-1809, July 2012

Nonlinear systems

We consider the following nonlinear system:

$$\Sigma\text{:}\ \dot{x}(t) = f(x(t), u(t))$$

where:

- $x(t) \in \mathbb{R}^n$ is the state at time t
- $u(t) \in \mathbf{U} \subseteq \mathbb{R}^m$ is the control input at time t and $\mathbf{U}$ is a finite set
- $u(.) \in \mathcal{U}$ where $\mathcal{U}$ is the class of piecewise continuous functions from $\mathbb{R}_0^+$ to $\mathbf{U}$

We denote by $\boldsymbol{x}(t, x_0, u)$ the state reached by Σ at time t with initial state x_0 and control input u

Incremental forward completeness

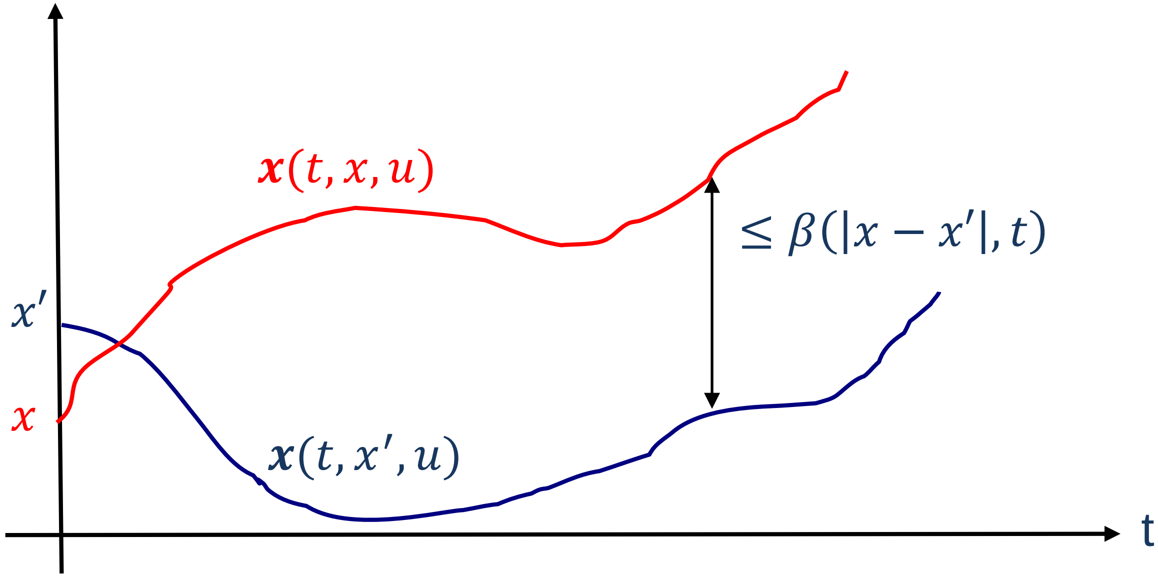
Definition

Nonlinear system Σ is incrementally forward complete (δ-FC) if it is forward complete and there exists a continuous function

$$\beta\colon \mathbb{R}_0^+ \times \mathbb{R}_0^+ \to \mathbb{R}^+$$

such that for every $s \in \mathbb{R}^+$, function $\beta(.,s)$ belongs to class K and for any $x, x' \in \mathbb{R}^n$ and any $u(.) \in \mathcal{U}$ the following inequality holds

$$|\boldsymbol{x}(t,x,u) - \boldsymbol{x}(t,x',u)| \leq \beta(|x - x'|, t)$$



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Remarks:

