Lecture Notes on Adaptive Control

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Contents

Chapter 1

Tools

1.1 Notation and Preliminaries

For vectors $x \in \mathbb{R}^n$, $z \in \mathbb{R}^m$, we let $\text{col}(x, z) := (x^{\mathrm{T}}\ z^{\mathrm{T}})^{\mathrm{T}}$.

Norms and Function Spaces

Norms on finite-dimensional vector spaces $\mathcal{X} \cong \mathbb{R}^n$ are denoted by a single bar, $|\cdot| : \mathcal{X} \to \mathbb{R}_{\geq 0}$. This notation encompasses vector norms, $|x|$ for $x \in \mathbb{R}^n$, and matrix norms, $|A|$ for $A \in \mathbb{R}^{m\times n}$. Due to the fact that norms on finite-dimensional vector spaces are all equivalent, norm types are left unspecified, unless explicitly noted. As a result, there is little loss of generality in assuming that $|\cdot| : \mathcal{X} \to \mathbb{R}_{\geq 0}$ is the euclidean norm on $\mathcal{X}$, that is:

$$|x| = |x|_2 := \sqrt{\sum_{i=1}^{n} x_i^2}\,,\ x \in \mathbb{R}^n\,, \qquad |A| = |A|_2 := \sup_{\substack{x\in\mathbb{R}^n \\ x\neq 0}} \frac{|Ax|_2}{|x|_2} = \sqrt{\lambda_{\max}\left(A^{\mathrm{T}}A\right)}\,,\ A \in \mathbb{R}^{m\times n}$$

where $\lambda_{\max}(M)$ denotes the largest eigenvalue of the matrix $M \in \mathbb{R}^{n\times n}$. Given a set $\mathcal{A} \subset \mathbb{R}^n$, the *norm on* $\mathbb{R}^n$ *with respect to* $\mathcal{A}$ is defined as the *point-to-set distance*

$$|x|_{\mathcal{A}} := \text{dist}(x, \mathcal{A}) := \inf_{\xi\in\mathcal{A}} |x - \xi|$$

Conversely, for function spaces we distinguish several types of functional norms, denoted by a double bar, $\|\cdot\| : \mathcal{U} \to \mathbb{R}_{\geq 0}$, where $\mathcal{U}$ is a suitable space of functions. Functions spaces that will be considered in this class are the spaces of k-times differentiable functions, with $k \in [0, 1, \ldots, \infty]$. Specifically,

$$\mathcal{C}^k_{\mathcal{I}}(\mathcal{X})\,, \quad k \in [0, \infty]\,,\ \mathcal{I} \subset (-\infty, \infty)\,,\ \mathcal{X} \cong \mathbb{R}^n$$

denotes the space of k-times differentiable functions of a scalar variable defined on the interval $\mathcal{I}$ of the real line, with codomain $\mathcal{X}$. For example, $\mathcal{C}^0_{[0,\infty)}(\mathbb{R}^m)$ denotes the space of continuous functions $u(\cdot) : t \mapsto u(t) \in \mathbb{R}^m$, $t \in [0, \infty)$, whereas $\mathcal{C}^1_{(0,\infty)}(\mathbb{R}^{m\times n})$ denotes the space of continuously differentiable matrix-valued functions $u(\cdot) : t \mapsto u(t) \in \mathbb{R}^{m\times n}$, $t \in (0, \infty)$. Note that $\mathcal{C}^{k+1}_{\mathcal{I}}(\mathcal{X}) \subset \mathcal{C}^k_{\mathcal{I}}(\mathcal{X})$, for $k = 0, 1, \ldots, \infty$. For economy of notation, unless confusion may arise, we shall omit to specify the codomain. The following norms will be used on $\mathcal{C}^k_{\mathcal{I}}(\mathcal{X})$:

- ∞-norm: $\|u(\cdot)\|_\infty := \sup_{t\in\mathcal{I}} |u(t)|$
- 2-norm: $\|u(\cdot)\|_2 := \sqrt{\int_{\mathcal{I}} |u(\tau)|^2 \mathrm{d}\tau}$

If, for a given $u(\cdot) \in \mathcal{C}^k_{\mathcal{I}}(\mathcal{X})$, $\|u(\cdot)\|_\infty < \infty$, then the function $u(\cdot)$ is said to be *bounded* on its domain. If $\|u(\cdot)\|_2 < \infty$, $u(\cdot)$ is said to have *finite energy* on its domain. Note that if $\mathcal{I}$ is a compact interval (that is, $\mathcal{I} = [a, b]$ for some $-\infty < a < b < \infty$), then both $\|u(\cdot)\|_\infty < \infty$ and $\|u(\cdot)\|_2 < \infty$ hold by continuity. Consequently, it makes sense to define

$$\mathcal{L}^\infty_{(-\infty,\infty)}(\mathcal{X}) := \left\{u(\cdot) \in \mathcal{C}^0_{(-\infty,\infty)}(\mathcal{X}) : \|u(\cdot)\|_\infty < \infty\right\}$$
$$\mathcal{L}^\infty_{[0,\infty)}(\mathcal{X}) := \left\{u(\cdot) \in \mathcal{C}^0_{[0,\infty)}(\mathcal{X}) : \|u(\cdot)\|_\infty < \infty\right\}$$

as the spaces of bounded functions defined over $(-\infty, \infty)$ or $[0, \infty)$, and

$$\mathcal{L}^2_{(-\infty,\infty)}(\mathcal{X}) := \left\{u(\cdot) \in \mathcal{C}^0_{(-\infty,\infty)}(\mathcal{X}) : \|u(\cdot)\|_2 < \infty\right\}$$
$$\mathcal{L}^2_{[0,\infty)}(\mathcal{X}) := \left\{u(\cdot) \in \mathcal{C}^0_{[0,\infty)}(\mathcal{X}) : \|u(\cdot)\|_2 < \infty\right\}$$

as the spaces of square-integrable functions defined over $(-\infty, \infty)$ or $[0, \infty)$. Again, the domain of definition and the codomain will be dropped from the notation whenever convenient and appropriate, and use the simpler notation $\mathcal{L}^\infty$ and $\mathcal{L}^2$. For a given function $u(\cdot) \in \mathcal{C}^k_{\mathcal{I}}(\mathcal{X})$, where either $\mathcal{I} = (-\infty, \infty)$ or $\mathcal{I} = [0, \infty)$ and $\tau \in \mathcal{I}$, we define the *truncation of* $u(\cdot)$ *over* $(-\infty, \tau]$ (or over $[0, \tau]$) as the function $u_\tau(\cdot) : \mathcal{I} \to \mathcal{X}$ defined as

$$u_\tau(t) = \begin{cases} u(t) & t \in \mathcal{I} \text{ and } t \leq \tau \\ 0 & t \geq \tau \end{cases}$$

On the basis of this definition, one defines the *extended* $\mathcal{L}^\infty$ *and* $\mathcal{L}^2$ *spaces* respectively as follows:

$$\mathcal{L}^{\infty,e}_{(-\infty,\infty)}(\mathcal{X}) := \left\{u_\tau(\cdot) \in \mathcal{C}^0_{(-\infty,\infty)}(\mathcal{X}) : \|u_\tau(\cdot)\|_\infty < \infty \text{ for all } \tau \in \mathbb{R}\right\}$$
$$\mathcal{L}^{\infty,e}_{[0,\infty)}(\mathcal{X}) := \left\{u_\tau(\cdot) \in \mathcal{C}^0_{[0,\infty)}(\mathcal{X}) : \|u_\tau(\cdot)\|_\infty < \infty \text{ for all } \tau \geq 0\right\}$$

and

$$\mathcal{L}^{2,e}_{(-\infty,\infty)}(\mathcal{X}) := \left\{u_\tau(\cdot) \in \mathcal{C}^0_{(-\infty,\infty)}(\mathcal{X}) : \|u_\tau(\cdot)\|_2 < \infty \text{ for all } \tau \in \mathbb{R}\right\}$$
$$\mathcal{L}^{2,e}_{[0,\infty)}(\mathcal{X}) := \left\{u_\tau(\cdot) \in \mathcal{C}^0_{[0,\infty)}(\mathcal{X}) : \|u_\tau(\cdot)\|_2 < \infty \text{ for all } \tau \geq 0\right\}$$

Clearly, $\mathcal{L}^\infty \subset \mathcal{L}^{\infty,e}$ and $\mathcal{L}^2 \subset \mathcal{L}^{2,e}$, but not vice versa.
GIven a signal $u(\cdot) \in \mathcal{C}^k_{(-\infty,\infty)}(\mathcal{X})$, its *asymptotic norm*, $\|u(\cdot)\|_a$, is defined as

$$\|u(\cdot)\|_a := \limsup_{t\to\infty} |u(t)|$$

Comparison Functions

Definition 1.1.1 (Class-$\mathcal{K}$ Functions) *A function $\alpha(\cdot) : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is said to be* of class-$\mathcal{K}$ *if it is continuous, strictly increasing and satisfies $\alpha(0) = 0$. A class-$\mathcal{K}$ function is said to be* of class-$\mathcal{K}_\infty$ *if, in addition, it satisfies* $\lim_{s\to\infty} \alpha(s) = +\infty$*.*

Definition 1.1.2 (Class-$\mathcal{N}$ Functions) *A function $\eta(\cdot) : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is said to be* of class-$\mathcal{N}$ *if it is continuous and non-decreasing. Note that a class-$\mathcal{N}$ function $\eta(\cdot)$ does not necessarily satisfy $\eta(0) = 0$.*

Definition 1.1.3 (Class-$\mathcal{KL}$ Functions) *A function $\beta(\cdot) : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is said to be* of class-$\mathcal{KL}$ *if $\beta(\cdot, r)$ is a class-$\mathcal{K}$ function for all $r \in \mathbb{R}_{\geq 0}$ and, for all $s \in \mathbb{R}_{\geq 0}$, $\beta(s, \cdot)$ is a continuous strictly decreasing function satisfying* $\lim_{r\to\infty} \beta(s, r) = 0$*.*

