Port Hamiltonian systems : modelling origins and control

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Motivation

Design of controlled physical systems using network -based modelling

- modular model building : network of subsystems
- physically consistent models : energy, balance equations, power flows ...
- physically consistent numerical simulation schemes
- physically consistent control design

Reminder about stability of dynamical systems

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Reminder about stability of dynamical systems

Consider a dynamical system defined on a domain $D \subset \mathbb{R}^n \ni x$ by the differential equation :

$$\frac{dx}{dt} = f(x)$$

where $f : D \to \mathbb{R}^n$ is a locally Lipschtitz function.
Denote by $\Phi(t, x_0)$ the solution $x(t)$ with initial condition $x_0 \in D$.

Assume that $x^* = 0 \in D$ is an equilibrium point. It is :

- stable if $\forall \varepsilon > 0, \exists \delta > 0$ tel que :
  $\|x_0\| < \delta \Rightarrow \|\Phi(x_0, t)\| < \varepsilon \quad \forall t \geq 0$
- unstable otherwise
- asymptotically stable if it is stable and $\exists \delta > 0$ such that :
  $\|x_0\| < \delta \Rightarrow \lim_{t \to +\infty} \Phi(x_0, t) = 0$

Lyapunov's second method of analysis of stability

Théorème

Let $V : D \to \mathbb{R}$ *be a continuously differentiable* (C^1) *function satisfying :*

(i) V *positive definite :* $V(0) = 0$ *and* $V(x) > 0,\ x \in D \setminus \{0\}$

(ii) $L_f V(x) \leq 0$

then $x^* = 0 \in D$ *is a stable equilibrium point.*

if :

(iii) $L_f V(x) < 0,\ x \in D \setminus \{0\}$

then $x^* = 0 \in D$ *is a asymptotically stable equilibrium point.*

Définition

The function $V(x)$ satisfying these assumptions, is called Lyapunov function.

Domain of attraction

Définition

The domain of attraction of the stable equilibrium point $x^* = 0 \in D$ is :

$$D_{x^*} = \left\{ x \in D \, / \lim_{t \to +\infty} \Phi(t, x) = x^* \right\}$$

Théorème

The level sets of the Lyapunov function :
$\Omega_c = \{x \in \mathbb{R}^n \, / \, V(x) < c\}$, *where* $c > 0$ *contained in the domain of definition D and bounded :*
(i) are positively invariant : $\forall x_0 \in \Omega_c \, : \, \phi(t, x_0) \in \Omega_c, \forall t \in \mathbb{R}_+^*$
(ii) give an estimate of the domain of attraction : $\Omega_c \subset D_{x^*}$

Global stability : theorem of Barbashin-Krasovskii

Définition

The equilibrium point $x^*$is said globally asymptotically stable if it is asymptotically stable with domain of attraction $D_{x^*} = \mathbb{R}^n$.

Théorème

Let $V : D \to \mathbb{R}$ be a continuously differentiable (C^1) function satisfying :
(i) V positive definite : $V(0) = 0$ and $V(x) > 0,\ x \in D \setminus \{0\}$
(ii) V is radially unbounded : $\lim_{\|x\| \to +\infty} V(x) = +\infty$
(iii) $L_f V(x) < 0$
then $x^* = 0 \in D$ is a globally asymptotically stable equilibrium point.

Unstability : Chetaev's theorem

Théorème

Let $V : D \to \mathbb{R}$ *be a continuously differentiable* (C^1) *function satisfying :*
(i) $V(0) = 0$
(ii) $\forall \varepsilon \in R_+^*,\ \exists x \in D \setminus \{0\},\ \|x\| < \varepsilon \text{ and } V(x) > 0$
and consider a positive real number $r > 0$ *and the set* $U_r = \{x \in D \,/\, \|x\| < r \text{ and } V(x) > 0\}$, *then* $L_f V(x) > 0, \forall x \in U_r$ *implies that* $x^* = 0 \in D$ *is an unstable equilibrium point.*

LaSalle's theorem

Théorème

Let $\Omega \subset D$ be a compact set, positively invariant. And let $V : D \to \mathbb{R}$ be a continuously differentiable (C^1) function satisfying such that $L_f V(x) \leq 0$ in Ω. Define
(i) the set of states $E = \{x \in \Omega ; L_f V(x) = 0\}$ and
(ii) $M \subset E$ the largest set in E which is positively invariant ($\forall x_0 \in M : \phi(t, x_0) \in M, \forall t \in \mathbb{R}_+^*$), then :

$$\lim_{t \to +\infty} dist(\Phi(x_0, t), M) = 0$$

Note that here $V(x)$ needs not to be positive !

The the set Ω may be the level set of a Lyapunov function if it is bounded (hence compact).

Construction of Lyapunov functions

It might be difficult to find a Lyapunov function.

Construction using the gradient method : one looks for a vector field $k(x)$

- deriving from a potential : $k(x) = \frac{\partial V}{\partial x}$, i.e. satisfying : $\frac{\partial k_i}{\partial x_j} = \frac{\partial k_j}{\partial x_i}$
- such that $L_f V(x) = k^t(x) f(x) < 0$

and compute

$$V(x) = \int_0^{x_1} k_1(\chi_1, 0.., , 0)\, d\chi_1 + \int_0^{x_2} k_1(x_1, \chi_2 0.., , 0)\, d\chi_2 + ..$$

finally check if $V(x)$ is positive definite.

Stabilization by control Lyapunov functions

Stabilization by control Lyapunov functions

Asymptotic stabilization of a stable control system

Consider a nonlinear control system affine in the inputs :

$$\frac{dx}{dt} = f(x) + \sum_{i=1}^{m} g_i(x)\, u_i$$

where the drift vector field $f : D \subset \mathbb{R}^n \to \mathbb{R}^n$ and the input vector fields $g_i(x),\, i = 1,..,m$ are smooth.

Assume that the drift vector field $f(x)$ has a stable equilibrium point $x^* = 0 \in D$ with Lyapunov function $V(x)$.

Théorème

The state feedback : $u_i(x) = -L_{g_i} V(x)$ locally asymptotically stabilizes the equilibrium point if :
(i) the function $V(x)$ satisfies $L_f V(x) \leq 0,\, x \in D$ and $\frac{\partial V}{\partial x}(x) \neq 0,\, x \in D$
(ii) the accessibility distribution
$\mathscr{D}(x) = span\{f(x),\, ad_f^k g_i(x),\, i = 1, \ldots, m,\, k \in \mathbb{N}\}$ satisfies : $dim \mathscr{D} = n$

Lyapunov functions and stabilization

In the previous results the nonlinear control modified the time-variation of the original Lyapunov function in closed-loop :

$$\frac{dV}{dt} = L_f V(x) - \sum_{i=1}^{m} [L_{g_i} V(x)]^2$$

and hence restrict the invariant set to subset of :

$$E = \{x \in D \,/\, L_f V(x) = 0 \quad \text{and} \quad L_{g_i} V(x) = 0, i = 1, .. m\}$$

However remains the problems :

- find a open-loop Lyapunov function
- design a closed-loop Lyapunov function in order to stabilize unstable points.

Optimal stabilizing control and inverse design

Optimal stabilizing control and inverse optimal control design

Optimal control problem

Control Lyapunov functions may be defined through the optimal control problem.

Définition

Consider the nonlinear system : $\dot{x} = f(x) + g(x)\,u$, find a feedback $u(x)$ such that :

(a) the closed-loop system is asymptotically stable at $x^* = 0$

(b) the feedback minimizes the cost functional :

$$J = \int_0^{+\infty} \left(l(x) + u^T R(x)\, u \right) dt$$

where $l(x) \geq 0$ and $R(x) = R^T(x) > 0$

Optimal stabilizing control

Théorème

Assume that there is a C^1 semi-definite real function $V(x)$ satisfies $V(0) = 0$ and the Hamilton-Jacobi-Bellman equation :

$$l(x) + L_f V(x) - \frac{1}{4} (L_g V(x))^t R(x)^{-1} (L_g V(x)) = 0$$

such that the feedback : $u^*(x) = -\frac{1}{2} R(x)^{-1} L_g V(x)$ stabilizes the system at $x^* = 0$. Then $u^*(x)$ solves the optimal control problem and $V(x)$ is the optimal value function.

The function $V(x)$ may often be checked to be a Lyapunov function, depending on the choice of the cost function J.

Inverse optimal control design

One may invert the design process by choosing :

- a desired closed-loop Lyapunov function $V(x)$
- a symmetric positive definite matrix $R(x)$

such that the control $u(x) = -\frac{1}{4} R(x)^{-1} L_g V(x)$ ensures :
$\frac{dV}{dt} = L_f V(x) - \frac{1}{4} (L_g V(x))^T R(x)^{-1} L_g V(x) \leq 0$.

amounts to defining the control :
$u^*(x) = 2u(x) = -\frac{1}{2} R(x)^{-1} L_g V(x)$ as the solution of some optimal control problem with :

- $l(x) = \frac{1}{4} (L_g V(x))^t R(x)^{-1} (L_g V(x)) - L_f V(x) \geq 0$

which remains to be checked.

Control Lyapunov function

The problem is to generate a control Lyapunov function allowing to stabilize the system.

Théorème

A continuously differentiable (C^1) positive definte [event. radially unbounded] function $V(x)$ is a control Lyapunov function for the system : $\dot{x} = f(x) + g(x)u$ *if :* $L_g V(x) = 0 \quad \Rightarrow \quad L_f V(x) < 0$

This means that its variation may be rendered negative definite by feedback.

There exist different method to construct such functions : backstepping ....

Dissipative control systems

Dissipative control systems

Dissipative control systems

Disipative control systems are :

- related to the models of physical systems in the sense that they explicitely write a balance equation
- the fundamental extension of Lyapunov-stable dynamical systems to control systems

Consider a nonlinear control system :

$$\Sigma \left\{ \begin{array}{rcl} \frac{dx}{dt} & = & f(x, u) \\ y & = & h(x, u) \end{array} \right.$$

with state $x(t) \in \mathbb{R}^n$, input $u(t) \in \mathbb{R}^m$, output $y(t) \in \mathbb{R}^m$,
$f \in C^\infty(\mathbb{R}^n \times \mathbb{R}^m, \mathbb{R}^n)$ and $h \in C^\infty(\mathbb{R}^n \times \mathbb{R}^m, \mathbb{R}^m)$
and denote by $\phi(x_0, u(.), t)$ the solution of the system with initial condition x_0 and input $u(t)$.

Dissipative (control) systems : definition

Définition

The nonlinear system Σ is a dissipative system with :

(i) storage function $S \in C^{\infty}(\mathbb{R}^n, \mathbb{R})$, a positive function

(ii) supply rate $s \in C^{\infty}(\mathbb{R}^m \times \mathbb{R}^m, \mathbb{R})$

iif it satifies the dissipation inequality for any solution $x(t) = \phi(x_0, u(.), t) \quad t > 0$ and time instants $t_1 \geq t_0$:

$$S(x(t_1)) \leq S(x(t_0)) + \int_{t_0}^{t_1} s(u(t), y(t))\, dt$$

It is said lossless if the equality is satisfied :

$$S(x(t_1)) = S(x(t_0)) + \int^{t_1} s(u(t), y(t))\, dt$$

Passive systems

Définition

Depending on the supply rate $s(u(t), y(t))$ the dissipative system is said :

(i) passive if : $s(u(t), y(t)) = y^t u$

(i) strictly input passive if : $s(u(t), y(t)) = y^t u - \delta \|u\|^2$

(i) strictly output passive if : $s(u(t), y(t)) = y^t u - \varepsilon \|y\|^2$

The physical interpretation is :

- inputs and outputs are impedance variables
- the disspativity inequality is associated with energy balance :

$$\underbrace{S(x(t_1))}_{\text{energy at } t_1} \leq \underbrace{S(x(t_0))}_{\text{energy at } t_0} + \int_{t_0}^{t_1} \underbrace{y^t u}_{\text{power}} \, dt$$

Dissipative systems with L_2-gain

Définition

A dissipative system is said to have L_2-gain $\leq \gamma$ if it admits the supply rate :

$$s(u(t), y(t)) = \frac{1}{2}\gamma \|u\|^2 - \|y\|^2$$

It is said inner if it is lossless with $s(u(t), y(t)) = \frac{1}{2}\|u\|^2 - \|y\|^2$.

It is related to the scattering variables and the balance equation :

$$\underbrace{S(x(t_1))}_{\text{energy at } t_1} \leq \underbrace{S(x(t_0))}_{\text{energy at } t_0} + \underbrace{\int_{t_0}^{t_1} \underbrace{\|u\|^2}_{\text{incoming power}} - \underbrace{\|y\|^2}_{\text{outgoing power}} dt}_{\text{supplied energy}}$$

Dissipative systems for control

Dissipative systems:

- may be stabilized using the storage function as Lyapunov function
- are closed under feedback interconnection, for instance with passive controller.

One may also construct a closed-loop Lyapunov function $V_c(x)$ by finding feedback such that:

$$V_c(x) = S(x) - S(x^*) - \int_{t_0}^{t_1} s(u(x), h(x))\, dt$$

Port Hamiltonian are endowed with additional structure which allows a methodology for the construction of such feedbacks.

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Use formulation as port Hamiltonian systems

In the sequel we shall use a decompostion of the drift vector field stemming from port-based modelling as a port Hamiltonian system is written: and the system:

$$\begin{aligned} \dot{x} &= \underbrace{(J(x) - R(x))\frac{\partial H_0}{\partial x}}_{=f(x)} + \sum_{i=1}^{m} u_i g_i(x) \\ y_i &= g_i(x)^t \frac{\partial H_0}{\partial x} \qquad \text{port conjugated outputs} \end{aligned}$$

- a smooth Hamiltonian function $H_0(x)$
- a skew-symmetric matrix $J(x) \in \mathbb{R}^{n \times n}$ and a positive symetric matrix $R(x) \in \mathbb{R}^{n \times n}$
- m inputs $u_i \in \mathbb{R}^p$ and outputs $y_i \in \mathbb{R}^p$
- m input *vector fields* $g_i(x) \in \mathbb{R}^n$

Port Hamiltonian systems for modelling and control

Port Hamiltonian systems for modelling and control

Context and motivation

Use physical insight explicitely in the :

- physically-based modelling making use of physical invariants
- physically-based control design : design control Lyapunov functions but also of flows of energy in the system
- simultaneous design of process an control

Synthesis of Irreversible Thermodynamics and Analytical Mathematics

Irreversible Thermodynamics : systems of balance equations (conservation laws with source terms) in interaction by phenomenological laws

Analytical Mechanics : geometry of points or configuration with variational formulation leads to Lagrangian and Hamiltonian formulations

The synthesis of the Thermodynamics and Analytical Mechanics :

- uses extensive variables as state variables (versus configurations and their derivatives)
- uses pairs of conjuguated intensive and (time derivative of) extensive variables
- gives a geometric formulation to the coupling relations between the phenomenological laws and conservation laws

Finally the coupling relations are related to network type of coupling

Lagrangian and symplectic Hamiltonian systems

Lagrangian and symplectic Hamiltonian systems

Lagrangian systems

It is a mechanical perspective to physics !

Definition

A Lagrangian system with external forces is defined by:
(i) configuration manifold $Q = \mathbb{R}^n \ni q$ of the generalized coordinates
(ii) manifold of generalized velocities $TQ = \mathbb{R}^{2n} \ni (q, \dot{q})$ its tangent manifold
(iii) Lagrangian function $L(q, \dot{q})$, from the tangent space TQ to $\mathbb{R}$
and the Lagrangian equations:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}(q, \dot{q})\right) - \frac{\partial L}{\partial q}(q, \dot{q}) = F$$

with $F \in \mathbb{R}^n$ is the vector of generalized forces.