- At present, a nonlinear system that is FC but not δ-FC is not known
- When β is KL, definition above boils down to δ-GAS from which

$$\delta\text{-GAS} \Rightarrow \delta\text{-FC}$$

while the converse implication is not true in general

Incremental forward completeness

Definition

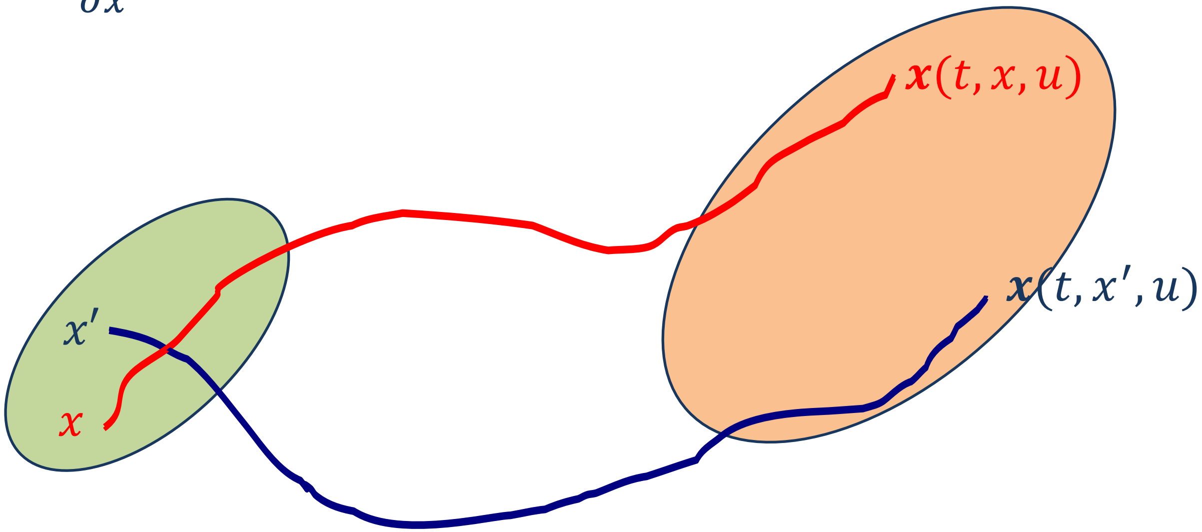
A smooth function

$$V: \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_0^+$$

is said to be a δ-FC Lyapunov function for Σ if there exist K_∞ functions α_1 and α_2 and $\kappa \in \mathbb{R}$ such that for any $x, x' \in \mathbb{R}^n$ and $u \in \mathbf{U}$

(i) $\alpha_1(|x - x'|) \leq V(x, x') \leq \alpha_2(|x - x'|)$,

(ii) $\frac{\partial V}{\partial x} f(x, u) + \frac{\partial V}{\partial x'} f(x', u) \leq \kappa V(x, x')$