1.2 Stability Definitions

In what follows, we consider nonlinear nonautonomous systems of the form

$$\begin{aligned} \dot{x} &= f(t, x) \\ x(t_0) &= x_0 \end{aligned} \tag{1.1}$$

with state $x \in \mathbb{R}^n$. The vector field $f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ is assumed to be at least continuous in $t \in \mathbb{R}$, and locally Lipschitz in $x \in \mathbb{R}^n$, uniformly in t (note that this implies that $\sup_{t\geq 0} |f(t, x)| < \infty$, for all x belonging to arbitrary compact sets $\mathcal{M} \subset \mathbb{R}^n$.) Often, we will add the further assumption that f is continuously differentiable, or even smooth. The assumption on local Lipschitz continuity (uniformly in t) of the vector field f guarantees that there exists a unique absolutely continuous solution $x(t; t_0, x_0)$ of (1.1), which can be extended over a maximal open interval $\mathcal{I}_{t_0,x_0} = (t_0 - \delta_{\min}, t_0 + \delta_{\max})$. If $\delta_{\max} = +\infty$, (respectively, $\delta_{\min} = +\infty$) for all initial conditions x_0 and all initial times t_0, we say that (1.1) is *forward complete* (respectively, *backward complete*). A system that is both backward and forward complete is said to be *complete*. On the other hand, if $\delta_{\max}$ (respectively, $\delta_{\min}$) is finite, then the trajectory $x(t; t_0, x_0)$ leaves any compact set $\mathcal{M}$ containing x_0 as $t \to t_0 + \delta_{\max}$ (respectively, $t \to t_0 - \delta_{\min}$.) It is assumed that the origin $x = 0$ is an equilibrium for (1.1), that is $f(t, 0) = 0$, for all $t \in \mathbb{R}$.

Definition 1.2.1 (Uniform Stability) *The origin of (1.1) is said to be* uniformly stable *if for any $\varepsilon > 0$ there exists $\delta_\varepsilon > 0$ such that for any $t_0 \in \mathbb{R}_{\geq 0}$ and any $|x_0| \leq \delta_\varepsilon$ the solution $x(t; t_0, x_0)$ satisfies $|x(t; t_0, x_0)| \leq \varepsilon$ for all $t \geq t_0$.*

Definition 1.2.2 (Uniform Global Stability) *The origin of (1.1) is said to be* uniformly globally stable *if there exists a class-$\mathcal{K}_\infty$ function $\gamma(\cdot)$ such that for each $(t_0, x_0) \in \mathbb{R}_{\geq 0} \times \mathbb{R}^n$ the solution $x(t; t_0, x_0)$ satisfies*

$$|x(t; t_0, x_0)| \leq \gamma(|x_0|), \qquad \forall\, t \geq t_0 .$$

Note that the definition of uniform stability is different from that of uniform global stability, as the latter embeds the notion of forward completeness, while the former does not.

Definition 1.2.3 (Uniform Global Attractivity) *The origin of (1.1) is said to be* uniformly globally attractive *if for any numbers* $R > 0$ *and* $\varepsilon > 0$ *there exists* $T > 0$ *(which depends only on* R *and* ε*) such that for any* $t_0 \in \mathbb{R}_{\geq 0}$ *and any* $|x_0| \leq R$

$$|x(t; t_0, x_0)| \leq \varepsilon\,, \qquad \forall\, t \geq t_0 + T\,.$$

Definition 1.2.4 (Uniform Global Asymptotic Stability) *The origin of (1.1) is said to be* uniformly globally asymptotically stable *if it is uniformly stable and uniformly globally attractive.*

A well known result is the following:

Proposition 1.2.5 *The origin of the system (1.1) is uniformly globally asymptotically stable if and only if there exists a class-*$\mathcal{KL}$ *function* $\beta(\cdot,\cdot)$ *such that all solutions of (1.1) satisfy*

$$|x(t; t_0, x)| \leq \beta(|x_0|, t - t_0)\,, \qquad \forall\, t \geq t_0$$

for all $(t_0, x_0) \in \mathbb{R}_{\geq 0} \times \mathbb{R}^n$*.*

Definition 1.2.6 (Exponential Convergence) *The trajectories of the system (1.1) are said to be* (locally) exponentially convergent *if there exists an open neighborhood of the origin* $\mathcal{D}$ *such that* for each *pair of initial conditions* $(t_0, x_0) \in \mathbb{R}_{\geq 0} \times \mathcal{D}$ *there exist constants* $\mu_0 > 0$*,* $\lambda_0 > 0$ *such that the solution* $x(t; t_0, x_0)$ *satisfies*

$$|x(t; t_0, x)| \leq \mu_0 |x_0| \mathrm{e}^{-\lambda_0 (t - t_0)}\,, \qquad \forall\, t \geq t_0\,. \tag{1.2}$$

The system (1.1) is said to be globally exponentially convergent *if* for each *pair of initial conditions* $(t_0, x_0) \in \mathbb{R}_{\geq 0} \times \mathbb{R}^n$ *there exist constants* $\mu_0 > 0$*,* $\lambda_0 > 0$ *such that (1.2) is satisfied.*

Definition 1.2.7 (Exponential Stability) *The origin of (1.1) said to be* (locally) exponentially stable *if there exist constants* $\mu > 0$*,* $\lambda > 0$ *and a neighborhood* $\mathcal{D}$ *of the origin such that for any initial condition* $(t_0, x_0) \in \mathbb{R}_{\geq 0} \times \mathcal{D}$ *the corresponding solutions satisfy*

$$|x(t; t_0, x)| \leq \mu |x_0| \mathrm{e}^{-\lambda (t - t_0)}\,, \qquad \forall\, t \geq t_0\,. \tag{1.3}$$

The system (1.1) is said to be globally exponentially stable *if there exist constants* $\mu > 0$*,* $\lambda > 0$ *such that (1.3) is satisfied for any* $(t_0, x_0) \in \mathbb{R}_{\geq 0} \times \mathbb{R}^n$*.*

1.3 Stability Theorems

In this section, we recall a few results on stability theory of the equilibrium of non-autonomous nonlinear systems of the form (1.1). Almost all the results presented in this section will be given without proof. The reader may refer to [5] for further details. The reader should be familiar with Lyapunov stability theory for autonomous nonlinear systems.

1.3.1 Lyapunov Theorems

Definition 1.3.2 (Lyapunov Function Candidates) *A $\mathcal{C}^1$ function $V : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ is said to be a* Lyapunov Function Candidate *for (1.1) if there exist class-$\mathcal{K}_\infty$ functions $\underline{\alpha}(\cdot)$, $\overline{\alpha}(\cdot)$, such that*

$$\underline{\alpha}(|x|) \leq V(t,x) \leq \overline{\alpha}(|x|) \tag{1.4}$$

for all $t \in \mathbb{R}$ and all $x \in \mathbb{R}^n$.

In particular, the lower bound in (1.4) establishes the fact that $V(t,x)$ is *positive definite* and *radially unbounded*, that is, $V(t,x) > 0$ for all $t \in \mathbb{R}$ and all $x \in \mathbb{R}^n - \{0\}$, and

$$\lim_{|x| \to \infty} V(t,x) = +\infty$$

Conversely, the upper bound establishes the property that $V(t,x)$ is *decrescent*. The classic Lyapunov Theorems for non-autonomous systems can be summarized as follows:

Theorem 1.3.3 *Assume that the $\mathcal{C}^1$ function $V : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ a Lyapunov Function Candidate for (1.1). Then, the equilibrium $x = 0$ of (1.1) is:*

- **Uniformly globally stable** *if*

  $$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t,x) \leq 0$$

  for all $t \geq 0$ and all $x \in \mathbb{R}^n$;

- **Uniformly globally asymptotically stable** *if there exists a class-$\mathcal{K}$ function $\alpha(\cdot)$ such that*

  $$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t,x) \leq -\alpha(|x|) \tag{1.5}$$

  for all $t \geq 0$ and all $x \in \mathbb{R}^n$;

- **Globally exponentially stable** *if (1.4) and (1.5) hold with quadratic functions, that is,*

  $$\underline{\alpha}(s) = \underline{a}\, s^2\,, \qquad \overline{\alpha}(s) = \overline{a}\, s^2\,, \qquad \alpha(s) = a\, s^2$$

  for some constants $0 < \underline{a} \leq \overline{a}$, $a > 0$;

- **Uniformly globally asymptotically and locally exponentially stable** *if (1.4) and (1.5) hold with locally quadratic functions, that is, there exist positive numbers δ, $\underline{a}$, $\overline{a}$, a such that*

  $$\underline{\alpha}(s) \geq \underline{a}\, s^2\,, \qquad \overline{\alpha}(s) \leq \overline{a}\, s^2\,, \qquad \alpha(s) \geq a\, s^2$$

  for all $s \in [0, \delta]$.

1.3.4 Converse Theorems and Related Results

We recall first some useful results on the inversion of the theorems of Lyapunov. Converse Lyapunov theorems play a crucial role in modern nonlinear control theory, as the existence of smooth Lyapunov functions is instrumental in assessing various forms of robustness with respect to vanishing and persistent perturbations. In regard to this, a fundamental example is given by the theorem of *total stability*, given later in the section. An introduction to the classic contributions by Massera and Kurzweil, can be found in the textbooks [5, 22]. For recent important results, the reader should consult [7] and [20], which also contain a very nice discussion of early work on the subject as well as detailed and precise bibliographic references. The first converse theorem is extremely important, and concerns the existence of a smooth Lyapunov function for locally Lipschitz systems possessing a UGAS equilibrium. For a proof, see [10] or [7].