Losslessness of Lagrangian systems with external force

A Lagrangian system with external forces satisfies the following power balance equation:

$$F^T \dot{q} = \frac{dH}{dt}$$

where the Hamiltonian H is obtained by the Legendre transformation of the Lagrangian function $L(q, \dot{q})$ with respect to the generalized velocity $\dot{q}$:

$$H(q,p) = \dot{q}^T p - L(q, \dot{q})$$

where p is the vector of generalized momenta:

$$p(q, \dot{q}) = \left(\frac{\partial L}{\partial \dot{q}}(q, \dot{q}) \right)$$

and the Lagrangian function is assumed to be hyperregular.

Lagrangian control system

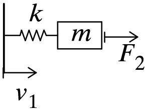
Definition

Consider a configuration space $Q = \mathbb{R}^n$ and its tangent space $TQ = \mathbb{R}^{2n}$, an input vector space $U = \mathbb{R}^p$.
A Lagrangian control systems is defined by a real function $L(q, \dot{q}, u)$ from $TQ \times U$ to $\mathbb{R}$, and the equations:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}(q, \dot{q}, u)\right) - \frac{\partial L}{\partial q}(q, \dot{q}, u) = 0 \qquad (1)$$

Lagrangian control system: example

Consider the harmonic oscillator and assume that the basis of the spring is moving with controlled position u_1 and that there is a force $u_2 = F_2$ exerted on the mass.



The generalized coordinate is the position $q \in \mathbb{R}$ of the mass with respect to an inertial frame and the Lagrangian function:

$$L(q, \dot{q}, u) = \frac{1}{2} m (\dot{q})^2 - \frac{1}{2} k (q - u_1)^2 + q u_2$$

one obtains the Lagrangian control system:

$$m\ddot{q} + k(q - u_1) - u_2 = 0$$

Losslessness of Lagrangian control systems

A Lagrangian control system satisfies the power balance equation:

$$u^T z = \frac{dE}{dt}$$

where:

$$z_i = -\sum_{i=1}^{n} \frac{\partial^2 H}{\partial q_j \partial u_i} \frac{\partial H}{\partial p_j} + \sum_{i=1}^{n} \frac{\partial^2 H}{\partial p_j \partial u_i} \frac{\partial H}{\partial q_j}$$

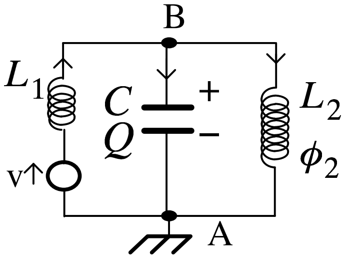
and the real function E is obtained by the Legendre transformation of the Lagrangian function $L(q, \dot{q})$ with respect to the generalized velocity $\dot{q}$ and the inputs and is defined by:

$$E(q, p, u) = H(q, p, u) - u^T \frac{\partial H}{\partial u}$$

with $H(q, p, u) = \dot{q}^T p - L(q, \dot{q}, u)$ where p is the vector of generalized momenta: $p(q, \dot{q}, u) = \left(\frac{\partial L}{\partial \dot{q}}(q, \dot{q})\right)$.

An LC circuit of order 3

Consider the following LC circuit:



what are the generalized coordinates ?

Multiple Lagragian formulations but all are not natural and in correspondence with electrical formulations !

Legendre transformation of a Lagrangian system

Consider a Lagrangian system with external forces and define the vector of generalized momenta:

$$p(q,\dot{q}) = \left(\frac{\partial L}{\partial \dot{q}}(q,\dot{q})\right) \in \mathbb{R}^n$$

and define the Legendre transformation with respect to $\dot{q}$ of the Lagrangian function, called Hamiltonian function:

$$H_0(q,p) = \dot{q}^T p - L(q,\dot{q})$$

then the Lagrangian system with external forces is equivalent to the following symplectic Hamiltonian system:

$$\begin{aligned} \dot{q} &= \tfrac{\partial H_0}{\partial p} \\ \dot{p} &= -\tfrac{\partial H_0}{\partial q} + F \end{aligned}$$

Control Hamiltonian system with external force

There is an alternative way of writing these equations as follows:

$$\begin{pmatrix} \dot{q} \\ \\ \dot{p} \end{pmatrix} = J_s \begin{pmatrix} \frac{\partial H_0(q,p)}{\partial q} \\ \\ \frac{\partial H_0(q,p)}{\partial p} \end{pmatrix} + \begin{pmatrix} 0_n \\ \\ I_n \end{pmatrix} F = J_s \begin{pmatrix} \frac{\partial H(q,p,u)}{\partial q} \\ \\ \frac{\partial H(q,p,u)}{\partial p} \end{pmatrix}$$

where $H(q,p,u) = H_0(q,p) - q^T F$ andJ_s is the following matrix, called symplectic matrix:

$$J_s = \begin{pmatrix} 0_n & I_n \\ -I_n & 0_n \end{pmatrix}$$

his symplectic matrix is the local representation, in canonical coordinates, of the symplectic Poisson tensor field.

Losslessness and Poisson bracket

A Hamiltonian system with external forces satisfies the following power balance equation: $F^T \dot{q} = \frac{dH_0}{dt}$ wich is computed as:

$$\frac{dH_0}{dt} = \left(\frac{\partial H_0(q,p)}{\partial q}, \frac{\partial H_0(q,p)}{\partial p} \right) \underbrace{J_s \begin{pmatrix} \frac{\partial H(q,p,u)}{\partial q} \\ \frac{\partial H(q,p,u)}{\partial p} \end{pmatrix}}_{\begin{pmatrix} \dot{q} \\ \dot{p} \end{pmatrix}}$$

which is the Poisson bracket between the functions H_0 and H:

$$\{H_0, H\}$$

It is the fundamental geometric structure of Hamiltonian systems and their control !.

Hamiltonian systems

Hamiltonian systems

Poisson bracket

Let $\mathscr{M}$ be n-dimensional differentiable manifold with space of smooth real function $C^\infty(\mathscr{M})$.

Definition

A Poisson bracket is a mapping:

$$\begin{array}{rccc} \{,\}: & C^\infty(\mathscr{M}) \times C^\infty(\mathscr{M}) & \mapsto & C^\infty(\mathscr{M}) \\ & (F,G) & \to & \{F,G\} \end{array} \quad \text{satisfying:}$$

bilinearity

skew-symmetry: $\{F,G\} = -\{G,F\}$

Jacobi identities:

$\{F,\{G,H\}\} + \{G,\{H,F\}\} + \{H,\{F,G\}\} = 0, \ \forall F,G,H$

Leibniz rule: $\{F, G\,H\} = \{F,G\}H + G\{F,H\}$

Poisson tensor

It may be shown that the Poisson bracket $\{F, G\}$ depends only on the differentials dF and dG such that:

$$\Lambda(\mathrm{d}F, \mathrm{d}G) = \{F, G\}$$

Extending this tensor to all 1-forms, one associates with any Poisson bracket the Poisson tensor (field):

$$\begin{array}{rcl} \Lambda : \Omega^1(\mathscr{M}) \times \Omega^1(\mathscr{M}) & \rightarrow & C^\infty(\mathscr{M}) \\ (\omega_1, \omega_2) & \mapsto & r \end{array}$$

is two times contravariant, skew-symmetric tensor field and satisfies the Jacobi identities.

Poisson bundle morphism

With the Poisson tensor, one may define a morphism of vector bundle:

$$\begin{array}{rcl} \Lambda^\sharp : T^*\mathscr{M} & \to & T\mathscr{M} \\ \omega & \mapsto & X = \Lambda^\sharp(\omega) \end{array}$$

such that:

$$\Lambda^\sharp(\omega)(\alpha) = \Lambda(\alpha, \omega), \forall \alpha \in \Omega^1(\mathscr{M})$$

Poisson structure matrix

Using some coordinates: $x_1, .., x_n$ the Poisson tensor is defined by the structure matrix J:

$$J_{ij} = \{x_i, x_j\}$$

which is:

1. skew-symmetric
2. satisfies the Jacobi identities, for any $i, j, k = 1, .., n$:

$$\sum_{l=1}^{n} \left(J_{lj}(x) \frac{\partial J_{ik}}{\partial x_l} + J_{li}(x) \frac{\partial J_{kj}}{\partial x_l} + J_{lk}(x) \frac{\partial J_{ji}}{\partial x_l} \right) = 0$$

Poisson bundle morphism in coordinates

Using the structure matrix $J(x)$ in some coordinates: $x_1, .., x_n$ and dual basis for $T\mathscr{M}$ and $T^*\mathscr{M}$, where:

- the 1-forms are: $\omega = \sum_{i=1}^n \omega_i \, \mathrm{d}x_i$ and
- the vector fields: $X = \sum_{i=1}^n \chi_i \frac{\partial}{\partial x_i}$

the bundle morphism $\Lambda^\sharp$ which defines the vector field $X = \Lambda^\sharp(\omega)$ is represented in coordinates by the structure matrix:

$$\begin{pmatrix} \chi_1 \\ .. \\ \chi_n \end{pmatrix}(x) = J(x) \begin{pmatrix} \omega_1 \\ .. \\ \omega_n \end{pmatrix}(x)$$

Jacobi identities

The Jacobi identities are integrability conditions.

The rank of the bracket at $x \in \mathscr{M}$ is: $\mathrm{rank} J(x)$ and is even: $\mathrm{rank} J(x) = 2k$.

If $n = 2k$, then the bracket is symplectic.

Then there exist canonical coordinates $(q_1, .., q_k, p_1, .., p_k, r_1, .., r_l)$ with $l = n - 2k$ such that:

$$J(x) = \begin{bmatrix} 0 & I_k & 0 \\ -I_k & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Casimir functions

The coordinates (functions) r generate Casimir functions $C(x) = \phi(r)$:

$$\{C, F\} = 0, \quad \forall F \in C^{\infty}(M)$$

Casimir functions are defined by the kernel of the Poisson bracket and in coordinates satisfy:

$$\frac{\partial C}{\partial x} \in \ker J(x)$$

If the Jacobi identities are not satisfied: $\{,\}$ is a pseudo-Poisson bracket: $\ker J(x)$ defines a co-distribution which is not integrable and there are no canonical coordinates.

Example: Lie-Poisson bracket for Euler-Poinsot problem

Consider as state variable the rotations in $\mathbb{R}^3$, called special orthogonal group $SO(3)$.

The angular velocities are represented by skew-symmetric matrices in $so(3)$ and endowed with

Lie bracket: $[\omega_1, \omega_2] = \omega_1\, \omega_2 - \omega_2\, \omega_1$.

There is a canonical Lie-Poisson bracket on

the momenta $p \in so^*(3)$: $\{F, G\}(p) = \langle p, [dF, dG]\rangle$

In Plücker coordinates, the structure matrix is:

$$J(p) = \begin{pmatrix} 0 & -p_z & p_y \\ p_z & 0 & -p_x \\ -p_y & p_x & 0 \end{pmatrix}$$

Casimir function is total momentum: $r(x) = p_x^2 + p_y^2 + p_z^2$

Hamiltonian systems w.r.t. Poisson brackets

A Hamiltonian system is defined by:

1. on a differentiable manifold $\mathscr{M} \ni x$ with Poisson bracket $\{,\}$ (and Λ the Poisson tensor)
2. the internal Hamiltonian function $H_0 \in C^\infty(\mathscr{M})$
3. and the differential equations:

$$\dot{x} = \{x, H_0\} = \Lambda^\sharp(\mathrm{d}H_0) \triangleq X_{H_0}$$

Using the structure matrix $J(x)$ in some coordinates $(x_1, .., x_n)$:

$$\frac{dx}{dt} = J(x) \frac{\partial H_0}{\partial x}$$

Invariants of Hamiltonian systems

There are two types of invariants due to the geometry (Poisson bracket):

- Hamiltonian function due to the *skew-symmetry* of the bracket:

$$\frac{dH_0}{dt} = \{H_0, H_0\} = 0$$

- the Casimir functions (non-symplectic case):

$$\frac{dC}{dt} = \{C, H_0\} = 0 \quad \forall H_0 \in C^{\infty}(\mathscr{M})$$

For physical (lossless) systems: the energy is generating function H_0 and is conserved.

Some references

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- V.Guillemin and S.Sternberg, **Symplectic Techniques in Physics**, Cambridge University Press, 1984
- P. Libermann and C.M. Marle, **Symplectic Geometry and Analytical Mechanics**, Reidel, Dordrecht, 1987
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A network origin of Poisson brackets

A network origin of Poisson brackets

A network origin of Poisson brackets

In Mathematical Physics Hamiltonian systems arize from :

- variational calculus, Lagrangian systems and their Legendre transformation : standard Hamiltonian systems defined with respect to symplectic Poisson bracket
- reduction of Hamiltonian systems with symmetries defined on Lie groups : Lie-Poisson bracket.

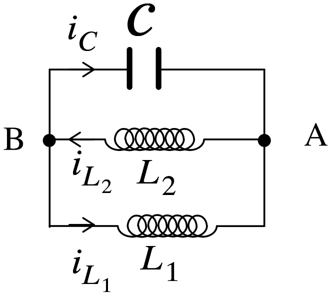
In Control Engineering the Poisson bracket are generated by interconnection structure :

- Kirchhoff's laws, kine-static models, stoichiometry
- coupling between reversible phenomena
- feedback interconnection

An LC circuit without elements in excess

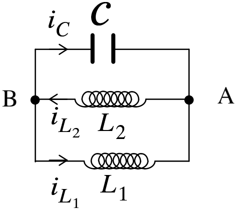
Consider the following LC-circuit:

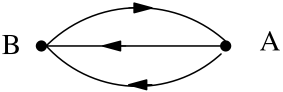
1. composed of 2 inductors and a capacitor in paralell
2. with total energy: $H_0 = \frac{Q_C^2}{2C} + \frac{\phi_{L_1}^2}{2L_1} + \frac{\phi_{L_2}^2}{2L_2}$



An LC circuit: Kirchoff's laws and Poisson bracket

Consider the spanning tree consiting of the capacitor $\{C\}$:





Kirchhoff's laws: $\begin{pmatrix} i_C \\ v_{L_1} \\ v_{L_2} \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}}_{J} \begin{pmatrix} v_C \\ i_{L_1} \\ i_{L_2} \end{pmatrix}$ define a constant skew-symmetric matrix J.

An LC circuit: Poisson bracket and Tellegen's theorem

Virtual power for any pair of co-energy variables:
skew-symmetric tensor:

$$\begin{aligned}\Lambda\left(\begin{pmatrix} v_C \\ i_{L_1} \\ i_{L_2}\end{pmatrix}, \begin{pmatrix} v'_C \\ i'_{L_1} \\ i'_{L_2}\end{pmatrix}\right) &= \left(v_C, i_{L_1}, i_{L_2}\right) J \begin{pmatrix} v'_C \\ i'_{L_1} \\ i'_{L_2}\end{pmatrix} \\ &= v_C\left(i'_{L_1} - i'_{L_2}\right) - i_{L_1} v'_C + i_{L_2} v'_C\end{aligned}$$

This is the foundation of Poisson bracket.