Incremental forward completeness

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Theorem

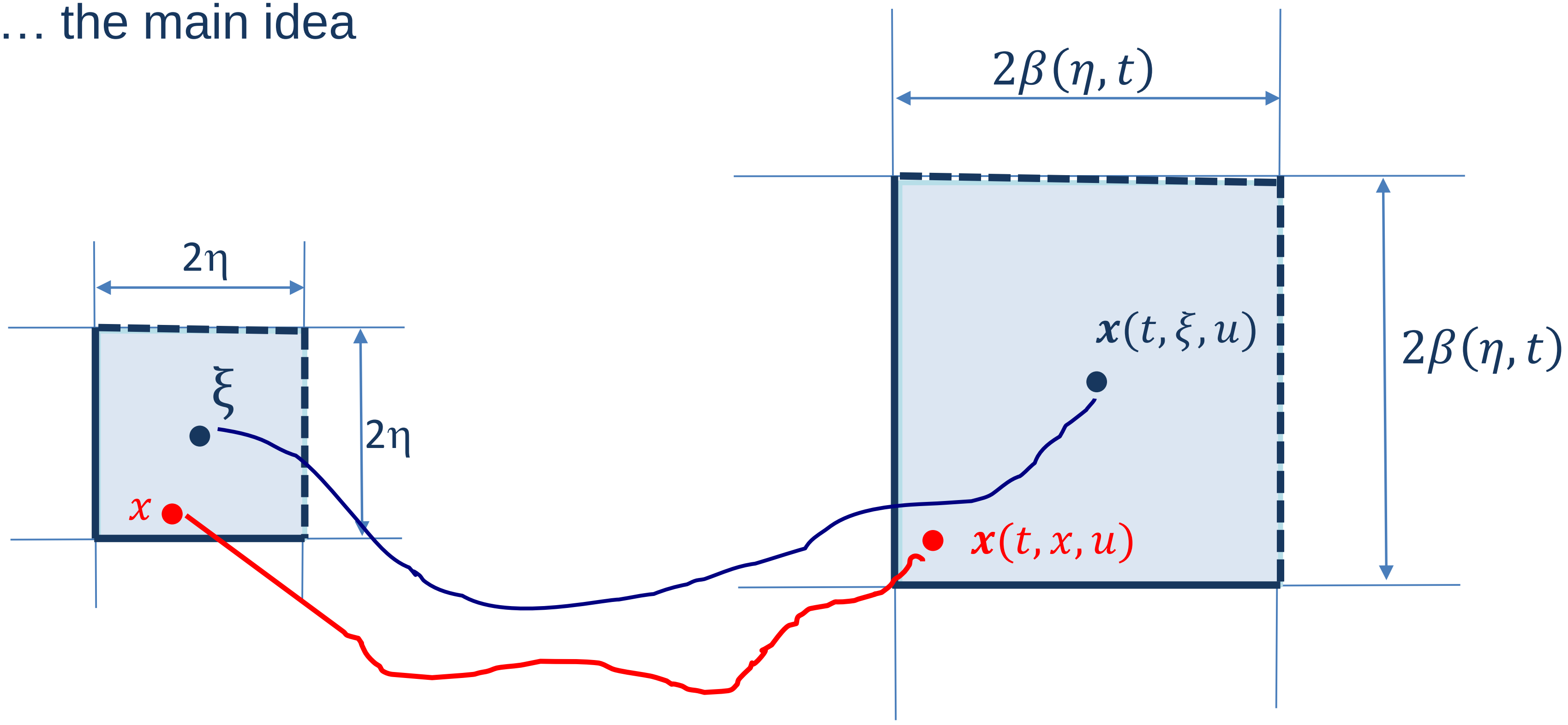
Nonlinear system is δ-FC if it admits a δ-FC Lyapunov function