Theorem 1.3.5 (Massera) *Assume that in (1.1) the vector field f is locally Lipschitz in $x \in \mathbb{R}^n$, uniformly in t. Assume that the equilibrium $x = 0$ is uniformly globally asymptotically stable. Then, there exists a* smooth *function $V : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \to \mathbb{R}_{\geq 0}$, class-$\mathcal{K}_\infty$ functions $\underline{\alpha}(\cdot)$, $\overline{\alpha}(\cdot)$, and a class-$\mathcal{K}$ function $\alpha(\cdot)$ such that*

$$\underline{\alpha}(|x|) \leq V(t,x) \leq \overline{\alpha}(|x|)$$

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t,x) \leq -\alpha(|x|)$$

for all $t \geq 0$ and all $x \in \mathbb{R}^n$.

Appropriate versions of the above converse theorem for uniform local asymptotic stability and local exponential stability read as follows:

Theorem 1.3.6 *Assume that in (1.1) the vector field f is continuously differentiable in $\mathbb{R}_{\geq 0} \times \mathcal{B}_r$, where $\mathcal{B}_r = \{x \in \mathbb{R}^n : |x| < r\}$, and that the Jacobian matrix $\partial f / \partial x$ is bounded on $\mathcal{B}_r$ uniformly in t. Assume that there exist a class-$\mathcal{KL}$ function $\beta(\cdot,\cdot)$ and a positive number r_0 satisfying $\beta(r_0, 0) < r$ such that the trajectories of* (1.1) *satisfy*

$$|x(t,t_0,x_0)| \leq \beta(|x_0|, t - t_0)\,, \qquad \forall\, x_0 \in \mathcal{B}_(r_0)\,,\ \forall\, t \geq t_0 \geq 0\,.$$

Then, there exists a continuously differentiable function $V : \mathbb{R}_{\geq 0} \times \mathcal{B}_{r_0} \to \mathbb{R}^n$ satisfying

$$\alpha_1(|x|) \leq V(t,x) \leq \alpha_2(|x|)$$

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t,x) \leq -\alpha_3(|x|)$$

$$\left|\frac{\partial V}{\partial x}\right| \leq \alpha_4(|x|)$$

for some class-$\mathcal{K}$ functions $\alpha_i(\cdot)$, $i = 1, \ldots, 4$, defined on $[0, r_0]$. *If, in addition, the system* (1.1) *is autonomous, the function V can be chosen independent of t.*

Theorem 1.3.7 *Assume that in (1.1) the vector field f is continuously differentiable in $\mathbb{R}_{\geq 0} \times \mathcal{B}_r$, where $\mathcal{B}_r = \{x \in \mathbb{R}^n : |x| < r\}$, and that the Jacobian matrix $\partial f / \partial x$ is bounded on $\mathcal{B}_r$ uniformly in t. Assume that there exist positive constants κ, λ, r_0, with $r_0 < r/\kappa$*

such that for any initial condition $(t_0, x_0) \in \mathbb{R}_{\geq 0} \times \mathcal{B}_{r_0}$ *the corresponding solution of (1.1) satisfies*

$$|x(t;t_0,x)| \leq \kappa |x_0| \mathrm{e}^{-\lambda(t-t_0)}, \qquad \forall\, t \geq t_0 .$$

Then, there exists a continuously differentiable function $V : \mathbb{R}_{\geq 0} \times \mathcal{B}_{r_0} \to \mathbb{R}^n$ *satisfying*

$$\begin{aligned} c_1|x|^2 \leq V(t,x) \leq c_2|x|^2 \\ \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t,x) \leq -c_3|x|^2 \\ \left|\frac{\partial V}{\partial x}\right| \leq c_4|x| \end{aligned}$$

for some positive constants c_i, $i = 1, \ldots, 4$, *for all* $t \geq 0$, *and all* $x \in \mathcal{B}_r$. *If, in addition, the equilibrium* $x = 0$ *is globally uniformly exponentially stable, the above inequalities hold on* $\mathbb{R}^n$. *Moreover, if the system is autonomous, the function* V *can be chosen independent of* t.

A proof of Theorem 1.3.6 and Theorem 1.3.7 can be found in [5] and [22]. It is worth noting that Theorem 1.3.5 and Theorem 1.3.7 can be combined, retaining the more restrictive assumptions on the regularity of the vector field f stated in Theorem 1.3.7, to obtain a converse Lyapunov theorem for UGAS and LES equilibria yielding a continuously differentiable Lyapunov function which is locally quadratic at the origin.

Theorem 1.3.8 *Assume that in (1.1) the vector field* f *is continuously differentiable in* $\mathbb{R}_{\geq 0} \times \mathbb{R}^n$, *and the Jacobian matrix* $\partial f / \partial x$ *is bounded on any compact set, uniformly in* t. *Then, the equilibrium* $x = 0$ *is uniformly globally asymptotically stable (UGAS) and locally exponentially stable (LES)* if and only if *there exist a continuously differentiable function* $V : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \to \mathbb{R}_{\geq 0}$, *class-*$\mathcal{K}_\infty$ *functions* $\underline{\alpha}(\cdot)$, $\overline{\alpha}(\cdot)$, *a class-*$\mathcal{K}$ *function* $\alpha(\cdot)$, *and positive numbers* δ, $\underline{a}$, $\overline{a}$, a *such that*

$$\begin{aligned} \underline{\alpha}(|x|) \leq V(t,x) \leq \overline{\alpha}(|x|) \\ \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t,x) \leq -\alpha(|x|) \end{aligned}$$

for all $t \geq 0$ *and all* $x \in \mathbb{R}^n$, *and*

$$\underline{\alpha}(s) \geq \underline{a}\, s^2 , \qquad \overline{\alpha}(s) \leq \overline{a}\, s^2 , \qquad \alpha(s) \geq a\, s^2$$

for all $s \in [0, \delta]$.

A proof of Theorem 1.3.8 can be obtained following the same lines, *mutatis mutandis*, of [4, Lemma 10.1.5]. For autonomous systems possessing a locally asymptotically stable equilibrium, the following theorem due to Kurzweil [6] establishes the existence of a smooth Lyapunov function which is *proper* on the domain of attraction, generalizing in a significant way the classic theorem by Zubov [24]. A nice self-contained proof can be found in [5, Theorem 4.17].

Theorem 1.3.9 (Kurzweil) *Assume that the system* (1.1) *is autonomous, and let* $f : \mathcal{D} \to \mathbb{R}^n$ *be locally Lipschitz on the domain* $\mathcal{D} \subset \mathbb{R}^n$ *containing the origin. Assume that the origin is a (locally) asymptotically stable equilibrium, and denote with* $\mathcal{A} \subset \mathcal{D}$ *its domain of attraction. Then, there exists a smooth, positive definite function* $V : \mathcal{A} \to \mathbb{R}_{\geq 0}$ *and a continuous, positive definite function* $W : \mathcal{A} \to \mathbb{R}_{\geq 0}$ *satisfying*

$$\lim_{x \to \partial \mathcal{A}} V(x) = +\infty$$

$$\frac{\partial V}{\partial x} f(x) \leq -W(x)\,, \qquad \forall\, x \in \mathcal{A}\,.$$

In particular, for any $c > 0$*, the level set* $\Omega_c = \{x \in \mathbb{R}^n : V(x) \leq c\}$ *is a positively invariant compact subset of* $\mathcal{A}$*.*

1.3.10 Stability of Perturbed Systems

The following theorem, known as the *Theorem of Total Stability*, establishes the fact that uniform asymptotic stability of an equilibrium of a nonlinear systems provides robustness against small non-vanishing perturbations. In particular, the theorem establishes boundedness of all trajectories of a perturbed system that originate sufficiently close to the equilibrium, if the perturbation is "sufficiently small" in a meaningful sense.

Theorem 1.3.11 (Total Stability)

Consider system (1.1), and assume that the vector field f *is continuously differentiable in* $\mathbb{R}_{\geq 0} \times \mathcal{B}_r$*, where* $\mathcal{B}_r = \{x \in \mathbb{R}^n : \ |x| < r\}$*, that the Jacobian matrix* $\partial f / \partial x$ *is bounded on* $\mathcal{B}_r$ *uniformly in* t*, and that* $f(t, 0) = 0$ *for all* $t \geq 0$*. Let* $g : \mathbb{R}_{\geq 0} \times \mathcal{B}_r \to \mathbb{R}^n$ *be such that* $g(t, x)$ *is piecewise continuous in* t *and locally Lipschitz in* x*, uniformly in* t*. Assume, in addition, that the equilibrium at the origin of (1.1) is (locally) uniformly asymptotically stable. Then, given any* $\varepsilon > 0$*, there exists* $\delta_1 > 0$ *and* $\delta_2 > 0$ *such that if*

$$\begin{aligned} &|x_0| \leq \delta_1 \\ &|g(t, x)| \leq \delta_2 \quad \forall\, t \geq 0 \quad \forall\, x \in \mathcal{B}_\varepsilon \end{aligned}$$

then the trajectory $x(t) = x(t; t_0, x_0)$ *of the perturbed system*

$$\begin{aligned} \dot{x} &= f(t, x) + g(t, x) \\ x(t_0) &= x_0 \end{aligned}$$

satisfies $|x(t)| \leq \varepsilon$ *for all* $t \geq t_0 \geq 0$*.*

Next, we restrict our attention to systems affected by bounded external disturbances, namely systems of the form

$$\begin{aligned} \dot{x} &= f(t, x, d) \\ x(t_0) &= x_0 \end{aligned} \tag{1.6}$$

where $d(\cdot) \in \mathcal{L}^\infty(\mathbb{R}^m)$. For these systems, we introduce and introduce a few important notions related to bounded-input bounded-state stability. The first such notion is improperly referred to as a stability property, as it does not entail contractivity of the forward trajectory over the semi-infinite interval (in terms of its $\mathcal{L}^\infty$-norm) with respect to the $\mathcal{L}^\infty$-norm of the disturbance. It is, however, an important property related to unifirm boundedness of trajectories in the face of bounded disturbances:

Definition 1.3.12 (Global Uniform Ultimate Boundedness) *System* (1.6) *is said to possess the* global uniform ultimate boundedness property (GUUB) with respect to d *if there exists a class-$\mathcal{N}$ function $\eta(\cdot)$ such that for any initial condition* $(t_0, x_0) \in \mathbb{R} \times \mathbb{R}^n$ *and any* $d(\cdot) \in \mathcal{L}^\infty(\mathbb{R}^m)$*, the forward solution* $x(t) := x(t; t_0, x_0, d_{[t_0,t)})$*,* $t \geq t_0$*, of* (1.6) *satisfies*

$$\|x(\cdot)\|_a \leq \eta(\|d(\cdot)\|_\infty), \qquad \|d(\cdot)\|_\infty := \sup_{t \geq t_0} |d(t)| \tag{1.7}$$

The GUUB property admits a (partial, as the converse statement does not hold) Lyapunov-like characterization, as follows:

Theorem 1.3.13 *Let* $V : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ *be a continuously differentiable function satisfying*

$$\underline{\alpha}(|x|) \leq V(t, x) \leq \overline{\alpha}(|x|)$$

for all $(t, x) \in \mathbb{R} \times \mathbb{R}^n$*, where* $\underline{\alpha}(\cdot)$ *and* $\overline{\alpha}(\cdot)$ *are class-$\mathcal{K}_\infty$ functions. Assume that there exists a class-$\mathcal{N}$-function $\chi(\cdot)$ such that for all $t \in \mathbb{R}$ and all $d \in \mathbb{R}^m$*

$$|x| > \chi(|d|) \implies \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t, x, d) < 0 \tag{1.8}$$

Then, system (1.6) *has the GUUB property with respect to d. Morover, the bound* (1.7) *holds with* $\eta(\cdot) = \underline{\alpha}^{-1} \circ \overline{\alpha} \circ \chi(\cdot)$*.*

The second notion is a generalization of the concept of *internal stability* of an LTI system, as it provides a complete characterization of bounded-input bounded-state behavior together with global uniform asymtotic stability of the origin when the disturbance is inactive:

Definition 1.3.14 (Input-to-State Stability) *System* (1.6) *is said to be* input-to-state stable (ISS) *if there exist class-$\mathcal{K}$ functions $\gamma_0(\cdot)$, $\gamma(\cdot)$ such that for any initial condition* $(t_0, x_0) \in \mathbb{R} \times \mathbb{R}^n$ *and any* $d(\cdot) \in \mathcal{L}^\infty(\mathbb{R}^m)$*, the forward solution* $x(t) := x(t; t_0, x_0, d_{[t_0,t)})$*,* $t \geq t_0$*, of* (1.6) *satisfies*

$$\begin{aligned} \|x(\cdot)\|_\infty &\leq \max\{\gamma_0(|x_0|), \gamma(\|d(\cdot)\|_\infty)\} \\ \|x(\cdot)\|_a &\leq \gamma(\|d(\cdot)\|_a) \end{aligned} \tag{1.9}$$

where $\|x(\cdot)\|_\infty := \sup_{t \geq t_0} |x(t)|$ *and* $\|d(\cdot)\|_\infty := \sup_{t \geq t_0} |d(t)|$*.*

The ISS property entails UGAS of the origin of the system when $d = 0$. The GUUB property admits a complete Lyapunov-like characterization, as follows:

Theorem 1.3.15 *System* (1.6) *ISS (with respect to d as an input) if and only if there exist a continuously differentiable function* $V : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}_{\geq 0}$*, class-$\mathcal{K}_\infty$ functions* $\underline{\alpha}(\cdot)$ *and* $\overline{\alpha}(\cdot)$*, and a class-$\mathcal{K}$-function $\chi(\cdot)$ such that*

$$\underline{\alpha}(|x|) \leq V(t, x) \leq \overline{\alpha}(|x|) \tag{1.10}$$

for all $(t, x) \in \mathbb{R} \times \mathbb{R}^n$ *and*

$$|x| > \chi(|d|) \implies \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t, x, d) < 0 \tag{1.11}$$

for all $t \in \mathbb{R}$ and all $d \in \mathbb{R}^m$ Morover, given (1.10) *and* (1.11)*, the bounds* (1.9) *hold with* $\gamma_0(\cdot) = \underline{\alpha}^{-1} \circ \overline{\alpha}(\cdot)$ *and* $\gamma(\cdot) = \underline{\alpha}^{-1} \circ \overline{\alpha} \circ \chi(\cdot)$*.*

1.3.16 Invariance-like Theorems

A peculiar characteristic of direct adaptive control techniques is that the Lie derivative of certain candidate Lyapunov functions is rendered negative semi-definite by design. For autonomous systems, this situation of usually handled resorting to La Salle's invariance principle and the theorem of Krasovskii and Barbashin. However, for non-autonomous systems, the situation is far more complicated, and available results are in general much weaker. The reason lies in the fact that ω-limit sets of bounded trajectories of non-autonomous systems are not necessarily invariant, as it is indeed the case for autonomous or periodic systems. Invariance of ω-limit sets is the fundamental technical result that enables a "reduction principle" in determining the behavior of solutions when restricted to the zeroing manifold for the derivative of a Lyapunov function candidate, and unfortunately this method of analysis can not be carried over to the non-autonomous case. However, a weaker extension of La Salle's invariance principle can be used to infer certain properties of the asymptotic behavior of systems for which a Lyapunov-like function admitting a negative semi-definite derivative can be found. We begin with a classic result, ubiquitous in the literature of adaptive control, which is a key technical lemma in establishing convergence of integrable signals.

Lemma 1.3.17 (Barbălat's lemma) *Let $\phi : \mathbb{R} \to \mathbb{R}$ be a uniformly continuous function over $[0, \infty)$. Assume also that $\lim_{t\to\infty} \int_0^t \phi(\tau)\mathrm{d}\tau$ exists and is finite. Then, $\lim_{t\to\infty} \phi(t) = 0$.*

The proof of Barbălat's lemma can be found in nearly every book on adaptive control, see for instance [3, Lemma 3.2.6], whereas a significant generalization has been recently suggested in [19].

The main result on "invariance-like" theorems for non-autonomous systems is due to Yoshizawa [23], and it is commonly referred to as the La Salle/Yoshizawa theorem.

Theorem 1.3.18 (La Salle/Yoshizawa) *Consider the nonautonomous system (1.1) where the vector field $f(t, x)$ is piecewise continuous in $t \in \mathbb{R}$, and locally Lipschitz in $x \in \mathbb{R}^n$ uniformly in t. Assume that $x = 0$ is an equilibrium point for (1.1), that is $f(t, 0) = 0$ for all t. Let $V : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ be a continuously differentiable function satisfying*

$$\underline{\alpha}(|x|) \leq V(t, x) \leq \overline{\alpha}(|x|) \tag{1.12}$$

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t, x) \leq -W(x) \tag{1.13}$$

for all $t \geq 0$, for all $x \in \mathbb{R}^n$, where $\underline{\alpha}(\cdot)$ and $\overline{\alpha}(\cdot)$ are class-$\mathcal{K}_\infty$ functions, and $W : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ is a continuous positive semi-definite function. Then, system (1.1) is uniformly globally stable, and satisfy

$$\lim_{t\to\infty} W(x(t)) = 0\,.$$

Proof. Since $f(t, x)$ is piecewise continuous in t and locally Lipschitz in x, the solution $x(t) := x(t : t_0, x_0)$ of (1.1) originating from any initial condition $(t_0, x_0) \in \mathbb{R}_{\geq 0} \times \mathbb{R}^n$ exists uniquely over a maximal interval $\mathcal{I}(t_0, x_0) = [t_0, t_0 + \delta_{\max})$. Next, we show that $x(t)$ is uniformly bounded over $\mathcal{I}(t_0, x_0)$, and thus $\mathcal{I}(t_0, x_0) = [t_0, \infty)$. Let

$$V(t) := V(t, x(t; t_0, x_0))$$

and note that since $\dot{V}(t) \leq 0$

$$V(t, x(t; t_0, x_0)) \leq V(t_0, x_0)\,, \qquad \forall\, t \in \mathcal{I}(t_0, x_0)\,.$$

Therefore

$$|x(t)| \leq (\underline{\alpha}^{-1} \circ \overline{\alpha})(|x_0|) =: B_{x_0}$$

for all $t \in \mathcal{I}(t_0, x_0)$, with $t_0 \in \mathbb{R}_{\geq 0}$ and $x_0 \in \mathbb{R}^n$ arbitrary. This shows that $\delta_{\max} = +\infty$, as otherwise $x(t)$ would leave the compact set $\{x : |x| \leq B_{x_0}\}$ as $t \to t_0 + \delta_{\max}$. Moreover, letting $\rho(\cdot) = (\underline{\alpha}^{-1} \circ \overline{\alpha})(\cdot)$, we obtain

$$|x(t; t_0, x_0)| \leq \rho(|x_0|)\,, \forall\, t_0 \geq 0\,,\ \forall\, t \geq t_0$$

hence global uniform stability of the origin is established. Since $V(t)$ is non increasing and bounded from below, $\lim_{t\to\infty} V(t) = V_\infty$ exists and is finite. Since $\dot{V}(t) \leq -W(x(t))$, it turns out that

$$\int_{t_0}^{t} W(x(\tau))\mathrm{d}\tau \leq V(t_0, x_0) - V(t)$$

and thus $\lim_{t\to\infty} \int_{t_0}^{t} W(x(\tau))\mathrm{d}\tau$ exists and is finite. Next, we show that $W(x(t))$ is a uniformly continuous function of t over $[t_0, \infty)$. Since, by definition,

$$x(t_2; t_0, x_0) = x(t_1; t_0, x_0) + \int_{t_1}^{t_2} f(\tau, x(\tau))\mathrm{d}\tau\,, \quad \forall\, t_2 \geq t_1 \geq t_0$$

and by virtue of the uniform local Lipschitz property there exist $L > 0$ such that

$$|f(t, x)| \leq L|x|\,, \qquad \forall\, t \geq t_0\,,\ \forall\, x : |x| \leq B_{x_0}$$

we obtain

$$|x(t_2) - x(t_1)| \leq \int_{t_1}^{t_2} L|x(\tau)|\mathrm{d}\tau \leq LB_{x_0}|t_1 - t_2|$$

for all $t_2 \geq t_1 \geq t_0$. Given any $\varepsilon > 0$, choose $\delta = \dfrac{\varepsilon}{LB_{x_0}}$ to obtain

$$|t_1 - t_2| < \delta \implies |x(t_1) - x(t_2)| < \varepsilon\,,$$

hence uniform continuity of $x(t)$ is established. Since $W(x)$ is a continuous function of x, it is uniformly continuous over the compact set $\{x : |x| \leq B_{x_0}\}$. Therefore, $W(x(t))$ is a uniformly continuous function of t, and the result of the theorem follows from Barbălat's lemma. □

It is worth noting that the La Salle/Yoshizawa theorem yields a much weaker result than its counterpart for autonomous systems (i.e., La Salle's invariance principle), as in the non-autonomous case the trajectory does not converge in general to an invariant set contained in $\mathcal{S} := \{x \in \mathbb{R}^n : W(x) = 0\}$. Furthermore, since convergence is established by means of Barbalăt's lemma, the set $\mathcal{S}$ is not guaranteed to be uniformly attractive.