Skew-symmetry is equivalent to Tellegen's theorem:

$$\left(v_C, i_{L_1}, i_{L_2}\right) \underbrace{J \begin{pmatrix} v_C \\ i_{L_1} \\ i_{L_2}\end{pmatrix}}_{\text{admissible variables}} = \left(v_C, i_{L_1}, i_{L_2}\right) \begin{pmatrix} i_C \\ v_{L_1} \\ v_{L_2}\end{pmatrix} = 0$$

An LC circuit: Hamiltonian formulation

Identifying the circuit variables:

$$\begin{pmatrix} i_C \\ v_{L_1} \\ v_{L_2} \end{pmatrix} = \frac{d}{dt}\begin{pmatrix} Q_C \\ \phi_{L_1} \\ \phi_{L_2} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} v_C \\ i_{L_1} \\ i_{L_2} \end{pmatrix} = \begin{pmatrix} \frac{\partial H_0}{\partial Q_C} \\ \frac{\partial H_0}{\partial \phi_{L_1}} \\ \frac{\partial H_0}{\partial \phi_{L_2}} \end{pmatrix}$$

The dynamics of the LC-circuit is a Hamiltonian system:

$$\frac{d}{dt}\begin{pmatrix} q_C \\ \phi_{L_1} \\ \phi_{L_2} \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}}_{J} \begin{pmatrix} \frac{\partial H_0}{\partial q_C} \\ \frac{\partial H_0}{\partial \phi_{L_1}} \\ \frac{\partial H_0}{\partial \phi_{L_2}} \end{pmatrix}$$

- with respect to the Poisson bracket with structure matrix J
- generated by the Hamiltonian: $H_0 = \frac{Q_C^2}{2C} + \frac{\phi_{L_1}^2}{2L_1} + \frac{\phi_{L_2}^2}{2L_2}$, the total electromagnetic energy

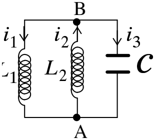
Circuit realization of the Poisson bracket

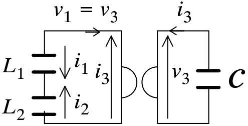
Using the thermodynamical classification of circuit variables:

$$\text{extensive variables: } \begin{pmatrix} i_C \\ v_{L_1} \\ v_{L_2} \end{pmatrix} = \frac{d}{dt} \begin{pmatrix} Q_C \\ \phi_{L_1} \\ \phi_{L_2} \end{pmatrix} \quad \text{intensive variables: } \begin{pmatrix} v_C \\ i_{L_1} \\ i_{L_2} \end{pmatrix} = \begin{pmatrix} \frac{\partial H_0}{\partial Q_C} \\ \frac{\partial H_0}{\partial \phi_{L_1}} \\ \frac{\partial H_0}{\partial \phi_{L_2}} \end{pmatrix}$$

The Poisson tensor $\begin{pmatrix} i_C \\ v_{L_1} \\ v_{L_2} \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}}_{J} \begin{pmatrix} v_C \\ i_{L_1} \\ i_{L_2} \end{pmatrix}$ has the

circuit realization:





An LC circuit: Casimir function and invariants

The structure matrix $J = \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$ has rank 2 , hence admits one generating Casimir function.

The kernel of the structure matrix J is : $\ker J = Span \left\{ \begin{matrix} 0 \\ 1 \\ 1 \end{matrix} \right\}$

A Casimir function satisfies : $\begin{pmatrix} \frac{\partial C}{\partial Q_C} \\ \frac{\partial C}{\partial \phi_{L_1}} \\ \frac{\partial C}{\partial \phi_{L_2}} \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$

for instance $C(Q, \Phi_1, \Phi_2) = \Phi_1 + \Phi_2$ which is the total magnetic flux through the inductors.

An LC circuit: canonical coordinates

Consider the change of coordinates:

$$\begin{pmatrix} q \\ p \\ r \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix}}_{=T} \begin{pmatrix} Q_C \\ \Phi_1 \\ \Phi_2 \end{pmatrix} \qquad \text{Casimir function } r(x) = \frac{1}{2}(\Phi_1 + \Phi_2)$$

hence: $\frac{d}{dt}\begin{pmatrix} q \\ p \\ r \end{pmatrix} = T\,\frac{d}{dt}\begin{pmatrix} Q_C \\ \Phi_1 \\ \Phi_2 \end{pmatrix}$ and $\begin{pmatrix} \frac{\partial H_0}{\partial Q_C} \\ \frac{\partial H_0}{\partial \phi_{L_1}} \\ \frac{\partial H_0}{\partial \phi_{L_2}} \end{pmatrix} = T^t \begin{pmatrix} \frac{\partial \bar{H}_0}{\partial g} \\ \frac{\partial \bar{H}_0}{\partial p} \\ \frac{\partial \bar{H}_0}{\partial r} \end{pmatrix}$ with:

$\bar{H}_0(q, p, r) = \frac{q^2}{2C} + \frac{(p+r)^2}{2L_1} + \frac{(r-p)^2}{2L_2}$

and the Hamiltonian system becomes:

$$\frac{d}{dt}\begin{pmatrix} q \\ p \\ r \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}}_{TJT^t} \begin{pmatrix} \frac{\partial \bar{H}_0}{\partial g} \\ \frac{\partial \bar{H}_0}{\partial p} \\ \frac{\partial \bar{H}_0}{\partial r} \end{pmatrix}$$

References and alternative formulations of LC-circuits

The Poisson bracket arize naturally from Kirchhoff's laws:
B.M. Maschke, A.J. van der Schaft and P.C. Breedveld, "*An intrinsic Hamiltonian formulation of network dynamics: non–standard Poisson structures and gyrators*", **Journal of the Franklin Institute**, Vol. 329, n. 5, pp. 923–966, 1992

Nevertheless there are alternative formulations:

- Lagrangian or standard Hamiltonian systems
  G.M. Bernstein and M.A. Lieberman, "*A method for obtaining a canonical Hamiltonian for nonlinear LC circuits,* **IEEE Trans. on Circuits and Systems**, CAS–35, 3, 411–420,1989
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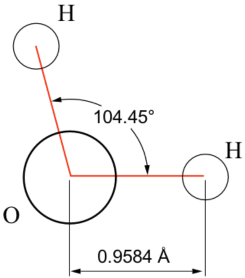
Control Hamiltonian systems

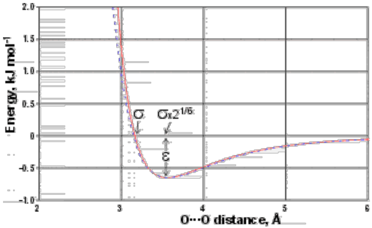
Control Hamiltonian systems

Interaction in Hamiltonian systems : a potential function

In Physics the interaction is defined by an interaction potential.

The water molecule may be considered as 3 mass points in interaction through e.g. the Lennard-Jones potential.





Interaction in Hamiltonian systems : a potential function

- The configuration are the positions of the mass points with respect to an inertial frame :
  $Q = (q_1, q_2, q_3) \in \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3 = \mathcal{Q}$,
- the state space is the cotangent bundle $T^*\mathcal{Q}$ with its canonical symplectic Poisson bracket
- the dynamics is a standard Hamiltonian system generated by Hamiltonian function :

$$H(Q, P) = \underbrace{\left(\frac{\|p_1\|^2}{2\,M_O} + \frac{\|p_2\|^2}{2\,M_H} + \frac{\|p_3\|^2}{2\,M_H} \right)}_{\text{kinetic energy}} + \underbrace{V(Q)}_{\text{interaction potential}}$$

Control Hamiltonian systems

For control Hamiltonian systems the interaction potential describes the interaction with the environment using a control variable $u(t)$!
It is defined by :

- a differentiable manifold $\mathscr{M} \ni x$ with Poisson bracket $\{,\}$
- m inputs u_i
- a Hamiltonian function: $H(x, u_i)$

Often the Hamiltonian function is the sum:

$$H(x, u_i) = H_0(x) + H_{\text{int}}(x, u_i)$$

where:

1. $H_0(x)$ defines the Hamiltonian drift vector field
2. $H_{\text{int}}(x, u_i)$ defines the interaction with the environment

In the sequel linear in control : $H_{\text{int}}(x, u_i) = \sum_{i=1}^{m} H_i(x)\, u_i(t)$.

Input-output Hamiltonian systems

An input-output Hamiltonian system is defined by:

- on a differentiable manifold $\mathscr{M} \ni x$ with Poisson bracket $\{,\}$ (and Λ the Poisson tensor)
- the internal Hamiltonian function $H_0 \in C^\infty(\mathscr{M})$
- m inputs u_i and outputs $\widetilde{y}_i$
- m interaction Hamiltonian functions $H_i \in C^\infty(\mathscr{M}),\, i \in \{1, \ldots, m\}$

and the system:

$$\Sigma_{io} \begin{cases} \dot{x}(t) &= X_{H_0}(x) - \sum_{i=1}^m u_i(t)\, J(x)\, X_{H_i}(x) \\ \tilde{y}_i(t) &= H_i(x(t)) \end{cases}$$

natural outputs

Input-output Hamiltonian systems : in coordinates

An input-output Hamiltonian system is defined by:

- a skew-symmetric structure matrix $J(x) \in \mathbb{R}^{n\times n}$ satisfying the Jacobi identities
- the internal Hamiltonian function $H_0 \in C^\infty(\mathscr{M})$
- m inputs u_i and outputs $\widetilde{y}_i$
- m interaction Hamiltonian functions $H_i \in C^\infty(\mathscr{M}),\, i \in \{1, \dots, m\}$

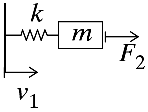
and the system:

$$\Sigma_{io} \left\{ \begin{array}{rcl} \dot{x}(t) & = & J(x)\frac{\partial H_0}{\partial x}(x) - \sum_{i=1}^{m} u_i(t)\, J(x)\frac{\partial H_i}{\partial x}(x) \\ \tilde{y}_i(t) & = & H_i(x(t)) \end{array} \right.$$

natural outputs

Example: mass-spring with mixed boundary conditions

Consider the mass-spring system with mixed boundary conditions:



where m is the mass and k the stiffness

- external force F applied on the mass
- controlled velocity v of the basis.

The dynamical system is:

$$\frac{d}{dt}\begin{pmatrix} q \\ p \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}}_{\text{interdomain coupling}} \underbrace{\begin{pmatrix} k\,q \\ \frac{p}{m} \end{pmatrix}}_{\text{driving force}} \underbrace{- v \begin{pmatrix} 1 \\ 0 \end{pmatrix} + F \begin{pmatrix} 0 \\ 1 \end{pmatrix}}_{\text{generalized external forces}}$$

Hamiltonian formulation of the mass-spring with mixed boundary conditions

The internal Hamiltonian is the total energy:
$H_0(q,p) = \frac{1}{2m} p^2 + \frac{1}{2} k\, q^2$.

Define the interaction Hamiltonians:

- $H_1(q,p) = p$ with the controlled velocity v of the basis
- $H_2(q,p) = q$ with the external force F applied on the mass

The formulation as input-output Hamiltonian system is:

$$\begin{cases} \frac{d}{dt}\begin{pmatrix} q \\ p \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}}_{\text{symplectic bracket}} \left[\underbrace{\begin{pmatrix} \frac{\partial H_0}{\partial q} \\ \frac{\partial H_0}{\partial p} \end{pmatrix}}_{\text{internal driving force}} \underbrace{-v \begin{pmatrix} \frac{\partial H_1}{\partial q} \\ \frac{\partial H_1}{\partial p} \end{pmatrix} + F \begin{pmatrix} \frac{\partial H_2}{\partial q} \\ \frac{\partial H_2}{\partial p} \end{pmatrix}}_{\text{external driving forces}} \right] \\ \begin{pmatrix} \tilde{y}_1 \\ \tilde{y}_1 \end{pmatrix} = \begin{pmatrix} p \\ q \end{pmatrix} \begin{matrix} \text{total momentum} \\ \text{relative displacement} \end{matrix} \end{cases}$$

Input-output Hamiltonian systems: power balance equation

For any interaction Hamiltonian H_j, $j \in \{1, \dots, m\}$:

$$\dot{\tilde{y}}_j = \frac{dH_j}{dt} = \{H_j, H_0\} - \sum_{i=1}^{m} u_i \{H_j, H_i\}$$

Using the *skew-symmetry* of the Poisson bracket ($\{H_j, H_i\} = -\{H_i, H_j\}$):

$$\frac{dH_0}{dt} = \{H_0, H_0\} - \sum_{j=1}^{m} u_j \{H_0, H_j\}$$

becomes the power balance equation:

$$\frac{dH_0}{dt} = \sum_{j=1}^{m} u_j \dot{\tilde{y}}_j$$

If the internal energy is bounded from below $H_0(x) \geq H_{min}$, the input-output Hamiltonian systems is *lossless*.

Concluding remarks and references

Input-output Hamiltonian systems enjoy numerous properties, however we shall see that they do not cover lot of engineering problems.

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Modelling origins and definition of Port Hamiltonian systems

Modelling origins and definition of Port Hamiltonian systems

Port Hamiltonian systems

Port Hamiltonian systems

Port Hamiltonian systems : definition

A port Hamiltonian system is defined by:

1. on a differentiable manifold $\mathscr{M} \ni x$ with *pseudo*-Poisson bracket $\{,\}$ (and Λ the *pseudo*-Poisson tensor)
2. the internal Hamiltonian function $H_0 \in C^\infty(\mathscr{M})$
3. m inputs $u_i \in \mathscr{U}$ and outputs $y_i \in \mathscr{U}^*$
4. m input *vector fields* g_i
5. and the system:

$$\Sigma_{phs} \left\{ \begin{array}{ccl} \dot{x} & = & X_{H_0} + \sum_{i=1}^{m} u_i g_i \\ y_i & = & \langle g_i, H_0 \rangle = L_{g_i} H_0 \quad \text{port conjugated outputs} \end{array} \right.$$

where $\langle X, \omega \rangle$ denotes the pairing between vector fields and 1-forms (and L_g the Lie derivative w.r.t. g).

Port Hamiltonian systems : in coordinates

In coordinates $(x_1, .., x_n)$ a port Hamiltonian system is written:

- a skew-symmetric structure matrix $J(x) \in \mathbb{R}^{n \times n}$
- a smooth Hamiltonian function $H_0(x)$
- m inputs $u_i \in \mathbb{R}^p$ and outputs $y_i \in \mathbb{R}^p$
- m input *vector fields* $g_i(x) \in \mathbb{R}^n$

and the system:

$$\begin{aligned} \dot{x} &= J(x)\frac{\partial H_0}{\partial x} + \sum_{i=1}^m u_i g_i(x) \\ y_i &= g_i(x)^t \frac{\partial H_0}{\partial x} \end{aligned} \qquad \text{port conjugated outputs}$$

Example: elementary storage of elastic energy

Consider a *translational spring*:

$$\begin{aligned} \dot{q} &= f \\ e &= \frac{\partial H}{\partial q}(q) \end{aligned} \tag{2}$$

with:

- q the displacement vector,
- f the velocity,
- e the elastic force of the spring with *potential* energy $H(q)$.

It is a port Hamiltonian system defined with respect to the Poisson structure matrix $J = 0$ and with $g = 1$.

Example: elementary storage of kinetic energy

Consider *point mass*:

$$\begin{aligned} \dot{p} &= f \\ e &= \frac{1}{m} p \end{aligned} \quad , \; p, f, e \in \mathbb{R}^3 \tag{3}$$

where

- p is the vector of momenta,
- m is the mass,
- f is the vector of external forces, and
- e denotes the velocity of the point mass.

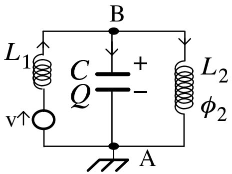
It is a port Hamiltonian system with Poisson structure matrix $J = 0$, $D = 0$ and $H(p) = \frac{1}{2m} \parallel p \parallel^2$ the *kinetic* energy.