Remark

A δ-FC Lyapunov function with $\kappa < 0$ is a δ-GAS Lyapunov function

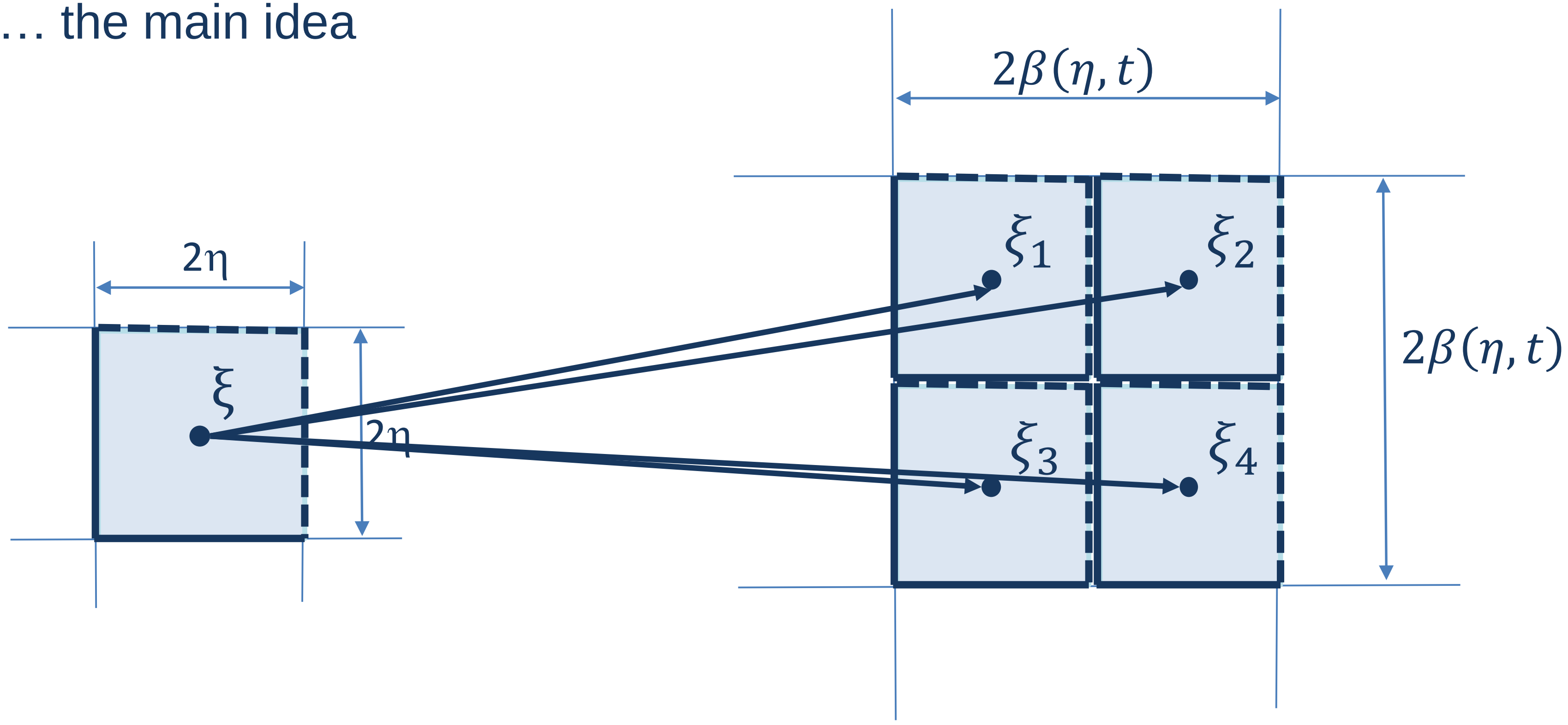
Symbolic models

… the main idea



Symbolic models

… the main idea



… translating instability
into nondeterminism
in the symbolic model !

Time-discretization

Given the nonlinear system Σ and a sampling time $\tau > 0$ consider the following transition system

$$T_\tau(\Sigma) = (X_\tau, X_{0\tau}, U_\tau, \longrightarrow_\tau, X_{m\tau}, Y_\tau, H_\tau)$$

where:

- $X_\tau = X_{0\tau} = X_{m\tau} = \mathbb{R}^n$
- $U_\tau =$ collection of constant control inputs from $[0, \tau[$ to $\mathbf{U}$
- $\xi \overset{u}{\longrightarrow}_\tau \xi'$, if $\boldsymbol{x}(\tau, \xi, u) = \xi'$
- $Y_\tau = \mathbb{R}^n$
- $H_\tau(\xi) = \xi$, for all $\xi \in \mathbb{R}^n$

Remark

$T_\tau(\Sigma)$ is metric, deterministic and alive

Symbolic models

Given

- a sampling time $\tau > 0$
- a state space quantization $\eta > 0$

define the following metric transition system

$$T_{\tau,\eta}(\Sigma) = (X_{\tau,\eta}, X_{0,\tau,\eta}, U_{\tau,\eta}, \longrightarrow_{\tau,\eta}, X_{m,\tau,\eta}, Y_{\tau,\eta}, H_{\tau,\eta})$$

where

- $X_{\tau,\eta} = X_{0,\tau,\eta} = X_{m,\tau,\eta} = [\mathbb{R}^n]_\eta$
- $U_{\tau,\eta} = U_\tau$
- $\xi \overset{u}{\longrightarrow}_{\tau,\eta} \xi'$, if $|\xi' - \boldsymbol{x}(\tau, \xi, u)| \leq \beta(\eta, \tau) + \eta$
- $Y_{\tau,\eta} = \mathbb{R}^n$
- $H_q(\xi) = \xi$, for all $\xi \in \mathbb{R}^n$

Question

Is $T_{\tau,\eta}(\Sigma)$ symbolic, metric, deterministic and alive?