To determine the behavior of the trajectory on the set $\mathcal{S}$, it may prove instrumental to use an additional *auxiliary function* $H : \mathbb{R} \times \mathbb{R}^n$, when appropriate conditions hold. The first result, due to Anderson and Moore [1], employs the auxiliary function

$$H(t, x) = \int_{t}^{t+\delta} \dot{V}(\tau, \chi(\tau, t, x))d\tau$$

where $\chi(\tau, t, x)$ is the solution of (1.1) originating from the initial condition x at time t, and $\delta > 0$ is a given constant. For a proof, see [1] or [5, Theorem 8.5].

Theorem 1.3.19 (Anderson and Moore) *Let the assumptions of Theorem 1.3.18 hold for the system* (1.1)*, with* (1.13) *replaced by the weaker condition*

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t,x) \leq 0 \qquad \forall\, (t,x) \in \mathbb{R} \times \mathbb{R}^n$$

and assume, in addition, that there exist $\delta > 0$ and $\lambda \in (0,1)$ such that

$$\int_t^{t+\delta} \dot{V}(t, \chi(\tau; t, x)) \mathrm{d}\tau \leq -\lambda V(t,x)$$

for all $t \in \mathbb{R}_{\geq 0}$ and all $x \in \mathbb{R}^n$, where $\chi(\tau; t, x)$ is the solution of (1.1) at time τ originating from the initial condition x at the initial time t. Then, the equilibrium $x = 0$ is uniformly globally asymptotically stable. Furthermore, if for some positive numbers a_1, a_2, and ρ the comparison functions $\underline{\alpha}(\cdot)$ and $\overline{\alpha}(\cdot)$ satisfy

$$\underline{\alpha}(s) \geq a_1 s^2\,, \qquad \overline{\alpha}(s) \leq a_2 s^2$$

for all $s \in [0, \rho)$, then the equilibrium $x = 0$ is uniformly globally asymptotically and locally exponentially stable.

The most important application of Theorem 1.3.19 regards the appropriate extension to the time-varying case of the familiar notion that an *observable* LTI system having a convergent output response under zero input is necessarily asymptotically stable.

Proposition 1.3.20 *Consider the linear time-varying system*

$$\begin{aligned} \dot{x} &= A(t)x \\ y &= C(t)x \end{aligned} \tag{1.14}$$

where the mappings $A : \mathbb{R}_{\geq 0} \to \mathbb{R}^{n \times n}$ and $C : \mathbb{R} \to \mathbb{R}^{m \times n}$ are continuous and bounded. Assume that (1.14) *is* uniformly completely observable[1]*, that is, there exist constants $\delta > 0$ and $\kappa > 0$ such that the* observability gramian

$$W(t_1, t_2) = \int_{t_1}^{t_2} \Phi^{\mathrm{T}}(\tau, t_1) C^{\mathrm{T}}(\tau) C(\tau) \Phi(\tau, t_1) d\tau\,, \quad t_1 \leq t_2$$

satisfies

$$\kappa I \leq W(t, t+\delta)\,, \qquad \forall\, t \geq 0\,.$$

Furthermore, assume that there exists a continuously differentiable, symmetric mapping $P : \mathbb{R} \to \mathbb{R}^{n \times n}$ solution of the differential equation

$$\dot{P}(t) + A^{\mathrm{T}}(t) P(t) + P(t) A(t) \leq -C^{\mathrm{T}}(t) C(t)$$

satisfying

$$c_1 I \leq P(t) \leq c_2 I$$

for all $t \geq 0$ and some $c_1 > 0$, $c_2 > 0$. Then, the origin is a uniformly (globally) asymptotically stable equilibrium of (1.14).

[1] The reader should be aware of the fact that the definition given here is only valid for *bounded* realizations. The reader should consult [2, 17] for the more general situation in which $A(\cdot)$ and $C(\cdot)$ are measurable and locally essentially bounded.

Proof. Consider the Lyapunov function candidate

$$V(t,x) = x^{\mathrm{T}} P(t) x$$

yielding, along trajectories of (1.14),

$$\dot{V}(t,x) \le -x^{\mathrm{T}} C^{\mathrm{T}}(t) C(t) x \le 0\,.$$

It is easy to see that the assumptions of Theorem 1.3.18 are satisfied, and thus trajectories of (1.14) are bounded, and satisfy $\lim_{t\to\infty} y(t) = 0$. Consider the auxiliary function

$$\begin{aligned} H(t,x) &= \int_t^{t+\delta} \dot{V}(\tau, \chi(\tau,t,x))d\tau \\ &\le -\int_t^{t+\delta} \chi^{\mathrm{T}}(\tau,t,x) C^{\mathrm{T}}(\tau) C(\tau) \chi^{\mathrm{T}}(\tau,t,x) d\tau \\ &= -x^{\mathrm{T}} \int_t^{t+\delta} \Phi^{\mathrm{T}}(\tau,t) C^{\mathrm{T}}(\tau) C(\tau) \Phi(\tau,t) d\tau\, x\,, \end{aligned}$$

as $\chi(\tau,t,x) = \Phi(\tau,t)x$. Therefore,

$$H(t,x) \le -\kappa |x|^2 \le -\frac{\kappa}{c_2} V(t,x)\,,$$

and, since c_2 can always be chosen to be such that $c_2 > \kappa$, the result follows directly from Theorem 1.3.19. □

The second result on uniform asymptotic stability of systems having a Lyapunov function with negative semi-definite derivative is due to Matrosov [11], and it is presented here in a simplified version. The interested reader is referred to [14] for the proof of a more general version, and to [9] for recent important generalizations.

Theorem 1.3.21 (Matrosov) *Consider the nonlinear system* (1.1)*, where f is continuous in t, and locally Lipschitz in x, uniformly in t. Assume that there exists a continuously differentiable function $V : \mathbb{R}_{\ge 0} \times \mathbb{R}^n \to \mathbb{R}_{\ge 0}$ such that the assumptions of Theorem 1.3.18 hold. Assume, in addition, that there exists a continuously differentiable function $H : \mathbb{R}_{\ge 0} \times \mathbb{R}^n \to \mathbb{R}$ with the following properties:*

i. For any fixed $x \in \mathbb{R}^n$, there exists a number $M > 0$ such that

$$|H(t,x)| \le M \qquad \forall\, t \ge 0\,.$$

i1. Let $\mathcal{E}$ be the set of all points $x \in \mathbb{R}^n$ such that $x \ne 0$ and $W(x) = 0$, that is, $\mathcal{E} = \{x : W(x) = 0\} \cap \{x \ne 0\}$. Assume that $\mathcal{E}$ is nonempty[2]*. Assume that the function $H(t,x)$ satisfies*

$$\dot{H}(t,x) := \frac{\partial H}{\partial t} + \frac{\partial H}{\partial x} f(t,x) > 0 \qquad \forall\, t \ge 0\,, \forall\, x \in \mathcal{E}\,.$$

Then, the equilibrium $x = 0$ is globally uniformly asymptotically stable

[2] Note that this rules out the possibility that $W(x)$ is positive definite.

1.4 Passivity

Passivity theory plays a fundamental role in the analysis and design of adaptive systems. Roughly speaking, the concept of passivity is a generalization of the notion of conservation of energy, in the sense that the rate of change of the energy stored in the system does not exceed the power supplied externally. Adaptive laws are usually designed to either exploit natural passivity properties of given plant models (as in the case of Euler-Lagrange or Hamiltonian systems) or to enforce passivity of the resulting closed-loop system. Passivity theory (or, more generally, the theory of dissipative systems) is usually formulated for autonomous systems, where, used in combination with La Salle's invariance principle and specific notions of observability, it yields a powerful tool to assess global asymptotic stability from Lyapunov functions admitting negative semi-definite derivatives. Furthermore, passivity theory and the related concept of finite $\mathcal{L}_2$-gain stability offer a natural extension to nonlinear systems of the concept of $\mathcal{H}_\infty$ norm of a stable transfer function, with all the advantages given by a Lyapunov-like characterization. The excellent monograph [21] provides a standard reference and a rewarding reading, while the reader interested in quickly grasping the fundamental concepts will find a lucid introduction in [4, Sections 10.7–10.9] and [5, Chapter 6]. Here, we will limit ourselves to giving only the most basic definitions and properties, extended to non-autonomous systems, that will be used in the sequel, adopting a simpler (albeit more restrictive) "differential" characterization of dissipativity.