LC circuit with voltage source

LC-circuit with voltage source

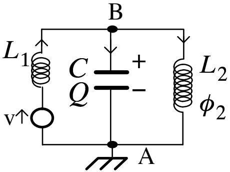
LC circuit with voltage source

Consider the following LC-circuit:



- with total energy: $H_0 = \frac{Q_C^2}{2C} + \frac{\phi_{L_1}^2}{2L_1} + \frac{\phi_{L_2}^2}{2L_2}$
- and voltage source v_s.

LC circuit with voltage source: Kirchhoff's laws

The circuit : 

admits the Kirchhoff's laws:

$$\begin{pmatrix} i_C \\ v_{L_1} \\ v_{L_2} \\ -i_s \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 1 & -1 & 0 \\ -1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}}_{\text{skew-symmetric}} \begin{pmatrix} v_C \\ i_{L_1} \\ i_{L_2} \\ v_s \end{pmatrix}$$

LC circuit with voltage source: port Hamiltonian system

The port Hamiltonian formulation of the electrical circuit with source

$$\left\{\begin{array}{lcl}
\dfrac{d}{dt}\begin{pmatrix} q_C \\ \phi_{L_1} \\ \phi_{L_2} \end{pmatrix} & = & \overbrace{\begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}}^{\text{Poisson bracket}} \overbrace{\begin{pmatrix} v_C \\ i_{L_1} \\ i_{L_2} \end{pmatrix}}^{\mathrm{d}H_0} + \underbrace{\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}}_{\text{input vector field}} v_s \\
i_s & = & \underbrace{(0,1,0)}_{\langle g,} \underbrace{\begin{pmatrix} v_C \\ i_{L_1} \\ i_{L_2} \end{pmatrix}}_{\mathrm{d}H_0\rangle}
\end{array}\right.$$

Spinning body with one actuating wrench

Spinning body with actuating wrench

$$\begin{cases} \dfrac{d}{dt}\begin{pmatrix} p_x \\ p_y \\ p_z \end{pmatrix} = \overbrace{\begin{pmatrix} 0 & -p_z & p_y \\ p_z & 0 & -p_x \\ -p_y & p_x & 0 \end{pmatrix}}^{\text{Lie}-\text{Poisson bracket}} \overbrace{\begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix}}^{\mathrm{d}K} + \underbrace{\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}}_{\text{input vector field}} \gamma \\ y = \underbrace{(0,1,0)}_{\langle g,} \underbrace{\begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix}}_{\mathrm{d}H_0\rangle} \end{cases}$$

where $K(p) = \frac{1}{2} p^t \mathbb{J}^{-1} p$ is the kinetic energy and $\mathrm{d}K(p)$ is the velocity.

Port Hamiltonian systems: skew-symmetry

The port Hamiltonian systems is defined by: pseudo-Poisson tensor and two dual input-output relations.

This extends the skew-symmetric map $\Lambda^\sharp$ to the map $\mathfrak{T}_{\Lambda^\sharp, g_i}$:

$$\begin{array}{ccc} T_x^*\mathscr{M} \times \mathbb{R}^n & \longrightarrow & T_x\mathscr{M} \times \mathbb{R}^n \\ \begin{pmatrix} dH_0(x) \\ u \end{pmatrix} & \longmapsto & \begin{pmatrix} X \\ -y \end{pmatrix} = \begin{pmatrix} \Lambda^\sharp(dH_0(x)) + \sum_{i=1}^m u_i g_i \\ -L_{g_i} H_0(x) \end{pmatrix} \end{array}$$

It is linear and skew-symmetric.

Port Hamiltonian systems: skew-symmetry

It is better seen in coordinates where the map $\mathfrak{T}_{\Lambda^\sharp, g_i}$ becomes:

$$\begin{array}{ccc} T_x^*\mathscr{M} \times \mathbb{R}^n & \longrightarrow & T_x\mathscr{M} \times \mathbb{R}^n \\ \begin{pmatrix} \frac{\partial H_0(x)}{\partial x} \\ u \end{pmatrix} & \longmapsto & \begin{pmatrix} X \\ -y \end{pmatrix} = \underbrace{\begin{pmatrix} J(x) & g(x) \\ -g^t(x) & 0 \end{pmatrix}}_{\text{skew-symmetric}} \begin{pmatrix} \frac{\partial H_0(x)}{\partial x} \\ u \end{pmatrix} \end{array}$$

Recall Kirchhoff's laws for the LC-circuit with voltage source:

$$\begin{pmatrix} i_C \\ v_{L_1} \\ v_{L_2} \\ -i_s \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 1 & -1 & 0 \\ -1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}}_{\text{skew-symmetric}} \begin{pmatrix} v_C \\ i_{L_1} \\ i_{L_2} \\ v_s \end{pmatrix}$$

Skew-symmetry and balance equation

The time derivative of the internal Hamiltonian function is:

$$\frac{dH_0}{dt} = \langle \mathrm{d}H_0, X_{H_0} \rangle + \sum_{j=1}^{m} u_j \langle \mathrm{d}H_0, g_i \rangle$$

which by skew-symmetry of the map $\mathfrak{T}_{\Lambda^\sharp, g_i}$ becomes the power balance equation:

$$\frac{dH_0}{dt} = \sum_{j=1}^{m} u_j y_j$$

If H_0 is bounded from below, the system is *lossless*.

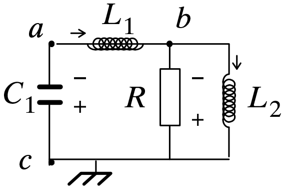
For any Casimir function $C(x)$ there is a balance equation: $\frac{dC}{dt} = \sum_{j=1}^{m} u_j \langle \mathrm{d}C, X_{H_0} \rangle$.

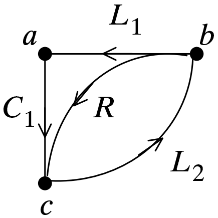
Dissipative Port Hamiltonian systems

Dissipative Port Hamiltonian systems

Example of an RLC circuit

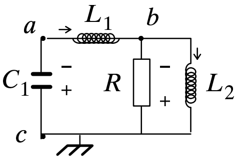
Consider the RLC circuit with its interconnection graph :

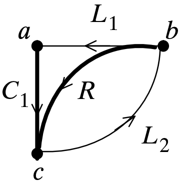




RLC circuit, spanning tree and Kirchhoff's laws

Consider the following spanning tree $\{C_1, R\}$:





admits the Kirchhoff's laws:

$$\begin{pmatrix} i_{C_1} \\ v_{L_1} \\ v_{L_2} \\ i_R \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 \\ 0 & -1 & 1 & 0 \end{pmatrix}}_{\text{skew-symmetric}} \begin{pmatrix} v_{C_1} \\ i_{L_1} \\ i_{L_2} \\ v_R \end{pmatrix}$$

RLC circuit as Port Hamiltonian system with dissipation

The circuit variables corresponding to the resistor are considered as port variables :

$$\left\{ \begin{array}{lcl} \dfrac{d}{dt}\begin{pmatrix} q_{C_1} \\ \phi_{L_1} \\ \phi_{L_2} \end{pmatrix} & = & \overbrace{\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}}^{\text{Poisson bracket}} \overbrace{\begin{pmatrix} v_{C_1} \\ i_{L_1} \\ i_{L_2} \end{pmatrix}}^{\mathrm{d}H_0} + \underbrace{\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}}_{\text{input vector field } g_R} v_R \\ i_R & = & \underbrace{(0, -1, 1)}_{\langle g_R,} \underbrace{\begin{pmatrix} v_{C_1} \\ i_{L_1} \\ i_{L_2} \end{pmatrix}}_{\mathrm{d}H_0\rangle} \end{array} \right.$$

subject to Ohm' law : $v_R = R\, i_R$

RLC circuit as dissipative Port Hamiltonian system

Eliminating the dissipative relation $v_R = R\, i_R$ one obtains :

$$\frac{d}{dt}\begin{pmatrix} q_{C_1} \\ \phi_{L_1} \\ \phi_{L_2} \end{pmatrix} = \underbrace{\overbrace{\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}}^{\text{Poisson bracket}} \overbrace{\begin{pmatrix} v_{C_1} \\ i_{L_1} \\ i_{L_2} \end{pmatrix}}^{dH_0}}_{\text{Hamiltonian vector field}} \underbrace{- \overbrace{\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} R\,(0, 1, -1)}^{\text{pseudo metric}} \overbrace{\begin{pmatrix} v_{C_1} \\ i_{L_1} \\ i_{L_2} \end{pmatrix}}^{dH_0}}_{\text{gradient vector field}}$$

the sum of a Hamiltonian and a gradient vector field.

As the generating functions are equal : $H_0\,(x)$, one may write :

$$\frac{d}{dt}\begin{pmatrix} q_{C_1} \\ \phi_{L_1} \\ \phi_{L_2} \end{pmatrix} = \overbrace{\begin{pmatrix} 0 & 1 & 0 \\ -1 & -R & R \\ 0 & R & -R \end{pmatrix}}^{\text{Leibniz bracket}} \overbrace{\begin{pmatrix} v_{C_1} \\ i_{L_1} \\ i_{L_2} \end{pmatrix}}^{dH_0}$$

called dissipative Hamiltonian system

Port Hamiltonian systems with dissipation in coordinates

Consider Port Hamiltonian system with structure matrix $J(x)$ and Hamiltonian function $H_0(x)$:

- port variables $(u, y) \in \mathbb{R}^m \times \mathbb{R}^m$
- dissipation port variables$(u^R, y^R) \in \mathbb{R}^p \times \mathbb{R}^p$:

$$\begin{cases} \dot{x} & = & J(x)\frac{\partial H_0}{\partial x} + \sum_{i=1}^m u_i g_i(x) + \sum_{i=1}^p u_i^R g_i^R(x) \\ y_i & = & g_i(x)\frac{\partial H_0}{\partial x} \\ y_j^R & = & g_j^R(x)^t \frac{\partial H_0}{\partial x} \end{cases}$$

and *dissipative closure equation* defined by a symmetric matrix $R^\sharp(x)$ and:

$$u^R = R^\sharp(x)\, y^R$$

Dissipative Port Hamiltonian systems

Eliminating the dissipation port variables, one obtains a dissipative Port Hamiltonian system:

Definition

A dissipative Port Hamiltonian system on $\mathbb{R}^n$ is defined by:
(i) a skew-symmetric structure matrix $J(x) = -J^t(x)$
(ii) a symmetric (positive) structure matrix $R(x) = R^t(x) \geq 0$
(iii) a Hamiltonian function $H_0(x)$
and the dynamical equations:

$$\left\{ \begin{array}{rcl} \dot{x} & = & [J(x) - R(x)] \frac{\partial H_0}{\partial x} + \sum_{i=1}^{m} u_i g_i(x) \\ y_i & = & g_i(x) \frac{\partial H_0}{\partial x} \end{array} \right.$$

where $R(x) = g^R R^\sharp \left(g^R\right)^t$.

Port Hamiltonian systems with dissipation: intrinsic formulation

Consider Port Hamiltonian system with

- port variables $(u, y) \in \mathscr{U} \times \mathscr{U}^*$ where $\mathscr{U}$ and $\mathscr{U}^*$ are vector bundle on $\mathscr{M}$
- dissipation port variables$(u^R, y^R) \in \mathscr{U}_R \times \mathscr{U}_R^*$where $\mathscr{U}_R$ and $\mathscr{U}_R^*$ are vector bundle on $\mathscr{M}$:

$$\begin{cases} \dot{x} & = & X_{H_0} + \sum_{i=1}^m u_i g_i + \sum_{j=1}^p u_i^R g_i^R \\ y_i & = & \langle \mathrm{d}H_0, g_i \rangle \\ y_j^R & = & \langle \mathrm{d}H_0, g_i^R \rangle \end{cases}$$

and *dissipative closure equation* $u^R = R_x^\sharp(y^R)$ defined by a symmetric (positive) contravariant positive tensor R with $R^\sharp$ is the associated vector bundle morphism.

Elimination of dissipation port variables and Leibniz bracket

Definition

The Leibniz bracket $[\,,\,]_{\Lambda, g^R, R}$ is defined by:

$$[F, G]_{\Lambda, g^R, R} = \Lambda(F, G) - \langle \mathrm{d}F, R(x) \langle \mathrm{d}G, g \rangle g \rangle$$

and has in coordinates the structure matrix:

$$J(x) - (g^R)^t \, R(x) \, g^R(x)$$

Neither skew-symmetric nor symmetric it defines a left bundle morphism:

$$\begin{array}{rcl} M_L^\sharp : T^*\mathscr{M} & \rightarrow & T\mathscr{M} \\ \omega & \mapsto & X = M_L^\sharp(\omega) \end{array}$$

such that: $M_L^\sharp(\omega)(\alpha) = [F, G]_{\Lambda, g^R, R}(\alpha, \omega)$, $\forall \alpha \in \Omega^1(\mathscr{M})$

Power balance equation

The port Hamiltonian system with dissipation:

$$\begin{cases} \dot{x} &= [x, H_0]_{\Lambda, g^R, R} + \sum_{i=1}^m u_i g_i \\ y_i &= \langle \mathrm{d}H_0, g_i \rangle \end{cases}$$

satisfies the power balance equation:

$$\frac{dH_0}{dt} = \sum_{j=1}^m u_j y_j - [H_0, H_0]_{\Lambda, g, R}$$

which depends only on the symmetric part of the bracket, in coordinates:

$$\frac{dH_0}{dt} = \sum_{j=1}^m u_j y_j - \frac{\partial H_0}{\partial x}^t (g^R)^t R(x) g^R(x) \frac{\partial H_0}{\partial x}$$

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- B. Brogliato, R. Lozano, B. Maschke and O. Egeland, **Dissipative Systems Analysis and Control**, Communications and Control Engineering Series, Springer Verlag, London, 2007, ISBN 10: 1-84628-516-X 2nd ed, 2007
- **Modeling and Control of Complex Physical Systems - The Port-Hamiltonian Approach**, Duindam, V., Macchelli, A., Stramigioli, S., Bruyninckx, H. (eds.), ISBN 978-3-642-03195-3.,Springer , Sept. 2009

Examples of Port-Hamiltonian systems

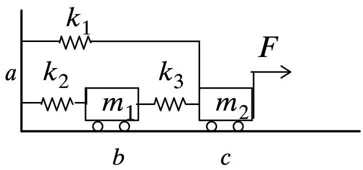
Examples of Port Hamiltonian systems

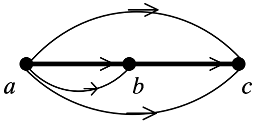
1-D mechanical systems

1-dimensional mechanical systems are analogous to circuits :

- relative velocities are cycle variables
- forces are cocycle variables

Consider the mechanical system :

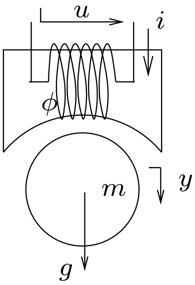




write the the port-Hamiltonian systems obtained from Lagrange and Euler modelling and compare.

The levitating ball : total energy

Consider the iron ball levitating in a magnetic field :



The total energy is : $H_0(x) = \frac{1}{2\,L(x_2)} {x_1}^2 + \frac{1}{2m} {x_3}^2 - mgx_2$
where : $x_1 = \phi$ is the total magnetic flux, $x_2 = y$ is the displacement of the ball and x_3 is the kinetic momentum of the ball, m is the mass, g the gravitational constant and $L(x_2) = L_\infty + \frac{k}{(\cdots)}$, $L_\infty > 0$, $a > 0$, $k > 0$.