Symbolic models

Theorem

Consider the nonlinear system Σ and a desired accuracy $\mu > 0$.
Suppose that Σ is δ-FC. Then for any $\tau > 0$ and $\eta \leq \mu$ we have

- Transition system $T_{\tau,\eta}(\Sigma)$ is alternatingly μ-simulated by $T_{\tau}(\Sigma)$
- Transition system $T_{\tau}(\Sigma)$ is μ-simulated by $T_{\tau,\eta}(\Sigma)$

or equivalently

$$T_{\tau,\eta}(\Sigma) \preccurlyeq_{\mu}^{alt} T_{\tau}(\Sigma) \preccurlyeq_{\mu} T_{\tau,\eta}(\Sigma)$$

Remarks

- If a control strategy enforcing a (logic) specification on $T_{\tau,\eta}(\Sigma)$ exists then a control strategy exists which enforces the same specification on $T_{\tau}(\Sigma)$ within any desired accuracy
- Do we have the «compleness property» in the control here?

Example: Robot motion planning

Robot motion planning

A widely used model in the literature is the unicycle model:

$$\Sigma : \begin{cases} \dot{x} = v_0 \dfrac{\cos(\alpha + \theta)}{\cos(\alpha)} \\ \dot{y} = v_0 \dfrac{\sin(\alpha + \theta)}{\cos(\alpha)} \\ \dot{\theta} = \dfrac{v_0}{b} \tan(\delta) \end{cases}$$

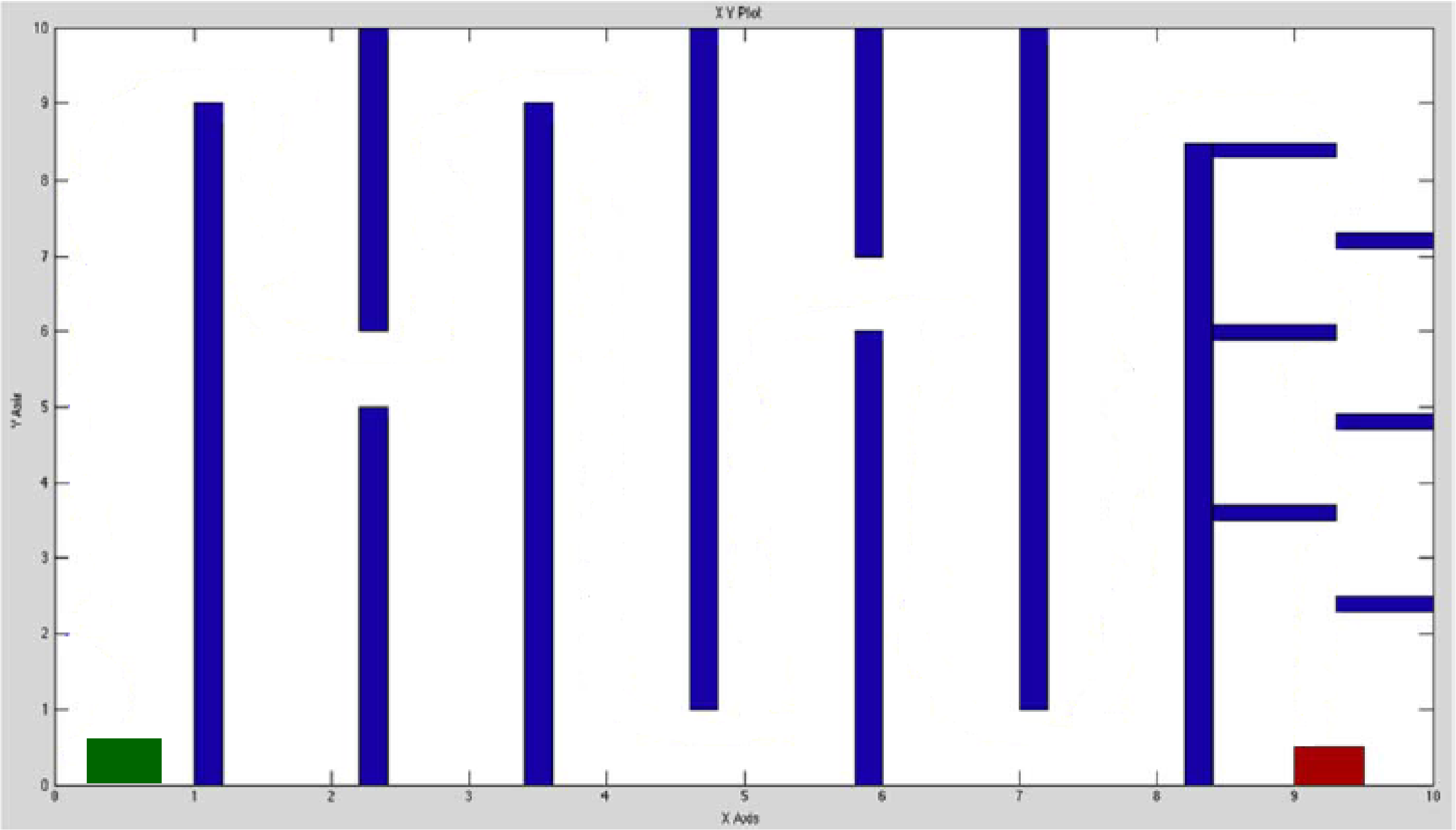
with $\alpha = arctan(a \cdot \tan(\delta)/b)$