Consider the following non-autonomous system in affine form

$$\begin{aligned} \dot{x} &= f(t,x) + g(t,x)u \\ y &= h(t,x) \end{aligned} \tag{1.15}$$

with state $x \in \mathbb{R}^n$, input $u \in \mathbb{R}^m$, and output $y \in \mathbb{R}^m$. It is assumed that $f(t,x)$, $g(t,x)$, and $h(t,x)$ are continuous in t and smooth in x. Also, assume that $f(t,0) = 0$ and $h(t,0) = 0$ for all t.

Definition 1.4.1 (Passivity) *System (1.15) is said to be* passive *if there exists a smooth nonnegative function $V : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ (usually called a* storage function*) satisfying*

$$\begin{aligned} &\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t,x) \leq 0 \\ &\frac{\partial V}{\partial x} g(t,x) = h^{\mathrm{T}}(t,x) \end{aligned}$$

for all $t \in \mathbb{R}_{\geq 0}$, and all $x \in \mathbb{R}^n$.

Definition 1.4.2 (Strict passivity) *System (1.15) is said to be* strictly passive *if there exists a smooth positive definite storage function $V : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}_{\geq 0}$, and a positive definite function $\alpha(\cdot)$ (called* dissipation rate*) satisfying*

$$\begin{aligned} &\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t,x) \leq -\alpha(x) \\ &\frac{\partial V}{\partial x} g(t,x) = h^{\mathrm{T}}(t,x) \end{aligned}$$

for all $t \in \mathbb{R}_{\geq 0}$, and all $x \in \mathbb{R}^n$.

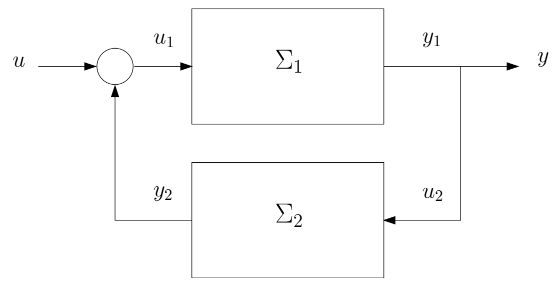

Figure 1.1: Feedback interconnection of passive systems

Corollary 1.4.3 *Assume that (1.15) is passive with respect to a positive definite and decrescent storage function $V(t, x)$, that is, such that*

$$W_1(x) \leq V(t, x) \leq W_2(x)$$

for all $t \in \mathbb{R}_{\geq 0}$ and all $x \in \mathbb{R}^n$, for some positive definite functions $W_1(\cdot)$, $W_2(\cdot)$. Then, the equilibrium $x = 0$ of the unforced system (that is, when $u = 0$) is uniformly stable.

Corollary 1.4.4 *Assume that (1.15) is strictly passive with respect to a positive definite, decrescent, and radially unbounded storage function $V(t, x)$, that is, such that*

$$\gamma_1(|x|) \leq V(t, x) \leq \gamma_2(|x|)$$

for all $t \in \mathbb{R}_{\geq 0}$ and all $x \in \mathbb{R}^n$, for some class-$\mathcal{K}_\infty$ functions $\gamma_1(\cdot)$ and $\gamma_2(\cdot)$. Then, the equilibrium $x = 0$ of the unforced system (that is, when $u = 0$) is uniformly globally asymptotically stable.

Among the desirable properties of passive systems, one of the most useful is the fact that passivity is preserved under negative feedback interconnection. Specifically, let systems Σ_1 and Σ_2 be described respectively by

$$\Sigma_1 : \begin{cases} \dot{x}_1 = f_1(t, x_1) + g_1(t, x_1)u_1 \\ y_1 = h_1(t, x_1) \end{cases}$$

$$\Sigma_2 : \begin{cases} \dot{x}_2 = f_2(t, x_2) + g_2(t, x_2)u_2 \\ y_2 = h_2(t, x_2) \end{cases}$$

where $x_1 \in \mathbb{R}^{n_1}$, $x_2 \in \mathbb{R}^{n_2}$, $u_1 \in \mathbb{R}^m$, $u_2 \in \mathbb{R}^m$, $y_1 \in \mathbb{R}^m$, and $y_2 \in \mathbb{R}^m$. Consider the negative feedback interconnection of Σ_1 and Σ_2, defined by the relations

$$\begin{aligned} u_1 &= -y_2 + u \\ u_2 &= y_1 \\ y &= y_1 \end{aligned} \tag{1.16}$$

where u and y are the overall input and output of the feedback system (see Figure 1.1).

Proposition 1.4.5 *Assume that Σ_1 is passive with storage function $V_1(t, x_1)$, and that Σ_2 is passive with storage function $V_2(t, x_2)$. Then, the negative feedback interconnection defined by (1.16) is passive, with storage function $V(t, x_1, x_2) = V_1(t, x_1) + V_2(t, x_2)$. If both subsystems are strictly passive, with dissipation rates given by $\alpha_1(x_1)$ and $\alpha_2(x_2)$ respectively, then the feedback interconnection defined by (1.16) is strictly passive, with storage function $V(t, x_1, x_2) = V_1(t, x_1) + V_2(t, x_2)$ and dissipation rate $\alpha(x_1, x_2) = \alpha_1(x_1) + \alpha_2(x_2)$.*

Proposition 1.4.6 *Assume that Σ_1 is strictly passive with positive definite, decrescent and radially unbounded storage function $V_1(t, x_1)$ and dissipation rate $\alpha_1(x_1)$. Let Σ_2 be passive with positive definite and decrescent storage function $V_2(t, x_2)$. Then, when $u = 0$, the negative feedback interconnection defined by (1.16) has a uniformly stable equilibrium at the origin $(x_1, x_2) = (0, 0)$. Moreover, if $V_2(t, x_2)$ is radially unbounded, then all trajectories are uniformly bounded, and satisfy $\lim_{t\to\infty} x_1(t) = 0$.*

For LTI systems of the form

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned} \tag{1.17}$$

with $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, and $y \in \mathbb{R}^m$, the following result applies.

Proposition 1.4.7 *Consider system* (1.17)*. Suppose there exist a symmetric positive definite matrix $P \in \mathbb{R}^{n\times n}$ and a symmetric positive semi-definite matrix $Q \in \mathbb{R}^{n\times n}$ such that*

$$A^{\mathrm{T}}P + PA \leq -Q$$

$$PB = C^{\mathrm{T}}.$$

Then, system (1.17) *is passive, and the pair (C, A) is detectable if and only if the pair (A, B) is stabilizable. If, in addition, $Q > 0$, then the system is strictly passive.*

Finally, we recall some useful results for LTI SISO systems with strictly proper transfer function. The reader is referred to [3, Section 3.5] and [5, Chapter 6] for further details. Consider again system (1.17), assume $u \in \mathbb{R}$, $y \in \mathbb{R}$, and let $G(s) = C(sI - A)^{-1}B$ denote its transfer function.

Definition 1.4.8 *A rational proper transfer function $G(s)$ is called* positive real *(PR) if*

(i) $G(s)$ *is real for real* s.

(ii) $\mathrm{Re}[G(s)] \geq 0$ *for all* $\mathrm{Re}[s] > 0$.

Furthermore, assume that $G(s)$ is not identically zero. Then, $G(s)$ is called strictly positive real *(SPR) if $G(s - \epsilon)$ is positive real for some $\epsilon > 0$.*

Lemma 1.4.9 *A rational proper transfer function $G(s)$ is PR if and only if*

(i) $G(s)$ *is real for real* s.

(ii) $G(s)$ *is analytic in* $\mathrm{Re}[s] > 0$*, and the poles on the $j\omega$-axis are simple and such that the associated residues are real and positive.*

(iii) For all real value ω for which $s = j\omega$ is not a pole of $G(s)$, one has $\mathrm{Re}[G(j\omega)] \geq 0$.

For a proof, see [3, Lemma 3.5.1]. The connection between (strict) positive realness of $G(s)$ and (strict) passivity of the realization (1.17) is given by the celebrated KYP lemma, and its subsequent variations:

Lemma 1.4.10 (Kalman, Yakubovich, Popov) *Assume that* (1.17) *is a minimal realization of $G(s)$. Then, $G(s)$ is PR if and only if there exist a symmetric positive definite matrix $P \in \mathbb{R}^{n \times n}$ and a vector $q \in \mathbb{R}^n$ such that*

$$\begin{aligned} &A^{\mathrm{T}} P + PA = -qq^{\mathrm{T}} \\ &PB = C^{\mathrm{T}} . \end{aligned}$$

Lemma 1.4.11 (Meyer, Lefschetz, Kalman, Yakubovich) *A necessary condition for the transfer function $G(s) = C(sI - A)^{-1}B$ to be SPR is that for any positive definite matrix $L \in \mathbb{R}^{n \times n}$ there exist a symmetric positive definite matrix $P \in \mathbb{R}^{n \times n}$, a scalar $\nu > 0$ and a vector $q \in \mathbb{R}^n$ such that*

$$\begin{aligned} &A^{\mathrm{T}} P + PA = -qq^{\mathrm{T}} - \nu L \\ &PB = C^{\mathrm{T}} . \end{aligned}$$

If (1.17) *is a minimal realization of $G(s)$, the above condition is also sufficient.*

The KYP and MLKY lemmas imply that for a minimal realization of a SISO system, positive realness of $G(s)$ (respectively, strictly positive realness) is equivalent to passivity (respectively, strict passivity). In case the realization is not minimal, but the matrix A is Hurwitz, strict positive realness implies strict passivity.

Chapter 2

Stability of Adaptive Systems

2.1 Introduction

In this chapter, we introduce fundamental issues concerning stability of equilibria for classes of systems that arise in direct adaptive control systems design. We start from a few motivating examples, and introduce a typical system structure that we regard as a *standard adaptive control problem*. We then specialize the tools introduced in Chapter 1 to deal with the stability analysis for the standard problem.