The levitating ball : the port Hamiltonian model

The port Hamiltonian model is obtained as :

$$\frac{dx}{dt} = \left(\underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}}_{J} - \underbrace{\begin{bmatrix} R & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{R} \right) \frac{\partial H_0}{\partial x}(x) + \underbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}_{g} u$$

with u the input voltage, R is the electric resistance in the coil and:

$$\frac{\partial H_0}{\partial x}(x) = \begin{pmatrix} \frac{x_1}{L(x_2)} \\ \frac{x_3}{m} \\ -\frac{1}{2}\frac{dL}{dx_2}(x_2)\left(\frac{x_1}{L(x_2)}\right)^2 - m\,g \end{pmatrix} \begin{matrix} \text{current through the coil } I \\ \text{velocity of the ball} \\ \text{electro-motive+gravity force} \end{matrix}$$

and conjugated output: $y = \frac{\partial H_0}{\partial x_1} = I$

The levitating ball : coupling through energy

The coupling between the mechanical and magnetic domain :

- does not occur through the structure matrices as :

$$J - R = \begin{bmatrix} -R & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \quad \begin{array}{l} \text{magnetic domain} \\ \text{mechanical potential domain} \\ \text{mechanical kinetic domain} \end{array}$$

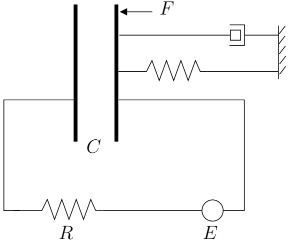
  is bloc-diagonal

- occur through the Hamiltonian which is not separated :

$$H_0(x) = \frac{1}{2\,L\,(x_2)} {x_1}^2 + \frac{1}{2m} {x_3}^2 - mgx_2$$

A microphone: the total energy

Consider the model of microphone:



The total energy is: $H(q, p, Q) = \frac{1}{2} k (q - \bar{q})^2 + \frac{1}{2m} p^2 + \frac{1}{2C(q)} Q^2$
where q is the displacement, p the momentum of the plate, Q the charge of the condensator and $C(q) = \varepsilon_0 \frac{S}{q}$, ε_0 is the permittivity of the air.

A microphone: port Hamiltonian model

The port-Hamiltonian model is:

$$\frac{d}{dt}\begin{pmatrix} q \\ p \\ Q \end{pmatrix} = \left(\underbrace{\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{J} - \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & \nu & 0 \\ 0 & 0 & \frac{1}{R} \end{bmatrix}}_{R^\sharp} \right) \frac{\partial H}{\partial x}(x) + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} F + \begin{bmatrix} 0 \\ 0 \\ \frac{1}{R} \end{bmatrix} E$$

where F is the force exerted by acoustic pressure and E is the input voltage, with conjugated outputs:

$$y_F = \frac{\partial H}{\partial p} = \dot{q} \qquad y_E = \frac{1}{R}\frac{\partial H}{\partial Q} = I$$

A microphone : coupling through energy

The coupling between the mechanical and magnetic domain :

1. does not occur through the structure matrices as :
$$J - R = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 - \frac{1}{R} \end{bmatrix} \quad \begin{array}{l} \text{mechanical potential domain} \\ \text{mechanical kinetic domain} \\ \text{electrical domain} \end{array}$$
is bloc-diagonal
2. occur through the Hamiltonian which is not separated :
$$H(q, p, Q) = \frac{1}{2} k (q - \bar{q})^2 + \frac{1}{2m} p^2 + \frac{1}{2C(q)} Q^2$$

Permanent Magnet Synchronous Motor

In the dq coordinates the dynamic model is:

$$\begin{aligned} L_d \frac{di_d}{dt} &= -R_s\, i_d + \omega L_q i_q + v_d \\ L_q \frac{di_q}{dt} &= -R_s\, i_q + \omega L_d \Phi + v_q \\ \mathbb{I} \frac{d\omega}{dt} &= n_P \left((L_d - L_q)\, i_d i_q + \Phi\, i_q - \tau_l \right) \end{aligned}$$

where:

1. i_q and i_p are the current in dq coordinates,
2. ω the angular velocity
3. n_P is the number of pole pairs, L_d and L_q are stator inductances in the dq frame, R_s is stator winding resistance, τ_l is a constant unknown load torque, Φ is back emf constant and $\mathbb{I}$ the moment of inertia.

P M S Motor: Hamiltonian formulation

The total energy is: $H_0(x) = \frac{1}{2}\frac{x_1^2}{L_d} + \frac{1}{2}\frac{x_2^2}{L_q} + \frac{1}{2}\frac{x_3^2}{(\mathbb{I}/n_P)}$
with conserved variables: $\phi_d = x_1$ and $\phi_q = x_2$ the magnetic fluxes in dq coordinates and the kinetic momentum p.

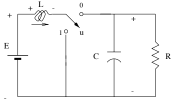
The PMSM admits a port-Hamiltonian formulation:

$$\frac{dx}{dt} = (J(x) - R)\frac{\partial H_0}{\partial x} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} v_d \\ v_q \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ -\frac{1}{n_P} \end{pmatrix}\tau_l$$

with structure matrices:

$$J(x) = \begin{pmatrix} 0 & 0 & x_2 \\ 0 & 0 & -(x_1 + \Phi) \\ -x_2 & (x_1 + \Phi) & 0 \end{pmatrix} \quad \textit{and} \quad R = \begin{pmatrix} R_s & 0 & 0 \\ 0 & R_s & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

A power converter: boost converter



Consider the boost converter:

The averaged model controlled by Pulse Width Modulation with slew rate $u \in [0, 1]$ may be written as Port Hamiltonian system :

$$\frac{dx}{dt} = [J(u) - R] \frac{\partial H_0}{\partial x} + \underbrace{\begin{pmatrix} 1 \\ 0 \end{pmatrix}}_{g} E \qquad y_E = \frac{\partial H_0}{\partial x_1} = \underbrace{\frac{x_1}{L}}_{\text{current in source}}$$

with :

- state variables : x_1 magnetic flux and x_2 the electrical charge
- Hamiltonian (the electro-magnetic energy) : $H_0(x) = \frac{1}{2}\frac{x_1^2}{L} + \frac{1}{2}\frac{x_2^2}{L}$
- structure matrices : $J(u) = \begin{pmatrix} 0 & -u \\ u & 0 \end{pmatrix}$ and $R = \begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{R} \end{pmatrix}$ where R is the output load resistance
- port input : the voltage source E

Properties of Port Hamiltonian systems

Properties of Port Hamiltonian systems

Comparison with input-output Hamiltonian systems

Comparison with input-output Hamiltonian systems

Comparison with input-output Hamiltonian systems

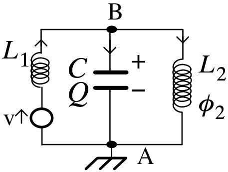
Port Hamiltonian systems weaken the structure compared to input-output Hamiltonian systems :

- the skew-symmetric bracket (structure matrix $J(x)$) may not satisfy the Jacobi identities
- the input vector fields are not Hamiltonian : $\nexists H_i(x)\ s.t.: g_i(x) = J(x)\frac{\partial H_i}{\partial x}$
- the bracket may include dissipation :$J(x) - R(x)$
- outputs are different

$$\underbrace{\begin{array}{rcl} \dot{x} & = & [J(x) - R(x)]\frac{\partial H_0}{\partial x} + \sum_{i=1}^{m} u_i g_i(x) \\ y_i & = & g_i(x)\frac{\partial H_0}{\partial x} \end{array}}_{\text{port Hamiltonian system}} \qquad \underbrace{\begin{array}{rcl} \dot{x}(t) & = & J(x)\frac{\partial H_0}{\partial x} + \sum_{i=1}^{m} u_i \left[J(x)\frac{\partial H_i}{\partial x}\right] \\ \tilde{y}_i(t) & = & H_i(x(t)) \end{array}}_{\text{Input-output Hamiltonian system}}$$

Input vector fields are not Hamiltonian for networks (1)

Consider the LC circuit wih voltage source :



Interconnection of Port Hamiltonian systems

Interconnection of Port Hamiltonian systems

Feedback interconnection of Port Hamiltonian Systems : in coordinates

Consider two Port Hamiltonian systems, $k = 1, 2$ with structure matrices J^k, input vector fields g_i^k, interconnection vector fieldsγ_i^k and Hamiltonian functions $H_0^k\left(x^k\right)$:

$$\begin{cases} \dot{x}^k & = J^k\left(x^k\right) \quad \frac{\partial H_0^k}{\partial x^k} + \sum_{i=1}^{m^k} u_i^k g_i^k\left(x^k\right) + \sum_{i=1}^{m} v_i^k \gamma_i^k\left(x^k\right) \\ y_i^k & = \quad \left(g_i^k\right)^t \frac{\partial H_0^k}{\partial x^k} \\ \iota_i^k & = \quad \left(\gamma_i^k\right)^t \frac{\partial H_0^k}{\partial x^k} \end{cases}$$

with feedback interconnection: $\begin{array}{rcl} \gamma_i^1 & = & \iota_i^2 \\ \gamma_i^2 & = & -\iota_i^1 \end{array}$

Note that it is power continuous: $\gamma_i^1 \iota_i^1 + \gamma_i^2 \iota_i^2 = 0$

Feedback interconnection of Port Hamiltonian Systems

Consider two Port Hamiltonian systems, $k = 1, 2$ with pseudo-Poisson tensor Λ^k, input vector fields g_i^k, interconnection vector fieldsγ_i^k and Hamiltonian functions $H_0^k\left(x^k\right)$:

$$\left\{\begin{array}{rcl} \dot{x}^k & = & X_{H_0^k} + \sum_{i=1}^{m^k} u_i^k g_i^k + \sum_{i=1}^{m} v_i^k \gamma_i^k \\ y_i^k & = & \langle \mathrm{d}H_0^k, g_i^k \rangle \\ \iota_i^k & = & \langle \mathrm{d}H_0^k, \gamma_i^k \rangle \end{array}\right.$$

with feedback interconnection: $\begin{array}{rcl} \gamma_i^1 & = & \iota_i^2 \\ \gamma_i^2 & = & -\iota_i^1 \end{array}$

Note that it is power continuous: $\gamma_i^1 \iota_i^1 + \gamma_i^2 \iota_i^2 = 0$

Composed Port Hamiltonian System

Composed port Hamiltonian system:

- on product manifold $\mathscr{M}^1 \times \mathscr{M}^2$
- endowed with composed pseudo-Poisson bracket $\{\,,\,\}$:

$$\{F, G\} = \Lambda^1 (\mathrm{d}_{x_1} F, \mathrm{d}_{x_1} G) + \Lambda^2 (\mathrm{d}_{x_2} F, \mathrm{d}_{x_2} G) + \langle \mathrm{d}_{x_1} F, \langle \mathrm{d}_{x_2} G, \gamma_i^2 \rangle \gamma_i^1 \rangle - \langle \mathrm{d}_{x_2} F, \langle \mathrm{d}_{x_1} G, \gamma_i^2 \rangle \gamma_i^1$$

- generated by total Hamiltonian: $H(x_1, x_2) = H_0^1(x_1) + H_0^2(x_2)$
- with product input vector fields $g = g^1 \otimes g^2$

Composed Port Hamiltonian System

Composed port Hamiltonian system generated by total Hamiltonian:

$$H_0\left(x_1, x_2\right) = H_0^1\left(x_1\right) + H_0^2\left(x_2\right)$$

is given by:

$$\begin{aligned}
\frac{d}{dt}\begin{pmatrix} x^1 \\ x^2 \end{pmatrix} &= \begin{pmatrix} J^1\left(x^1\right) & \gamma^1\left(x^1\right)\left(\gamma^2\right)^t\left(x^2\right) \\ -\gamma^2\left(x^2\right)\left(\gamma^1\right)^t\left(x^1\right) & J^2\left(x^2\right) \end{pmatrix}\begin{pmatrix} \frac{\partial H_0}{\partial x^1} \\ \frac{\partial H_0}{\partial x^2} \end{pmatrix} \\
&+ \sum_{i=1}^{m^1} u_i^1 \begin{pmatrix} g_i^1\left(x^1\right) \\ 0 \end{pmatrix} + \sum_{i=1}^{m^1} u_i^2 \begin{pmatrix} 0 \\ g_i^2\left(x^2\right) \end{pmatrix} \\
y_i^1 &= \left(g_i^1\right)^t \frac{\partial H_0}{\partial x^1} \\
y_i^2 &= \left(g_i^2\right)^t \frac{\partial H_0}{\partial x^2}
\end{aligned}$$

Conclusion

Port Hamiltonian systems are an extension of Hamiltonian systems:

- extend the Poisson bracket from generalized velocities/forces to include a pair of conjugated port variables
- relax the integrability conditions : Jacobi identities and Hamiltonian input vector fields
- allows to write balance equations including energy flows from the environment
- allows for interconnection (or composition) of Hamiltonian systems

Control of Port Hamiltonian systems

Control of Port Hamiltonian systems

Rationale

Rationale

Stabilization using the port Hamiltonian structure : 2 routes

In the sequel we present two ways of using the port Hamiltonian structure for nonlinear regulation :

1. Control by interconnection : derive the state feeedback $u = u(x)$ from the feedback interconnection of the PHS with a virtual controller as PHS and reduction using Casimir functions.
2. Interconnection and Damping Assignment-Passivity-Based Control (IDA-PBC) : find state feeedback $u = u(x)$ such that closed-system is port Hamiltonian with assigned structure matrices and Hamiltonian :

$$\frac{dx}{dt} = (J_d(x) - R_d(x)) \frac{\partial H_d}{\partial x}(x)$$

Some general references

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3. R. Ortega, A. van der Schaft, I. Mareels, and B. Maschke, *Putting energy back in control*, **IEEE Control Syst. Mag.**, vol. 21, no. 2, pp. 18–33, Apr. 2001
4. R. Ortega, A. van der Schaft, B. Maschke, and G. Escobar, *Interconnection and damping assignment passivity-based control of port-controlled Hamiltonian systems*, **Automatica**, vol. 38, no. 4, pp. 585–596, Apr. 2002.
5. R. Ortega, A. van der Schaft, F. Castaños, and A. Astolfi, *Control by Interconnection and Standard Passivity-Based Control of Port-Hamiltonian Systems*, **IEEE Control Syst. Mag.**, vol. 53, no. 11, 2008

Some general references (more)

Control by constant interconnection

Control by constant interconnection

Embedding by interconnection

Consider a Port Hamiltonian System on $x(t) \in \mathbb{R}^n$ with structure matrices $J(x) = -J^t(x)$, $R(x) = R^t(x) \geq 0$, inputs $u(t) \in \mathbb{R}^m$ and outputs $y(t) \in \mathbb{R}^m$:

$$\Sigma \begin{cases} \dot{x} &= (J(x) - R(x)) \frac{\partial H_0}{\partial x} + g(x)\, u \\ y &= g(x) \frac{\partial H_0}{\partial x} \end{cases}$$

and a controller system being a simple integrator $x_c \in \mathbb{R}^m$ with Hamiltonian $H_c(x_c)$:

$$\Sigma_c \begin{cases} \dot{x}_c &= u_c \\ y_c &= \frac{\partial H_c}{\partial x_c} \end{cases}$$

with feedback interconnection : $\begin{pmatrix} u \\ u_c \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} y \\ y_c \end{pmatrix}$

Embedding system

The embedding system is :

$$\frac{d}{dt}\begin{pmatrix} x \\ x_c \end{pmatrix} = \begin{pmatrix} J(x) - R(x) & g(x) \\ -g^t(x) & 0 \end{pmatrix}\begin{pmatrix} \frac{\partial H_{cl}}{\partial x} \\ \frac{\partial H_{cl}}{\partial x_c} \end{pmatrix} + \begin{pmatrix} g(x) \\ 0 \end{pmatrix} u$$

$$y = \left(g(x)^t, 0\right)\begin{pmatrix} \frac{\partial H_{cl}}{\partial x} \\ \frac{\partial H_{cl}}{\partial x_c} \end{pmatrix}$$

with closed-loopHamiltonian $: H_{cl}(x, x_c) = H_0(x) + H_c(x_c)$.