We suppose:

- a = 0.5, b = 1
- u=(v_0, δ) $\in$ U = [-1,1] x [-1,1]
- Control signals are piecewise constant
- (x, y, θ) $\in$ X = [0, 10] x [0, 10] x [-π, π]

Control problem

Specification: Starting from the green box, reach the red box while avoiding the blue obstacles



Desired accuracy: $\mu = 0.25$

Construction of the symbolic model

- **Check the δ-FC property (the unicycle model is not δ-GAS!)**

It can be shown that the unicycle model is δ-FC with

$$\beta(r, s) = (1 + 1.267s)r$$

Construction of the symbolic model

- **Check the δ-FC property (the unicycle model is not δ-GAS!)**

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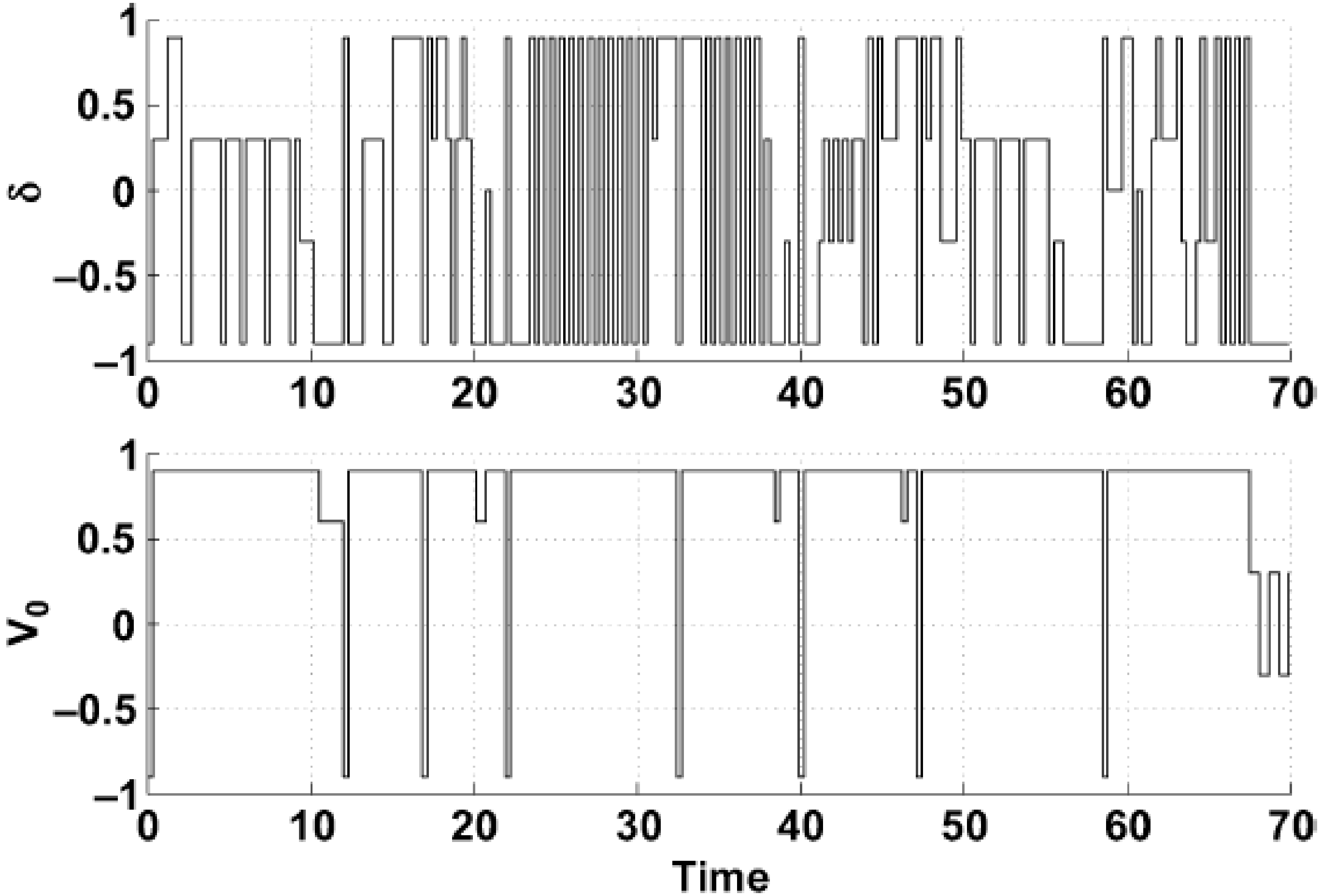
- **Design of quantization parameters**

  For $\mu = 0.25$ and $\tau = 0.3$ one can pick $\eta = 0.2$

- By using tool PESSOA, a symbolic model has been constructed which consists of
  - 91,035 states,
  - 49 inputs,
  - 34,020,088 transitions