2.1.1 Adaptive Stabilization of Nonlinear Systems in Normal Form

Suppose we are given a parameterized family of nonlinear time-invariant systems of the form

$$\dot{x} = f(x, \mu) + g(x, \mu)u \tag{2.1}$$

with state $x \in \mathbb{R}^n$, control input $u \in \mathbb{R}$ and unknown constant parameter vector $\mu \in \mathbb{R}^q$. We make the usual assumptions on smoothness of the vector fields, and assume that the origin $x = 0$ is an equilibrium of the unforced system, i.e., $f(0, \mu) = 0$ for all $\mu \in \mathbb{R}^q$.

The problem we want to address is the design of controllers of fixed structure that enforces certain properties for the trajectories of the closed-loop system, regardless of the actual value of the unknown parameter vector. The simplest (and most fundamental) problem that can be carved out from the above setup is the design of a (possibly dynamic) state-feedback controller, that is, a system of the form

$$\begin{aligned} \dot{\xi} &= \alpha(\xi, x) \\ u &= \beta(\xi, x) \end{aligned} \tag{2.2}$$

that renders the origin of the closed-loop system (2.1)-(2.2) a *globally uniformly asymptotically stable* equilibrium, *robustly* with respect to μ. Note that it is explicitly assumed that the entire state vector is available for feedback. Clearly, a general solution of the above problem is not available unless more structure is specified for the plant model. In particular, the problem can be considerably simplified if additional properties hold, namely the existence of a globally defined normal form in which the uncertain parameters enter linearly. Specifically, we make the following (quite restrictive) assumption:

Assumption 2.1.2 *There exists a globally-defined diffeomorphism*[1] $\Phi : \mathbb{R}^n \to \mathbb{R}^n$, *which preserves the origin, such that the system in the new coordinates* $z = \Phi(x)$ *reads as*

$$\begin{aligned} \dot{z} &= \frac{\partial \Phi}{\partial x} f(\Phi^{-1}(z), \mu) + \frac{\partial \Phi}{\partial x} g(\Phi^{-1}(z), \mu) u \\ &= A_b z + B_b \left[\phi^{\mathrm{T}}(z)\theta + u \right] \end{aligned} \tag{2.3}$$

where A_b, B_b *are in Brunovsky form, i.e.,*

$$A_b = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \quad B_b = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix},$$

the function $\phi(\cdot) : \mathbb{R}^n \to \mathbb{R}^p$ *is known, and* $\theta \in \mathbb{R}^p$, *with* $p \geq q$, *is a re-parametrization of the vector* μ, *that is, a continuous map* $\theta : \mu \mapsto \theta(\mu)$.

If this is the case, if the actual value of the parameter vector θ was available, the obvious memoryless control law that globally asymptotically (and exponentially) stabilizes the origin would be given by

$$u = Kz - \phi^{\mathrm{T}}(z)\theta \tag{2.4}$$

with $K \in \mathbb{R}^{1\times n}$ chosen in such a way that $A_b + B_b K$ is Hurwitz. Since θ is unknown, one may resort to the principle of *certainty equivalence*, and substitute θ in (2.4) with an estimate $\hat{\theta}$, and apply the control

$$u = Kz - \phi^{\mathrm{T}}(z)\hat{\theta}(t)$$

instead. The design must then be completed by a suitable *update law*

$$\dot{\hat{\theta}} = \varphi(\hat{\theta}, z) \tag{2.5}$$

that guarantees stability of the closed-loop system, and, hopefully, convergence of $z(t)$ to the origin, and of $\hat{\theta}(t)$ to θ. To find such an update law, let P be the symmetric, positive definite solution of the Lyapunov matrix equation

$$P(A_b + B_b K) + (A_b + B_b K)^{\mathrm{T}} P = -I$$

and consider the Lyapunov function candidate

$$V(z, \tilde{\theta}) = z^{\mathrm{T}} P z + \tfrac{1}{\gamma}\tilde{\theta}^{\mathrm{T}}\tilde{\theta}$$

where $\gamma > 0$ is a positive constant that plays the role of an *adaptation gain*, and $\tilde{\theta} = \theta - \hat{\theta}$ is a change of coordinates that shifts the origin of the coordinate system for the state of (2.5)

[1] That is, a continuously differentiable map whose inverse exists and is continuously differentiable as well.

to the “true” value of the parameter vector. Evaluating the derivative of V along solutions of (2.3)-(2.4) yields

$$\dot V(z,\tilde\theta) = -|z|^2 + 2z^{\mathrm T} P B_b \phi^{\mathrm T}(z)\tilde\theta + \tfrac{2}{\gamma}\tilde\theta\dot{\tilde\theta}$$

from which, keeping in mind that $\dot{\tilde\theta} = -\dot{\hat\theta}$, one obtains

$$\dot V(z,\tilde\theta) = -|z|^2 + \tfrac{2}{\gamma}\left[\gamma\phi(z)B_b^{\mathrm T} P z - \varphi(\hat\theta, z)\right] .$$

The obvious choice

$$\dot{\hat\theta} = \gamma\phi(z)B_b^{\mathrm T} P z$$

yields

$$\dot V(z,\tilde\theta) = -|z|^2 , \tag{2.6}$$

and this renders the equilibrium $(z,\tilde\theta) = (0,0)$ *uniformly globally stable*, as for any initial condition $(z_0,\tilde\theta_0) \in \mathbb{R}^n \times \mathbb{R}^p$ the corresponding trajectory of the closed-loop system

$$\begin{array}{rcl} \dot z &=& (A_b + B_b K)z + B_b\phi^{\mathrm T}(z)\tilde\theta \\ \dot{\tilde\theta} &=& -\gamma\phi(z)B_b^{\mathrm T} P z \end{array} \tag{2.7}$$

satisfies

$$V(z(t),\tilde\theta(t)) \le V(z_0,\tilde\theta_0) , \qquad \forall\, t \ge 0$$

and thus

$$|(z(t),\tilde\theta(t))| \le a\,|(z_0,\tilde\theta_0)| , \qquad \forall\, t \ge 0$$

for some $a > 0$ which depends only on the given choice of the controller parameters K and γ. The asymptotic properties of the trajectories of (2.7), on the other hand, can be determined by a simple application of La Salle's invariance principle, as (2.7) is an *autonomous* system. In particular, trajectories converge to the largest invariant set $\mathcal{M}$ contained in the set $\mathcal{S} = \{(z,\tilde\theta) \in \mathbb{R}^n \times \mathbb{R}^p : \ \dot V = 0\}$. It is easy to see that any trajectory $(z^\star(t), \tilde\theta^\star(t))$ which originates in $\mathcal{M}$ remains in $\mathcal{M}$ for all $t \ge 0$ (recall that (2.7) is forward complete) and satisfies

$$z^\star(t) \equiv 0 , \qquad \tilde\theta^\star(t) = \tilde\theta^\star = \text{const} .$$

As a result, the set $\mathcal{M}$ is given by

$$\mathcal{M} = \{(z,\tilde\theta) \in \mathbb{R}^n \times \mathbb{R}^p : \ z = 0 ,\ \phi^{\mathrm T}(0)\tilde\theta = 0\} .$$

Note that $\mathcal{M}$ is a closed set, but in general not compact. As a matter of fact, the only case in which $\mathcal{M}$ is compact is when $p = 1$ and $\phi(0) \neq 0$, and thus $\mathcal{M} = \{(0,0)\}$. As a result, it is not possible to conclude that the origin is an asymptotically stable equilibrium of (2.7), apart from the rather trivial case discussed above. The only conclusions that can be drawn are the following:

a.) The origin is a (uniformly) globally stable equilibrium of (2.7).

b.) The closed set $\mathcal{M}$ is globally attractive[2].

[2] It is worth noting that convergence to $\mathcal{M}$ is not guaranteed to be uniform, since $\mathcal{M}$ is not compact.

Assuming that $p > 1$, the closed-loop system achieves boundedness of all trajectories and regulation of $z(t)$ in place of global asymptotic stability of the origin in $\mathbb{R}^{n+p}$, which may still seem a reasonable outcome. In the literature, this result is referred to as "partial stabilization," that is, regulation of a certain subset of the state variables to zero, while preserving boundedness of all trajectories. The problem is that this is not enough to guarantee that (2.7) possesses even the mildest form of robustness ensured by the theorem of total stability. In particular, trajectories of (2.7) may grow unbounded in presence of arbitrarily small non-vanishing perturbations, as will be shown later in this chapter.

The question is whether (2.6) can be used to assess uniform global asymptotic stability of the equilibrium set $\mathcal{M}$, as opposed to asymptotic stability of the equilibrium at the origin. As a matter of fact, it is true that the Lyapunov function candidate V admits a class-$\mathcal{K}_\infty$ estimate from below which is a function of the point-to-set distance from $\mathcal{M}$ alone, as

$$V(z, \tilde{\theta}) \geq z^{\mathrm{T}} P z \geq \lambda_{\min}(P)|z|^2 = \lambda_{\min}(P)|(z, \tilde{\theta})|^2_{\mathcal{M}},$$

and that obviously the estimate

$$\dot{V}(z, \tilde{\theta}) \leq -|(z, \tilde{\theta})|^2_{\mathcal{M}}$$

holds for the derivative of V along (2.7). However, the function V does not admit a class-$\mathcal{K}_\infty$ estimate *from above* which is a function of $|z|$ alone, thus missing a *crucial* ingredient in the Lyapunov characterization of global uniform asymptotic stability with respect to a set. The following counterexample shows that, indeed, equation (2.6) does not imply stability of $\mathcal{M}$ in the sense of Lyapunov, and thus, for the system (2.7), a Lyapunov function with respect to the set $\mathcal{M}$ does not exist.