Note that $J_e = \begin{pmatrix} J(x) & g(x) \\ -g^t(x) & 0 \end{pmatrix}$ is the extension of the pseudo-Poisson bracket defined by $J(x)$ including the port-variables !

Reduction by Casimir function : shaping the Hamiltonian

Assume that the extended structure matrix :

$$J_e = \begin{pmatrix} J(x) - R(x) & g(x) \\ -g^t(x) & 0 \end{pmatrix}$$

admits m (left-)Casimir functions $C_i(x, x_c)$:

$$\left(\frac{\partial C}{\partial x}^t, \frac{\partial C}{\partial x_c}^t \right) J_e = 0$$

and assume that they are of the type :

$$C(x, x_c) = F(x) - x_c$$

Interpretation of the Casimir function of the augmented bracket

A Casimir function $C(x, x_c)$ satisfies :

$$\left(\frac{\partial F}{\partial x}^t, -I_m\right)\begin{pmatrix} J(x) - R(x) & g(x) \\ -g^t(x) & 0 \end{pmatrix} = 0$$

or satisfies :

1. the input vector fields are $g(x) = (J(x) + R(x))\frac{\partial F}{\partial x}$ dissipative Hamiltonian w.r.t. $(J(x) + R(x))$
2. transversality condition with respect to input distribution : $\frac{\partial F}{\partial x}^t g(x) = L_g F(x) = 0$

Reduction by Casimir function : shaping the Hamiltonian

Then on the invariant submanifold $C(x, x_c) = 0$ the dynamics is :

$$\begin{aligned}
\frac{dx}{dt} &= (J(x) - R(x))\frac{\partial H_0}{\partial x} + g(x)\frac{\partial H_c}{\partial x_c}(F(x)) \\
&= (J(x) - R(x))\frac{\partial H_0}{\partial x} + \overbrace{(J(x) + R(x))\frac{\partial F}{\partial x}}^{=g(x)}\frac{\partial H_c}{\partial x_c}\circ F(x) \\
&= (J(x) - R(x))\frac{\partial}{\partial x}(H_0 + H_c \circ F) + 2R(x)\frac{\partial}{\partial x}(H_c \circ F)
\end{aligned}$$

If $R(x)\frac{\partial F}{\partial x} = 0$, then the reduced dynamics is :

$$\frac{dx}{dt} = (J(x) - R(x))\frac{\partial H_d}{\partial x}$$

and the state feedback : $u(x) = \frac{\partial H_c}{\partial x_c}\circ F(x)$ shapes the Hamiltonian : $H_d(x) = H_0(x) + H_c \circ F(x)$

Equivalent conditions : dissipation obstacle

The previous conditions are equivalent to :

1. the input vector fields are $g(x) = (J(x) + R(x))\frac{\partial F}{\partial x}$ dissipative Hamiltonian w.r.t. $(J(x)+R(x))$
2. transversality condition with respect to input distribution : $\frac{\partial F}{\partial x}^t g(x) = L_g F(x) = 0$
3. the dissipation obstacle is not present $R(x)\frac{\partial F}{\partial x} = 0$: the function $C(x, x_c)$ is a left and right Casimir function

Design of the stabilizing controller

Assuming that $(J(x) - R(x))$ admits a (left) Casimir function,

design the control $u(x) = \frac{\partial H_c}{\partial x_c} \circ F(x) + v$:

1. choose H_c such that $H_d(x) = H_0(x) + H_c \circ F$ admits a minimum at some state x^* and is a closed-loop Lyapunov function
2. use dissipative feedback :
$v = -R_c\, g(x)^t \frac{\partial H_d}{\partial x} \left(= -R_c\, g(x)^t \frac{\partial H_0}{\partial x} \right)$

then the power balance equation is :

$$\begin{aligned} \frac{dH_d}{dt} &= - \quad \frac{\partial H_d}{\partial x}^t R \frac{\partial H_d}{\partial x} - \frac{\partial H_d}{\partial x} g(x)\, R_c\, g^t(x) \frac{\partial H_0}{\partial x} \\ &= \quad -\frac{\partial H_d}{\partial x}^t \left(R + g(x)\, R_c\, g^t(x)\right) \frac{\partial H_d}{\partial x} \end{aligned}$$

and apply Lasalle's theorem.

Discussion

1. The feedback modifies the closed-loop Hamiltonian but not the structure matrices :

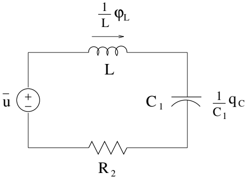
$$\frac{dx}{dt} = (J(x) - R(x)) \frac{\partial}{\partial x} (H_0 + H_c \circ F)$$

2. It resembles completely to the feedback of input-output systems :

$$u(x) = \frac{\partial H_c}{\partial x_c} \circ H_i(x) + v$$

where $H_i(x)$ is the interaction Hamiltonian but generalizes to Leibniz bracket : $(J(x) - R(x))$

Example : series RLCS circuit (model)



Consider the circuit :
with port Hamiltonian model with state variables $x = [q_C, \phi_L]^\top$ the charge in the capacitor and the flux in the inductance :

$$\frac{d}{dt}\begin{pmatrix} q_C \\ \phi_l \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & -R \end{pmatrix}\begin{pmatrix} \frac{\partial H_0}{\partial q_c} \\ \frac{\partial H_0}{\partial \phi_L} \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u$$

$$y = \begin{pmatrix} 0 & 1 \end{pmatrix}\begin{pmatrix} \frac{\partial H_0}{\partial q_c} \\ \frac{\partial H_0}{\partial \phi_L} \end{pmatrix}$$

with Hamiltonian: $H_0(q_C, \phi_L) = \frac{q_C^2}{2C} + \frac{\phi_L^2}{2L}$, the total electromagnetic

Example : series RLCS circuit (energy shaping)

The conditions for energy shaping by interconnection are :

1. the input vector field is dissipative Hamiltonian :
$$g\left(q_C, \phi_L\right) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & -R \end{pmatrix} \begin{pmatrix} \frac{\partial F}{\partial q_c} \\ \frac{\partial F}{\partial \phi_L} \end{pmatrix} \text{ with :}$$
$$F\left(q_C, \phi_L\right) = -q_C$$
.
2. transversality condition satisfied :
$$\begin{pmatrix} \frac{\partial F}{\partial q_c} & \frac{\partial F}{\partial \phi_L} \end{pmatrix} g\left(q_C, \phi_L\right) = \begin{pmatrix} -1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = 0$$
3. the dissipation obstacle is not present
$$R\left(q_C, \phi_L\right) \begin{pmatrix} \frac{\partial F}{\partial q_c} \\ \frac{\partial F}{\partial \phi_L} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & -R \end{pmatrix} \begin{pmatrix} -1 \\ 0 \end{pmatrix} = 0$$

Example : series RLCS circuit (stabilization)

Hence the Hamiltonian may be shaped to :

$$H_d(q_C,\phi_L) = H_0(xq_C,\phi_L) + H_c \circ F(q_C,\phi_L) = \frac{q_C^2}{2C} + \frac{\phi_L^2}{2L} + H_c(q_c)$$

and has a minimum at $(q_C,\phi_L)^* = (q_C^*,0)$ by choosing :

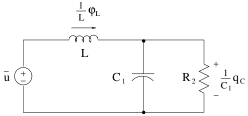
$$H_c(q_C) = \frac{1}{2C_a}q_C^2 - \left(\frac{1}{C}+\frac{1}{C_a}\right)q_C^*\,q_C + \kappa \quad C_a > -C\,;\ \kappa \in \mathbb{R}$$

and apply dissipative control:

$$v = -R_c\begin{pmatrix} 0 & 1 \end{pmatrix}\begin{pmatrix} \left(\frac{1}{C}+\frac{1}{C_a}\right)(q_C - q_C^*) \\ \frac{\phi_L}{L} \end{pmatrix} = -R_c\frac{\phi_L}{L}$$

only changes value of Ohm's dissipation to $R + R_c$ but stabilizes.

Example : paralell RLCS circuit (model)



Consider the circuit :
with port Hamiltonian model with state variables $x = [q_C, \phi_L]^\top$ the charge in the capacitor and the flux in the inductance :

$$\frac{d}{dt}\begin{pmatrix} q_C \\ \phi_l \end{pmatrix} = \begin{pmatrix} -\frac{1}{R} & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} \frac{\partial H_0}{\partial q_c} \\ \frac{\partial H_0}{\partial \phi_L} \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u$$

$$y = \begin{pmatrix} 0 & 1 \end{pmatrix}\begin{pmatrix} \frac{\partial H_0}{\partial q_c} \\ \frac{\partial H_0}{\partial \phi_L} \end{pmatrix}$$

with Hamiltonian: $H_0(q_C, \phi_L) = \frac{q_C^2}{2C} + \frac{\phi_L^2}{2L}$, the total electromagnetic energy.

Example : series RLCS circuit (dissipation obstacle)

The conditions for energy shaping by interconnection are :

1. the input vector field is dissipative Hamiltonian :

$$g\left(q_C,\phi_L\right)=\begin{pmatrix}0\\1\end{pmatrix}=\begin{pmatrix}-\frac{1}{R} & 1\\-1 & 0\end{pmatrix}\begin{pmatrix}\frac{\partial F}{\partial q_c}\\\frac{\partial F}{\partial \phi_L}\end{pmatrix}\Longleftrightarrow\begin{cases}\frac{\partial F}{\partial q_c}=-1\\\frac{\partial F}{\partial \phi_L}=-\frac{1}{R}\end{cases}\Longleftrightarrow$$

2. transversality condition not satisfied :

$$\begin{pmatrix}\frac{\partial F}{\partial q_c} & \frac{\partial F}{\partial \phi_L}\end{pmatrix}g\left(q_C,\phi_L\right)=\begin{pmatrix}-1 & -\frac{1}{R}\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix}\neq 0$$

3. the dissipation obstacle is present

$$R\left(q_C,\phi_L\right)\begin{pmatrix}\frac{\partial F}{\partial q_c}\\\frac{\partial F}{\partial \phi_L}\end{pmatrix}=\begin{pmatrix}-\frac{1}{R} & 0\\0 & 0\end{pmatrix}\begin{pmatrix}-1\\-\frac{1}{R}\end{pmatrix}\neq 0$$

Example : pendulum (model)

Consider the pendulum without damping

$$\begin{aligned} \dot{q} &= \frac{p}{m} \\ \dot{p} &= -mg\sin q + u \end{aligned}$$

with state variables $x = [q, p]^T$ with q the configuration and p the momentum.
The port Hamiltonian model is:

$$\begin{aligned} \frac{d}{dt}\begin{pmatrix} q \\ p \end{pmatrix} &= \underbrace{\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}}_{J}\begin{pmatrix} \frac{\partial H_0}{\partial q} \\ \frac{\partial H_0}{\partial p} \end{pmatrix} + \underbrace{\begin{pmatrix} 0 \\ 1 \end{pmatrix}}_{g} u \\ y &= (0\,1)\begin{pmatrix} \frac{\partial H_0}{\partial q} \\ \frac{\partial H_0}{\partial p} \end{pmatrix} = \frac{p}{m} \end{aligned}$$

with Hamiltonian :$H_0(q, p) = mg(1 - \cos q) + \frac{1}{2m}p^2$

Example : pendulum (extended Casimir functions)

Recall : we look for Casimir functions such as :

$$C(q, p, x_c) = F(q, p) - x_c$$

that is functions $F(q, p)$ satisfying :

1. input vector field is Hamiltonian :
$$g = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = J(x)\,\frac{\partial F}{\partial x} = \begin{pmatrix} \frac{\partial F}{\partial p} \\ -\frac{\partial F}{\partial q} \end{pmatrix}$$
2. the gradient of F is transversal to g :
$$L_g F(x) = 0 = \frac{\partial F}{\partial x}^t g(x) = \frac{\partial F}{\partial p}$$

hence a generating function is : $F(q, p) = -q$

Example : pendulum (control)

For the controller design we choose a function $H_C(x_c)$ such that $H_d(x) = H_0(x) + H_c \circ F$ has a minimum at the desired equilibrium $x_* = (x_1^*, 0)$.
The simplest choice is given by

$$H_C(x_c) = \cos x_c + \frac{1}{2}(x_c + x_1^*)^2$$

The control is finally obtained is:

$$u = -\frac{\partial H_C}{\partial x_c}(x_c)\,|_{x_c=-q} = \sin q - (q - x_1^*)$$

which is the well-known "proportional plus gravity compensation control"

Control by assignment of structure matrices and energy

Control by assignment of structure matrices and energy :
IDA-PBC

Objectives of IDA-PBC : closed-loop PHS

Consider a Port Hamiltonian System on $x(t) \in \mathbb{R}^n$ with structure matrices $J(x) = -J^t(x)$, $R(x) = R^t(x) \geq 0$, inputs $u(t) \in \mathbb{R}^m$ and outputs $y(t) \in \mathbb{R}^m$:

$$\Sigma \left\{ \begin{array}{rcl} \dot{x} & = & (J(x) - R(x)) \frac{\partial H_0}{\partial x} + g(x)\, u \\ y & = & g(x) \frac{\partial H_0}{\partial x} \end{array} \right.$$

find a static state-feedback control $u = \beta(x)$ such that the closed-loop dynamics is a Port Hamiltonian system with dissipation:

$$\Sigma_d \; : \; \dot{x} = [J_d(x) - R_d(x)] \frac{\partial H_d}{\partial x}(x) + g(x)\, v$$

with skew-symmetric matrix $J_d(x)$, any positive-semidefinite matrix $R_c(x)$.

Objectives of IDA-PBC : matching equation

Set :

$$J_a(x) = J_d(x) - J(x) \quad \text{and} \quad R_a(x) = R_d(x) - R(x)$$

and :

$$K(x) = \frac{\partial H_d}{\partial x}(x) - \frac{\partial H_0}{\partial x}(x) = \frac{\partial H_a}{\partial x}(x)$$

then the desired closed-loop behaviour leads to the matching equation :

$$-(J_a - R_a)\frac{\partial H_0}{\partial x}(x) + g(x)\beta(x) = [(J(x) + J_a(x)) - (R(x) + R_a(x))]\frac{\partial H_a}{\partial x}(x)$$

with design parameters $J_a(x)$, $R_a(x)$ and $H_a(x)$.