Design of the controller

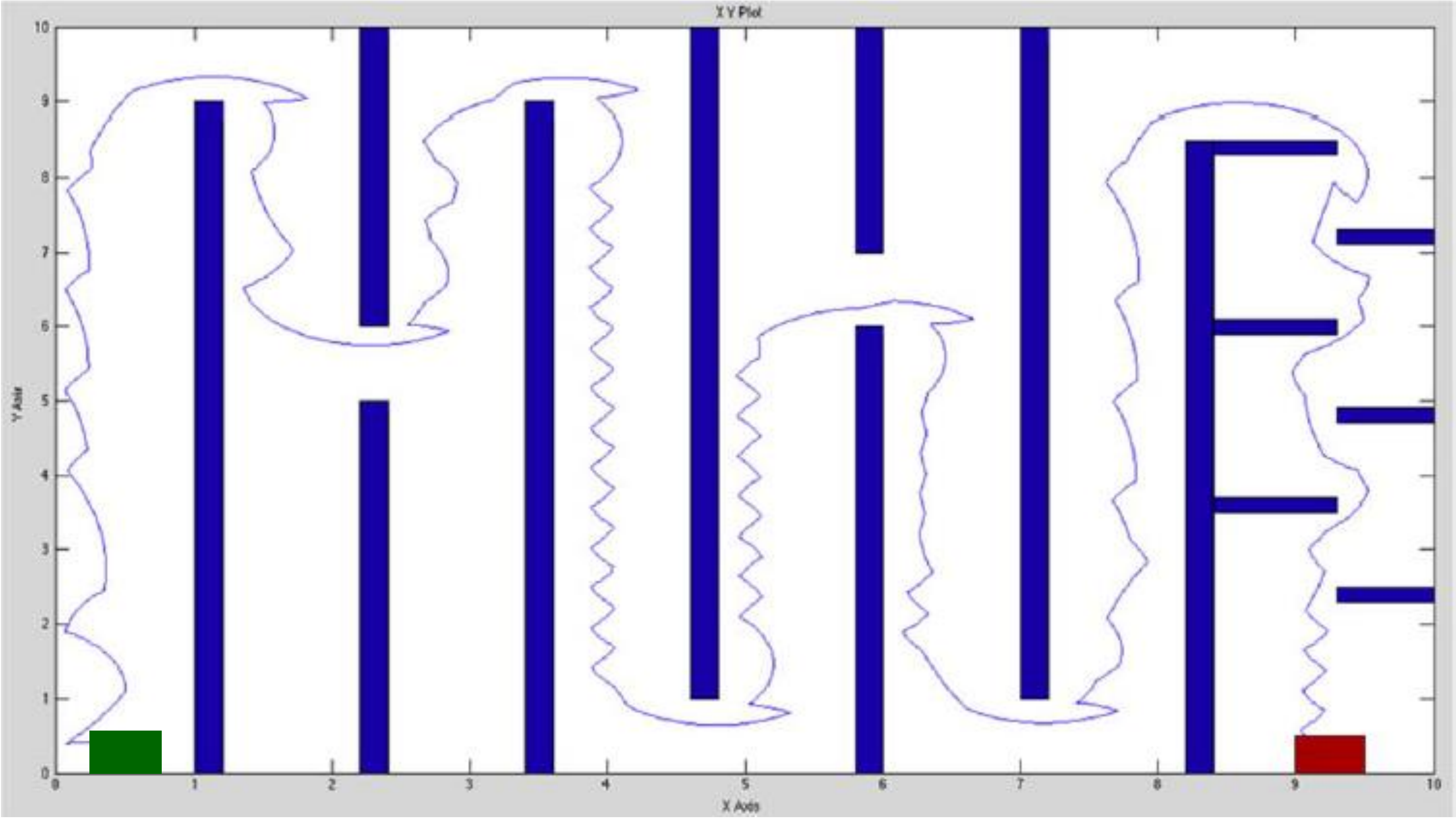
Controller synthesized by using standard algorithms in game theory:



Time of computation for the symbolic model and the controller is 535.987s on a laptop with CPU Intel Core 2 Duo @ 2.4 GHz

Validation

Let us apply the control strategy synthesized on the symbolic model to the original continuous system:



... specification fulfilled!

Conclusions

- We proposed symbolic models for nonlinear systems that are δ-FC
- At present a nonlinear system that is FC but not δ-FC is not known
- If a control strategy enforcing a (logic) specification on $T_{\tau,\eta}(\Sigma)$ exists then a control strategy exists which enforces the same specification on $T_{\tau}(\Sigma)$ within any desired accuracy
- The «compleness property» in the control is in general lost here
- Here, no disturbance inputs and state time-delays are considered However, in

[Pola et al., CDC16] Pola, G., Borri, A., Di Benedetto, M.D., On Symbolic Control Design of Discrete–Time Nonlinear Systems with State Quantized Measurements, 55th IEEE Conference on Decision and Control, Las Vegas, USA, December 2016, pp. 6571-6576

also disturbances are considered