Optimization Methods for Decision Making over Networks
PhD School SIDRA 2018
Homework

Problem 1

Consider the optimization problem

$$\begin{aligned} \min_{x_1, x_2} \quad & -x_1 \\ \text{subj. to} \quad & x_1 \geq 0 \\ & x_2 \geq 0 \\ & x_1 + x_2 \leq 2. \end{aligned}$$

(i) Draw the feasible set for the given optimization problem.

(ii) Derive and solve the Karush-Kuhn-Tucker optimality conditions for the given problem.

(iii) Derive the dual function and its domain. Show that the domain is convex and the dual function is concave on its domain (without using the proposition).

(iv) Write and solve the dual problem.

Problem 2

Consider the following cost-coupled quadratic optimization problem:

$$\min_{x \in \mathbb{R}^d} \sum_{i=1}^{m} f_i(x),$$

with $f_i(x) = x^\top Q_i x + r_i^\top x$, where Q_i are positive definite matrix in $\mathbb{R}^{d \times d}$, r_i are nonzero vectors in $\mathbb{R}^d$, $d \in \mathbb{N}$ representing the number of scalar decision variables and $m \in \mathbb{N}$ being the number of agents. Consider an instance of the problem with $d \geq 2$ and $m \geq 10$.

(i) Simulate (e.g., in Matlab or Python) a (centralized) gradient method for its solution. Denote by $x^\star$ the optimal solution and by $f^\star = \sum_{i=1}^{m} f_i(x^\star)$ the optimal cost.

(ii) Derive the operators defining the gradient method for the given optimization problem.

(iii) Simulate either one of the parallel methods (parallel dual decomposition, parallel ADMM) or one of the distributed methods (distributed subgradient, distributed proximal minimization, distributed dual decomposition, distributed ADMM) shown in the course for solving the same problem. In the case of a distributed method choose a suitable fixed, undirected, connected communication graph, e.g., an undirected ring.

Let $x_i(t)$ denote the solution estimate at iteration t associated with agent i. Provide plots highlighting the behavior of the parallel/distributed algorithm and compare it with the (centralized) problem solution, e.g., show the evolution throughout the algorithm iterations of the following quantities:

- disagreement error: $\displaystyle \frac{1}{m} \sum_{i=1}^{m} \|x_i(t) - \bar{x}(t)\|$, with $\displaystyle \bar{x}(t) \triangleq \frac{1}{m} \sum_{i=1}^{m} x_i(t)$;
- cost error: $\displaystyle \left| \sum_{i=1}^{m} f_i(x_i(t)) - f^\star \right|$.