IDA-PBC with assigned structure matrices

In this procedure one :

1. fixes some structure matrices $J_a(x)$ and $R_a(x)$
2. solves a PDE in $H_a(x)$ obtained by pre-multiplying the matching equation by a left annihilator $g^{\perp}(x)$ of $g(x)$ (full -rank $m \times n$ matrix satisfying $g^{\perp}(x)\, g(x) = 0$):

$$-g^{\perp}(x)(J_a - R_a)\frac{\partial H_0}{\partial x}(x) = g^{\perp}(x)\left[(J(x) + J_a(x)) - (R(x) + R_a(x))\right]\frac{\partial H_a}{\partial x}(x)$$

3. computes the control:

$$\begin{aligned}\beta(x) = \quad & \left[g^t(x)\, g(x)\right]^{-1} g^t(x) \\ & \left\{\left[(J(x) + J_a(x)) - (R(x) + R_a(x))\right]\frac{\partial H_a}{\partial x}(x) + (J_a - R_a)\frac{\partial H_0}{\partial x}(x)\right\}\end{aligned}$$

Objectives of IDA-PBC : stabilization

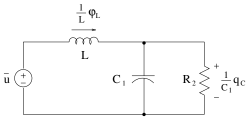
Assume that the control objective is to stabilize the system at the state $x^*$and that the IDA-PBC has a solution such that :

1. $H_d(x)$ is a Lyapunov function with strict minimum at x^*
2. the largest invariant set in closed-loop contained in $\left\{ x \in R^n / \frac{\partial H_d}{\partial x}^t (x) R_d(x) \frac{\partial H_d}{\partial x}(x) = 0 \right\}$ is $\{x^*\}$

then the point x^* is asymptotically stable in closed-loop

with estimated domain of attraction by larget bounded level set of $H_d(x)$.

Example : parallel RLCS circuit (model)



Consider the circuit :
with port Hamiltonian model with state variables $x = [q_C, \phi_L]^\top$ the charge in the capacitor and the flux in the inductance :

$$\begin{aligned} \frac{d}{dt}\begin{pmatrix} q_C \\ \phi_l \end{pmatrix} &= \begin{pmatrix} -\frac{1}{R} & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} \frac{\partial H_0}{\partial q_c} \\ \frac{\partial H_0}{\partial \phi_L} \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u \\ y &= \begin{pmatrix} 0 & 1 \end{pmatrix}\begin{pmatrix} \frac{\partial H_0}{\partial q_c} \\ \frac{\partial H_0}{\partial \phi_L} \end{pmatrix} \end{aligned}$$

with Hamiltonian: $H_0(q_C, \phi_L) = \frac{q_C^2}{2C} + \frac{\phi_L^2}{2L}$, the total electromagnetic energy.

Parallel RLCS circuit : matching equation

Choice of structure matrices and associated added Hamiltonian $H_a(x)$:

1. choose added structure matrices

$$J_a(x) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad R_a(x) = \begin{pmatrix} 0 & 0 \\ 0 & R_a \end{pmatrix} \; R_a > -R$$

2. solve a PDE in $H_a(x)$ using the left annihilator $g^{\perp}(x) = \begin{pmatrix} 1 & 0 \end{pmatrix}$ of $g(x) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, the matching equation :

$$-g^{\perp}(x)(J_a - R_a)\frac{\partial H_0}{\partial x}(x) = g^{\perp}(x)\left[(J(x) + J_a(x)) - (R(x) + R_a(x))\right]\frac{\partial H_a}{\partial x}(x)$$

becomes :

$$0 = \begin{pmatrix} -\frac{1}{R} & 1 \end{pmatrix} \begin{pmatrix} \frac{\partial H_a}{\partial q_c} \\ \frac{\partial H_a}{\partial \phi_L} \end{pmatrix} = -\frac{1}{R}\frac{\partial H_a}{\partial q_c} + \frac{\partial H_a}{\partial \phi_L}$$

Parallel RLCS circuit : closed-loop equilibria

The solutions of the matching equation are :

$$H_a(q_c, \phi_L) = H(R\, q_c + \phi_L) \qquad H \in C^\infty(\mathbb{R})$$

hence the closed-loop Hamiltonian is

$$H_d(q_C, \phi_L) = H_0(q_C, \phi_L) + H(R\, q_c + \phi_L) = \frac{q_C^2}{2C} + \frac{\phi_L^2}{2L} + H(R\, q_c + \phi_L)$$

The equilibrium in closed-loop is given by:

$$R\,\frac{\phi_L^*}{L} - \frac{q_C^*}{C} = 0 \quad and \quad \frac{\phi_L^*}{L} + \frac{\partial H}{\partial \xi}\left(\left[1 + \frac{R^2 C}{L}\right]\phi_L^*\right) = 0$$

Parallel RLCS circuit : two possible H

1. Case 1 : $H(\xi) = k\frac{\xi^2}{2}$ then :

$$\phi^* \in \mathbb{R} \quad \text{and} \quad k = -\frac{(L + R^2 C)}{L^2} < 0$$

   and $H(\xi)$ is concave !

2. Case 2 : $H(\xi) = k\frac{\xi^4}{4}$ then :

$$\phi^* = \pm\frac{1}{\sqrt{(-k)(L + R^2 C)}} \quad \text{and} \quad k = \in \mathbb{R}^*_-$$

   and $H(\xi)$ is again concave !

Parallel RLCS circuit : local positivity of H_d

Consider the Hessian of H_d at the quilibrium :

$$\frac{\partial^2 H_d}{\partial q_c, \phi_L^2} = \begin{pmatrix} \frac{1}{C} + R^2 \frac{\partial^2 H}{\partial \xi^2}\left(Rq_c^* + \phi_L^{*2}\right) & R\frac{\partial^2 H}{\partial \xi^2}\left(Rq_c^* + \phi_L^{*2}\right) \\ R\frac{\partial^2 H}{\partial \xi^2}\left(Rq_c^* + \phi_L^{*2}\right) & \frac{1}{L} + \frac{\partial^2 H}{\partial \xi^2}\left(Rq_c^* + \phi_L^{*2}\right) \end{pmatrix}$$

is definite positive iff :

1. either : $\frac{1}{C} + R^2 \frac{\partial^2 H}{\partial \xi^2}\left(Rq_c^* + \phi_L^2\right) > 0$ or :
   $\frac{1}{L} + \frac{\partial^2 H}{\partial \xi^2}\left(Rq_c^* + \phi_L^{*2}\right) > 0$
2. and det $\frac{\partial^2 H_d}{\partial q_c, \phi_L^2} > 0$ i.e. :
   $\frac{1}{LC}\left(1 + \left[L + R^2 C\right] \frac{\partial^2 H}{\partial \xi^2}\left(Rq_c^* + \phi_L^{*2}\right)\right) > 0$

which reduces to : $\frac{\partial^2 H}{\partial \xi^2}\left(Rq_c^* + \phi_L^2\right) > -\frac{1}{(R^2 C + L)}$

Parallel RLCS circuit : two possible H

Check for the two exemples the condition :
$\frac{\partial^2 H}{\partial \xi^2}\left(Rq_c^* + \phi_L^2\right) > -\frac{1}{(R^2C+L)}$

1. Case 1 : $H(\xi) = k\,\frac{\xi^2}{2}$ the condition reduces to :

$$\left(L + R^2 C\right)^2 < L^2$$

   which is wrong !
2. Case 2 : $H(\xi) = k\,\frac{\xi^4}{4}$ then :

$$(-k) < \frac{1}{(L+R^2C)\left(\frac{\phi_L^*}{L}\right)^2}$$

   which leads to a solution !

Parallel RLCS circuit : IDA-PBC control

The control law is given by :

$$\beta(x) = \left[g^t(x)\,g(x)\right]^{-1} g^t(x) \left\{\left[(J(x)+J_a(x)) - (R(x)+R_a(x))\right]\frac{\partial H_a}{\partial x}(x) + (J_a - R_a)\frac{\partial H_0}{\partial x}(x)\right\}$$

which becomes :

$$\beta(q_C,\phi_L) = 1\begin{pmatrix} 0 & 1 \end{pmatrix}\left[\begin{pmatrix} -\frac{1}{R} & 1 \\ -1 & -R_a \end{pmatrix}\begin{pmatrix} R\,\frac{\partial H}{\partial \xi}(R\,q_C, +\phi_L) \\ \frac{\partial H}{\partial \xi}(R\,q_C, +\phi_L) \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0 & R_a \end{pmatrix}\begin{pmatrix} \frac{q_C}{C} \\ \frac{\phi_L}{L} \end{pmatrix}\right]$$

or :

$$\beta(q_C,\phi_L) = (R - R_a)\frac{\partial H}{\partial \xi}(R\,q_C, +\phi_L) - R_a\frac{\phi_L}{L}$$

The levitating ball : the port Hamiltonian model

Consider the iron ball levitating in a magnetic field :

$$\frac{dx}{dt} = \left(\underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}}_{J} - \underbrace{\begin{bmatrix} R & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{R} \right) \frac{\partial H_0}{\partial x}(x) + \underbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}_{g} u$$

$$y = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \frac{\partial H_0}{\partial x}(x) = \frac{\partial H_0}{\partial x_1} = I$$

with total energy is : $H_0(x) = \frac{1}{2\,L(x_2)} {x_1}^2 + \frac{1}{2m} {x_3}^2 - mgx_2$
where : $x_1 = \phi$ is the total magnetic flux, $x_2 = y$ is the displacement of the ball and x_3 is the kinetic momentum of the ball, m is the mass, g the gravitational constant and $L(x_2) = L_\infty + \frac{k}{(a+x_2)}$, $L_\infty > 0$, $a > 0$, $k > 0$.

The levitating ball : open-loop equilibria

The equilibria are given by (x^*, u^*) such that:

$$(J-R)\,\frac{\partial H_0}{\partial x}(x^*)+g\,u^* = \begin{pmatrix} R\,\frac{x_1^*}{L(x_2^*)}+u^* \\ \frac{x_3^*}{m} \\ -\frac{1}{2}\,\frac{dL}{dx_2}\,(x_2^*)\left(\frac{x_1^*}{L(x_2^*)}\right)^2 - m\,g \end{pmatrix} = 0$$

and may be parametrized by the current: $y^* = \frac{\partial H_0}{\partial x_1}(x^*) = I^*$:

$$\begin{pmatrix} x_1^* \\ x_2^* \\ x_3^* \end{pmatrix} = \begin{pmatrix} L_\infty y^* + \sqrt{2mg} \\ -a + \sqrt{\frac{k}{2mg}}\quad y^* \\ 0 \end{pmatrix} \quad \text{and} \quad u^* = R\,y^*$$

are unstable: see linearized system.

The levitating ball : matching equation

Choice of structure matrices and associated added Hamiltonian $H_a(x)$:

1. choose added structure matrices :

$$J_a(x) = \begin{pmatrix} 0 & 0 & \alpha \\ 0 & 0 & 0 \\ -\alpha & 0 & 0 \end{pmatrix} \quad \text{and} \quad R_a(x) = 0_3$$

2. Use the left annihilator $g^{\perp}(x) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, establish the PDE in H_a :

$$-\begin{pmatrix} 0 & 0 & 0 \\ -\alpha & 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{\partial H_0}{\partial x_1} \\ \frac{\partial H_0}{\partial x_2} \\ \frac{\partial H_0}{\partial x_3} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ -\alpha & -1 & 0 \end{pmatrix} \begin{pmatrix} \frac{\partial H_a}{\partial x_1} \\ \frac{\partial H_a}{\partial x_2} \\ \frac{\partial H_a}{\partial x_3} \end{pmatrix}$$

The levitating ball : added Hamiltonian

From : $\frac{\partial H_a}{\partial x_3} = 0$ the added potential is a function $H_a(x_1, x_2)$ which does not depend on the velocity and satisfies :

$$\alpha \frac{\partial H_0}{\partial x_1} = -\alpha \frac{\partial H_a}{\partial x_1} - \frac{\partial H_a}{\partial x_2}$$

with solution :

$$H_a(x_1, x_2) = -\int_0^{x_1} \frac{\chi}{L\left(x_2 - \frac{(\chi - x_1)}{\alpha}\right)} d\chi + H\left(x_2 - \frac{x_1}{\alpha}\right)$$

For instance, if $L(x_2) = \frac{k}{(x_2+a)}$ then
$H_a(x_1, x_2) = \frac{1}{2k}\left(\frac{x_1^3}{3\alpha} - x_1^2(x_2+a)\right) + H\left(x_2 - \frac{x_1}{\alpha}\right)$

The levitating ball : IDA-PBC control

The control law is given by :

$$\beta(x) = \underbrace{1 \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}}_{=[g^t g]^{-1} g^t} \left[\begin{bmatrix} -R & 0 & \alpha \\ 0 & 0 & 1 \\ -\alpha & -1 & 0 \end{bmatrix} \frac{\partial H_a}{\partial x}(x) + \begin{pmatrix} 0 & 0 & \alpha \\ 0 & 0 & 0 \\ -\alpha & 0 & 0 \end{pmatrix} \frac{\partial H_0}{\partial x}(x) \right]$$

or :

$$\begin{aligned} \beta(x) &= -R \frac{\partial H_a}{\partial x_1}(x) + \alpha \frac{\partial H_a}{\partial x_3}(x) + \alpha \frac{\partial H_0}{\partial x_3}(x) \\ &= R \frac{x_1}{L(x_2)} + \alpha \frac{x_3}{m} + \frac{R}{\alpha} H' \left(x_2 - \frac{x_1}{\alpha} \right) \end{aligned}$$

Conclusion

Hamiltonian methods for the control of physical systems:

1. use the interconnection of Port Hamiltonian systems and Casimir functions
2. assign closed-loop Hamiltonian function and structure matrices.

A control synthesis based on insight of desired physical behaviour in closed-loop :

1. design directly interconnection of the system with environment and indirectly the controller
2. design the closed-loop port Hamiltonian behaviour and deduce the controller

Irreversible Port Hamiltonian systems

Irreversible Port Hamiltonian systems

Introduction and motivation

Introduction and motivation

Structured models for thermodynamics based control

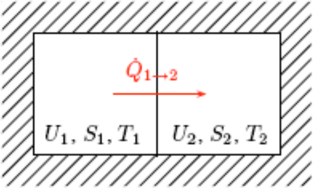
The dynamics of the CSTR and more generally irreversible thermodynamic systems has been the object of numerous suggestions concerning their systematic formulation in view of systems' and control.

This formulations embeds both the energy and the entropy balance equations and often lead to consider systems defined as the sum of Hamiltonian and gradient systems such as GENERIC etc ...[Hangos, K.M.,1999, Favache and Dochain, 2010; Favache et al., 2011, Grmela and Öttinger, 1997; Öttinger and Grmela, 1997; Mushik et al., 2000; Hoang et al., 2011, 2012; Ramirez et al., 2009; Johnsen et al., 2008]

Here we suggest a quasi-Hamiltonian formulation making appear explicitely the stoichiometric matrix and the entropy

Two cells exchanging heat flow

The model



Thermodynamic model given by Gibbs' relation : $dU_i = T_i\, dS_i$ where $T_i = \frac{\partial U_i}{\partial S_i}(S_i)$, $i = 1, 2$

Heat flux due to conducting wall : $\dot{Q}_{1\to 2} = \lambda\,(T_1 - T_2)$ with λ the heat conduction coefficient

Continuity of heat flux : $\dot{Q}_{1\to 2} = -T_1 \frac{dS_i}{dt} = T_2 \frac{dS_2}{dt}$

Entropy balance equations

Entropy balance equations for each cell : a quasi-Hamiltonian formulation

The Hamiltonian-like formulation :

$$\frac{d}{dt}\begin{pmatrix} S_1 \\ S_2 \end{pmatrix} = \underbrace{\lambda\left(\frac{1}{T_2}-\frac{1}{T_1}\right)\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}}_{J(T)}\begin{pmatrix} T_1 \\ T_2 \end{pmatrix}$$

with $T_i = \frac{\partial(U_1+U_2)}{\partial S_i}(S) = \frac{\partial U_i}{\partial S_i}(S_i)$.

1. $J(T)$ is skew-symmetric but depend on the temperature and not the entropy.
2. the map from the gradient of the internal energy $\frac{\partial(U_1+U_2)}{\partial S}$ to the generalized velocities **is not linear** !

