SIDRA 2017
Formal Methods for the Control of Large-scale Networked Nonlinear Systems with Logic Specifications

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Homeworks for ECTS accreditation

Give the solution to at least one homework between homeworks no. 4 and no. 8 and at least one homework among the remaining ones below:

Homework no. 1: Consider the system described by the following scalar ODE

$$\dot{x}(t) = -3x(t) - tanh(x(t)) + u(t). \tag{1}$$

By using Definition 4 (ISS Definition) and Theorem 6 (ISS Characterization) in lecture notes L2-L9, prove that the system described by (1) is ISS.

Hint: Make use of Lyapunov function $V : \mathbb{R} \to \mathbb{R}_0^+$ defined, for $x \in \mathbb{R}$, as $V(x) = x^2$, make use of inequality $2ab \leq a^2 + b^2$, $a, b \in \mathbb{R}$, and take into account that $x \cdot tanh(x) \geq 0$, $x \in \mathbb{R}$.

Homework no. 2: Consider the system described by the scalar ODE (1). By using Definition 3 (δ-GAS Definition) and Theorem 4 (δ-GAS Characterization) in lecture notes L2-L9, prove that the system described by (1) is δ-GAS.

Hint: Make use of Lyapunov function $V : \mathbb{R} \times \mathbb{R} \to \mathbb{R}_0^+$ defined, for $x, y \in \mathbb{R}$, as $V(x, y) = (x-y)^2$, and take into account that $(x-y)(tanh(x)-tanh(y)) \geq 0$, $x, y \in \mathbb{R}$.

Homework no. 3: Consider the system described by the scalar ODE (1). By using Definition 5 (δ-ISS Definition) and Theorem 7 (δ-ISS Characterization) in lecture notes L2-L9, prove that the system described by (1) is δ-ISS.

Hint: Make use of Lyapunov function $V : \mathbb{R} \times \mathbb{R} \to \mathbb{R}_0^+$ defined, for $x, y \in \mathbb{R}$, as $V(x, y) = (x - y)^2$, take into account that $(x - y)(tanh(x) - tanh(y)) \geq 0$, $x, y \in \mathbb{R}$, make use of inequality $2ab \leq a^2 + b^2$, $a, b \in \mathbb{R}$, and follow the same reasoning as in Example 6 of lecture notes L2-L9.

Homework no. 4: Construct a transition system T that is nonblocking and not co–accessible.

Hint: Use the definitions of nonblocking and co–accessible transition systems given in lecture L3.

Homework no. 5: Given the regular expression:

$$(a+b)(c+d)^*(e+f)$$

find:

- a regular language L corresponding to the regular expression above;
- a symbolic transition system

$$T = (X, X_0, U, \longrightarrow, X_m, Y, H),$$

with marked input language $\mathcal{L}^u_m(T) = L$;
- construct the dual transition system of T.

Hint: Just generalize Example 1 of lecture L7a.

Homework no. 6: Consider the system described by the following scalar RFDE

$$\dot{x}(t)) = -4x(t) + x(t-\Delta) - x^3(t) + u(t). \tag{2}$$

By using Definition 6 (0-GAS Definition for RFDEs), Definition 9 (functional derivative) and Theorem 9 (0-GAS Characterization for RFDEs) in lecture notes L2-L9, prove that the system described by (2), with $u(\cdot) \equiv 0$, is 0-GAS.

Hint: Rewrite system (2) in the form given in (7) of lecture notes L2-L9, make use of the Lyapunov-Krasovskii functional $V : \mathcal{C} \to \mathbb{R}^+_0$ defined, for $\phi \in \mathcal{C}$, as $V(\phi) = \phi^2(0) + \int_{-\Delta}^{0} \phi^2(\theta)d\theta$, compute the functional derivative recalling standard theorems on integration (most terms have limit 0), make use of Young's inequality $2ab \leq a^2 + b^2$, $a, b \in \mathbb{R}$.

Homework no. 7: Let $\mathbf{Y}$ be the space of constant functions from $[a,b]$ to $\mathbb{R}$ with $a, b \in \mathbb{R}$ and $a < b$. For any $\lambda > 0$ consider the space $\mathcal{A}_{\mathbf{Y}}(\lambda)$ of constant functions from $[a,b]$ to $2\lambda\mathbb{Z}$. Show that $\mathcal{A}_{\mathbf{Y}}$ is a countable approximation of $\mathbf{Y}$.

Hint: Just apply definition in slide no. 6 of lecture L10.

Homework no. 8: Consider a linear system in the standard form:

$$\Sigma : \begin{cases} \dot{x} = Ax + Bu, \\ x(t) \in \mathbb{R}^n, \\ u(t) \in \mathbb{R}^m, \\ t \in \mathbb{R}^+_0 \end{cases} \tag{3}$$

Show that Σ is incrementally forward complete (δ–FC).

Hint: Make use of the explicit solution of the differential equation in (3) and follow the definition of δ–FC given in slide no. 3 of lecture L11.