Balance equations of the total energy and the total entropy

Energy and entropy balance equations

When considering the thermal domain, the non-linearity is unavoidable :

1. **Conservation of the total internal energy** due to the skew-symmetry of $J(T)$:

$$\frac{d}{dt}(U_1 + U_2) = (T_1,\, T_2)\,\lambda\left(\frac{1}{T_2} - \frac{1}{T_1}\right)\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} T_1 \\ T_2 \end{pmatrix} = 0$$

2. **Increase of the total entropy** due to the non-linear map :

$$\frac{d}{dt}(S_1+S_2) = (1,1)\,\lambda\left(\frac{1}{T_2} - \frac{1}{T_1}\right)\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} T_1 \\ T_2 \end{pmatrix} = \lambda\,\frac{(T_1 - T_2)^2}{T_1 T_2} \geq 0$$

Irreversible Hamiltonian systems

The quasi-PHS is defined by the following dynamic equation

$$\dot{x} = R\left(x, \frac{\partial U}{\partial x}, \frac{\partial S}{\partial x}\right) J \frac{\partial U}{\partial x}(x)$$

where

1. $x \in \mathbb{R}^n$ is the vector of extensive variables,
2. generated bythe total energy $U(x) : \mathscr{C}^\infty(\mathbb{R}^n) \to \mathbb{R}$
3. a constant skew-symmetric matrix $J \in \mathbb{R}^n \times \mathbb{R}^n$
4. $R = R\left(x, \frac{\partial U}{\partial x}, \frac{\partial S}{\partial x}\right)$ is function depending on the total entropy $S(x)$ and total energy $U(x)$:

$$R\left(x, \frac{\partial U}{\partial x}, \frac{\partial S}{\partial x}\right) = \gamma\left(x, \frac{\partial U}{\partial x}\right) \left(\frac{\partial S}{\partial x}(x)^T J \frac{\partial U}{\partial x}(x)\right) \tag{4}$$

with $\gamma(x, \frac{\partial U}{\partial x})$, a positive definite function.

2 cells as Port Hamiltonian system

The quasi-Hamiltonian formulation is :

$$\begin{aligned}
\frac{d}{dt}\begin{pmatrix} S_1 \\ S_2 \end{pmatrix} &= \lambda\left(\frac{1}{T_1}-\frac{1}{T_2}\right)\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} T_1 \\ T_2 \end{bmatrix} \\
&= \underbrace{\frac{\lambda}{T_1 T_2}}_{=\gamma}\underbrace{(T_1-T_2)}_{=\{S,U\}_J}\underbrace{\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}}_{=J}\begin{bmatrix} \frac{\partial U}{\partial S_1} \\ \frac{\partial U}{\partial S_2} \end{bmatrix}
\end{aligned}$$

with :

- internal energy U, co-energy variables : $\frac{\partial U}{\partial S_i} = T_i(S_i)$
- entropy function : $\mathscr{S} = S_1 + S_2$ and
$$\frac{\partial \mathscr{S}}{\partial x}^\top J \frac{\partial U}{\partial x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}^\top \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} T_1 \\ T_2 \end{bmatrix} = T_1 - T_2$$
- positive function : $\gamma\left(\frac{\partial U}{\partial S}\right) = \frac{\lambda}{T_1 T_2}$

Irreversible port Hamiltonian systems : add conjugate external variables

The Irreversible Port Hamiltonian System augments the dynamic equation with a pair of port variables (u, y)

$$\dot{x} = R\left(x, \frac{\partial U}{\partial x}, \frac{\partial S}{\partial x}\right) J \frac{\partial U}{\partial x}(x) + W\left(x, \frac{\partial U}{\partial x}\right) + g\left(x, \frac{\partial U}{\partial x}\right) u(t)$$

and the conjugated output

$$y = g(x)^\top \frac{\partial U}{\partial x}(x)$$

Energy balance and entropy balance with irreversible entropy creation.

$$\boxed{\frac{dU}{dt} = \frac{\partial U}{\partial x}^\top (W + gu), \qquad \frac{dS}{dt} = \boldsymbol{\sigma} + \frac{\partial U}{\partial x}^\top (W + gu)}$$

2 cells interacting with a thermostat

The pseudo-Hamiltonian formulation is :

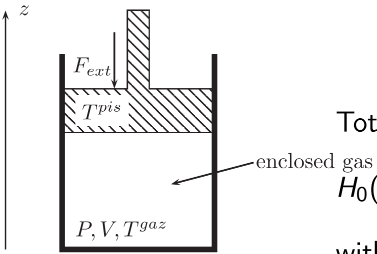
$$\begin{aligned}
\frac{d}{dt}\begin{pmatrix} S_1 \\ S_2 \end{pmatrix} &= \lambda\left(\frac{1}{T_1}-\frac{1}{T_2}\right)\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} T_1 \\ T_2 \end{bmatrix} + \lambda_e \begin{bmatrix} 0 \\ \frac{u(t)-T_2(S_2)}{T_2(S_2)} \end{bmatrix} \\
&= \underbrace{\frac{\lambda}{T_1 T_2}}_{=\gamma}\underbrace{(T_1 - T_2)}_{=\{S,U\}_J}\underbrace{\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}}_{=J}\begin{bmatrix} \frac{\partial U}{\partial S_1} \\ \frac{\partial U}{\partial S_2} \end{bmatrix} \underbrace{-\lambda_e \begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{W} + \underbrace{\frac{\lambda_e}{\frac{\partial U}{\partial x_2}}\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{g} u
\end{aligned}$$

with $u(t)$, the temperature of the thermostat and λ_e the heat conduction coefficient of the external wall.

The gas - piston system

The gas - piston system

The gas - piston system: energy



Total energy

$$H_0(x) = U((S, V)) + H_{mec}(z, p)$$

with mechnical energy

$$H_{mec}(z, p) = \frac{1}{2m} p^2 + mgz$$

and internal energy

$$dU = T\, dS - PdV$$

The gas - piston system: dynamical equations

$$\begin{aligned}
\frac{dS}{dt} &= \frac{1}{T} F_r v = \frac{1}{T} \nu v^2 \triangleq \sigma_{int} \geq 0 && \text{entropy balance} \\
\frac{dV}{dt} &= A v && \text{volume balance} \\
\frac{dz}{dt} &= v && \text{kinematic equation} \\
\frac{dp}{dt} &= -mg + AP - \nu v && \text{momentum balance}
\end{aligned}$$

in matrix form

$$\frac{d}{dt}\begin{bmatrix} S \\ V \\ z \\ p \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 0 & 0 & \frac{\nu v}{T} \\ 0 & 0 & 0 & A \\ 0 & 0 & 0 & 1 \\ -\frac{\nu v}{T} & -A & -1 & 0 \end{bmatrix}}\begin{bmatrix} T \\ -P \\ F \\ v \end{bmatrix} \tag{5}$$

The gas - piston system: Irreversible PHS

$\dot{x} = J_{irr}(x) \frac{\partial H_0}{\partial x}(x)$

where $J_{irr}(x) = J_0 + \sum_{i=1}^{p} \gamma_i \left(x, \frac{\partial H_0}{\partial x}\right) \left(\frac{\partial S}{\partial x}^\top J_i \frac{\partial H_0}{\partial x}\right) J_i$

with structure matrices

$$J_0 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & A \\ 0 & 0 & 0 & 1 \\ 0 & -A & -1 & 0 \end{bmatrix} \quad \text{and} \quad J_1 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}$$

and entropy function $S(x) = x_1 = S$ and function $\gamma_1(x) = \frac{\nu}{T(S)}$

and driving force $\left(\frac{\partial S}{\partial x}^\top J_1 \frac{\partial U}{\partial x}\right) = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} J_1 \begin{bmatrix} T \\ -P \\ F \\ v \end{bmatrix} = v$

The CSTR: model formulation

The CSTR: model formulation

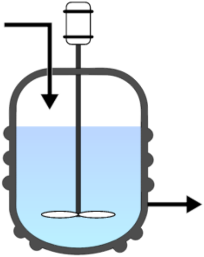
The CSTR: single reaction case

For simplicity consider a single reaction

$$\nu_1 A_1 + \ldots + \nu_{m-1} A_{m-1} \rightleftharpoons \nu_m A_m$$

$$\begin{array}{ll} \nu_1, \ldots, \nu_m : & \text{stoichiometric coefficients} \\ A_1, \ldots, A_m : & \text{chemical species} \end{array}$$

and denote the reaction rate: $r(\mathscr{A}, T) = r_f(\mathscr{A}_f, T) - r_r(\mathscr{A}_r, T)$



with $\mathscr{A}$ the affinity vector of the reaction.

The CSTR: mass and entropy balance equations

The balance equations

$$\underbrace{\dot{n}_i = F_{ei} - F_{si} + r_i V}_{\text{mass}}$$

$$\underbrace{\dot{S} = \sum_{i=1}^{m} (F_{ei} s_{ei} - F_{si} s_i) + \frac{u(t)}{T_w} + \sigma,}_{\text{entropy}}$$

with

- F_{ei}, F_{si} input and output molar flows, s_{ei}, s_i molar entropies,
- T_w jacket temperature and
- $u(t)$ input heat flow flux.

The CSTR: as an Irreversible Port Hamiltonian system

$$\dot{x} = \underbrace{\gamma\{S,U\}_J J \frac{\partial U}{\partial x}(x)}_{\text{quasi-Hamiltonian}} \underbrace{+ W\left(x, \frac{\partial U}{\partial x}\right) + g\left(x, \frac{\partial U}{\partial x}\right) u(t)}_{\text{external flow}}$$

with $U =$ the internal energy

$$\underbrace{J = \begin{bmatrix} 0 & \dots & 0 & \bar{v}_1 \\ 0 & \dots & 0 & \vdots \\ 0 & \dots & 0 & \bar{v}_m \\ -\bar{v}_1 & \dots & -\bar{v}_m & 0 \end{bmatrix}}_{\text{stoichiometric matrix}}, \underbrace{W = \begin{bmatrix} F_{e1} - F_{s2} \\ \vdots \\ F_{em} - F_{sm} \\ \frac{1}{T}\sum_{i=1}^{m}(F_{ei}s_{ei} - F_{si}s_i) \end{bmatrix}}_{\text{Mass transfer}}, gu = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \frac{1}{T},$$

$\{S,U\}_J$ is the reactions driving force, the affinity of reaction $\mathscr{A}$

$$\boxed{\{S,U\}_J = \mathscr{A} = -\sum_{i=1}^{m} \bar{v}_i \mu_i} \quad \gamma = \frac{rV}{T\mathscr{A}} \geq 0, \quad \begin{array}{ll} \mu_1, \dots, \mu_m : & \text{chemical pot.} \\ rV : & \text{molar flow} \end{array}$$

An alternative pseudo-gradient-Hamiltonian representation of the CSTR

$$\dot{x} = \frac{1}{T}\left(\underbrace{\mathscr{J}_f(x_1, T)}_{\text{skew-sym.}} + \underbrace{\eta_1\left(x_1, \frac{\partial S}{\partial x_1}\right) M}_{\text{symmetric}}\right)\frac{\partial S}{\partial x}(x) + g\left(\frac{\partial S}{\partial x}\right) u$$

- with vector of extensive variables $x_1 = [n_1, \ldots, n_m, U]^\top = [\mathbf{n}^\top, U]^\top$ and
- the entropy function $S(\mathbf{n}, U)$ is used as generating function. Its gradient is then $\frac{\partial S}{\partial x_1} = \left[\left(-\frac{\mu_1}{T}\right), \ldots, \left(-\frac{\mu_m}{T}\right), \frac{1}{T}\right]^\top$

An alternative pseudo-gradient-Hamiltonian representation of the CSTR

structure matrices:

$$\mathscr{J}_f = \begin{bmatrix} 0 & 0 & 0 & f_{n1} \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & f_{nm} \\ -f_{n1} & \dots & -f_{nm} & 0 \end{bmatrix}, \; M = \begin{bmatrix} 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 \\ 0 & \dots & 0 & 1 \end{bmatrix}$$

with $f_{ni} = F_{ei} - F_{si}(\mathbf{n}, T) + V\bar{v}_i r(\mathbf{n}, T)$and

$$\eta_1\left(\mathbf{n}, \frac{\mu}{T}, \frac{1}{T}\right) = T\sum_{i=1}^{m}\left(F_{ei}s_{ei} - F_{si}s_i - \frac{\mu_i}{T}\bar{v}_i r(\mathbf{n}, T)\, V\right),$$

Thermodynamics-based control of the CSTR

Thermodynamics-based control of the CSTR

Energy based availability function

Numerous stabilizing controllers are based on the use of the convexity of the entropy function (for single phase systems) in Chemical Engineering [Alonso & Ydstie (1996); Ydstie & Alonso (1997); Alonso & Ydstie (2001); Ydstie (2002); Hoang et al. (2011, 2012)]

Here we use the convexity of the internal energy function and use the energy-based availability function

$$A(x,x^*) = U(x) - \left[U(x^*) + \frac{\partial U}{\partial x}(x^*)^\top (x - x^*) \right] \geq 0$$

where x^* is a reference and possibly a desired equilibrium.

Stabilization by assignment of entropy source and structure matrix: control objective

The desired closed-loop dynamic is

$$\dot{x} = \left(-\sigma_d M + R_d J_d \right) \frac{\partial A}{\partial x}$$

with $M(x) \geq 0$ and $J_d(x) = -J_d^\top(x)$, scalar functions $\gamma_d > 0$,
$\sigma_d = \gamma_d \{S, A\}_{J_d}^2$ and $R_d = \gamma_d \{S, A\}_{J_d}$
for which the balance equation of the availability function is

$$\frac{dA(x,x^*)}{dt} = -\sigma_d \left(\frac{\partial A}{\partial x}^\top M \frac{\partial A}{\partial x} \right) \leq 0$$

.

Stabilization by assignment of entropy source and structure matrix: control law

The desired closed-loop dynamic achieved by the state- feedback

$$u(x) = g^{\dagger}(x)\Big(R_d J_d - \sigma_d M\Big)\left(\frac{\partial U}{\partial x}(x) - \frac{\partial U}{\partial x}(x^*)\right) - g^{\dagger}(x) R J \frac{\partial U}{\partial x}(x),$$

with pseudo-inverse $g^{\dagger}(x) = [g^{\top}(x) g(x)]^{-1} g^{\top}(x)$

if the following matching equation is satisfied

$$g^{\perp}(x)\Big(R_d J_d - \sigma_d M\Big)\left(\frac{\partial U}{\partial x}(x) - \frac{\partial U}{\partial x}(x^*)\right) - g^{\perp}(x) R J \frac{\partial U}{\partial x}(x) = 0$$

Stabilization of the CSTR: a simple solution

Choose : $J_d = J, M = \mathrm{diag}(0,\ldots,0,1)$,and a positive function $\gamma_d(x)$,the only the entropy balance equation is changed

$$\dot{S} = -\gamma_d \sum_{i=1}^{m} \nu_i \left(\mu_i(x) - \mu_i(x^*)\right) - \sigma_d (T - T^*).$$

then the matching equation is equivalent to the expression of De Donder's extent of reaction

$$\frac{n_{0_i} - n_i}{\bar{\nu}_i} = \xi.$$

And the balance equation of the energy-based avalaibility becomes

$$\begin{aligned}\frac{dA}{dt} &= -\sigma_d (T - T^*)^2, \\ &= -\gamma_d (\mathscr{A} - \mathscr{A}^*)^2 (T - T^*)^2.\end{aligned}$$

Conclusion

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