Example 2.1.3 Consider the simple problem of global asymptotic stabilization of the origin of the scalar system

$$\dot{x} = \mu x^2 + u$$

where $\mu > 0$ is an unknown parameter. The origin is semi-globally stabilizable by means of the simple high-gain feedback $u = -kx$, $k > 0$, meaning that the origin is rendered locally asymptotically stable, with domain of attraction given by the open interval $\mathcal{A} = (-\infty, k/\mu)$. However, it is clear that global asymptotic stabilization is not attainable by linear feedback alone. Applying the principle of certainty equivalence, a candidate controller is given by the control law

$$u = -kx - \hat{\theta}x^2, \qquad k > 0$$

with update law

$$\dot{\hat{\theta}} = \gamma x^3, \qquad \gamma > 0$$

obtained using the obvious Lyapunov function candidate $V(x, \tilde{\theta}) = x^2 + \gamma^{-1}\tilde{\theta}^2$, where $\tilde{\theta} = \hat{\theta} - \mu$. Clearly, in this case we have adopted the trivial re-parametrization $\theta(\mu) = \mu$ for the unknown plant parameter.

Application of La Salle's invariance principle shows that trajectories of the closed-loop system

$$\begin{aligned} \dot{x} &= -kx - x^2\tilde{\theta} \\ \dot{\tilde{\theta}} &= \gamma x^3 \end{aligned} \tag{2.8}$$

are bounded, and converge asymptotically to the equilibrium set $\mathcal{M} = \{0\} \times \mathbb{R}$. We will first show that, while the set $\mathcal{M}$ is obviously attractive, it is not stable in the sense of Lyapunov. Recall that, for the set $\mathcal{M}$ to be stable in the sense of Lyapunov, for any $\epsilon > 0$ there must exist $\delta > 0$ so that for any initial condition $(x_0, \tilde{\theta}_0)$ satisfying $|(x_0, \tilde{\theta}_0)|_{\mathcal{M}} \leq \delta$ the corresponding trajectory $(x(t), \tilde{\theta}(t))$ satisfies $|(x(t), \tilde{\theta}(t))|_{\mathcal{M}} \leq \epsilon$ for all $t \geq 0$.

Fix $k > 0$ and $\gamma > 0$, and consider the set $\mathcal{S}_1 = \{(x, \tilde{\theta}) : x \geq 0,\, x\,\tilde{\theta} \geq -k\}$. This set is forward invariant for the closed-loop system (2.8), as the lower boundary $\{x = 0\}$ is made of trajectories (the set $\mathcal{M}$, which is an equilibrium set), whereas on the boundary $\{x\,\tilde{\theta} = -k\}$ the vector fields point inward (note that $\dot{x} = 0$ and $\dot{\tilde{\theta}} > 0$ on $\{x\,\theta = -k\}$.) On the other hand, the set $\mathcal{S}_2 = \{(x, \tilde{\theta}) : x \geq 0,\, x\,\tilde{\theta} < -k\}$ is backward invariant. Choose, arbitrarily, $\epsilon > 0$ and an initial condition $p(0) = (x(0), \tilde{\theta}(0))$ such that $|p(0)|_{\mathcal{M}} > \epsilon$. Without loss of generality, assume that $p(0)$ lies on the first quadrant, so that $x\,\tilde{\theta} > 0$ (see Figure 2.1). Since $\mathrm{d}x/\mathrm{d}\tilde{\theta} < 0$ on $\mathcal{S}_1$, the trajectory $p(t)$ originating from $p(0)$ remains in $\mathcal{S}_1$ and converge asymptotically to $\mathcal{M}$. Note also that the trajectory in question approaches $\mathcal{M}$ along the normal direction to the set, since $\mathrm{d}x/\mathrm{d}\tilde{\theta} \to -\infty$ as $x \to 0$. Integrating the system backward from the initial condition p moves the trajectory towards the boundary $\{x\,\tilde{\theta} = -k\}$, since in this case

$$\frac{\mathrm{d}x}{\mathrm{d}(-t)} = kx + x^2\tilde{\theta} \quad \text{and} \quad \frac{\mathrm{d}\tilde{\theta}}{\mathrm{d}(-t)} = -\gamma x^3$$

Since $x(t)$ is increasing in backward time, it is bounded away from zero, and so is $\mathrm{d}\tilde{\theta}/\mathrm{d}(-t)$. As a result, there exists a finite time $-\tau$ such that $x(-\tau)\tilde{\theta}(-\tau) = -k$. At the boundary, the vector field of the backward system points inward $\mathcal{S}_2$. Once the backward trajectory has entered the invariant set $\mathcal{S}_2$, the sign of $\mathrm{d}x/\mathrm{d}(-t)$ is reversed, and thus $\lim_{t\to-\infty} x(t) = 0$. This implies that for any $0 < \delta < \epsilon$ there exists $T > 0$ such that $|p(-T)|_{\mathcal{M}} < \delta$. Therefore, the *forward* trajectory originating from $p(-T)$ leaves the ball $\{p : |p|_{\mathcal{M}} \leq \epsilon\}$ in finite time. By virtue of the fact that δ is arbitrary, this implies that the set $\mathcal{M}$ is not stable in the sense of Lyapunov. □

The Role of Passivity

The structure of the closed-loop system (2.7) lends itself to an interpretation that is of fundamental importance in the analysis of adaptive systems: system (2.7) can be seen as the negative feedback interconnection, shown in Figure 2.2, between the system

$$\Sigma_1 : \begin{cases} \dot{z} &= Az + B\phi^{\mathrm{T}}(z)u_1 \\ y_1 &= \phi(z)Cz\,, \end{cases}$$

where $A = (A_b + B_b K)$, $B = B_b$, and $C = B_b^{\mathrm{T}} P$, and the system

$$\Sigma_2 : \begin{cases} \dot{\tilde{\theta}} &= \gamma\, u_2 \\ y_2 &= \tilde{\theta}\,. \end{cases}$$

Note that, by construction, the triplet (A, B, C) is strictly positive real, since it possesses the KYP property,[3] and that the system Σ_1 is strictly passive, with positive definite and

[3] See Lemma 3.5.2, Lemma 3.5.3, and Lemma 3.5.4 in [3].

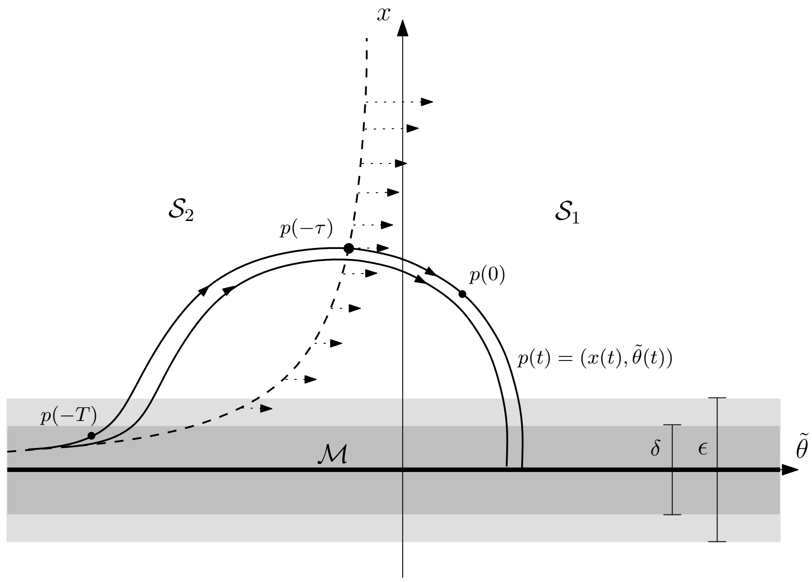

Figure 2.1: Example 2.8

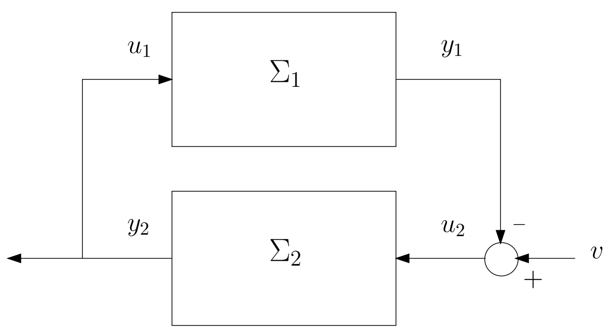

Figure 2.2: Adaptive feedback loop

proper storage function given precisely by $V_1(z) = z^{\mathrm{T}} P z$. Also, the system Σ_2 is readily seen to be passive, with a positive definite and proper storage function given by $V_2(\tilde{\theta}) = \gamma^{-1}\tilde{\theta}^{\mathrm{T}}\tilde{\theta}$. By virtue of Proposition 1.4.5, the feedback interconnection between Σ_1 and Σ_2 shown in Figure 2.2 is passive with respect to the input/output pair (v, y_2), and when $v = 0$ the state trajectories of Σ_1 converge to the origin by virtue of Proposition 1.4.6.

2.1.4 Model-Reference Adaptive Control of Scalar Linear Systems

As a second example, consider the SISO linear system defined by the I/O representation

$$\bar{y}(s) = \frac{b}{s+a}\bar{u}(s)$$

or, equivalently, by the state-space realization

$$\dot{y} = -ay + bu\,,\ \ y(0) = y_0 \tag{2.9}$$

with $y, u \in \mathbb{R}$. It is assumed that the parameter vector $\theta = \mathrm{col}(a, b)$ is unknown; however, the sign of b is known. In particular, without loss of generality, we let $b \geq b_0$ for some $b_0 > 0$. Note that the system (2.9) has unitary relative degree.

The problem we want to address is the following: Given an exponentially stable *reference model* of the form

$$\dot{y}_m = -a_m y_m + b_m u_r \,, \; y_m(0) = 0 \tag{2.10}$$

where $a_m, b_m > 0$ and $u_r(\cdot) \in \mathcal{L}^\infty_{[0,\infty)}$, find a control law for (2.9) to achieve *asymptotic model matching* between the two systems, that is, to let $\lim_{t\to\infty} |y(t) - y_m(t)| = 0$, regardless of the unknown value of the model parameter vector θ. To solve the problem, we appeal once again to the *certainty equivalence principle*, and first devise the solution under the assumption that θ is known. To this end, we postulate the following structure for the controller

$$u = k_1 y + k_2 r \tag{2.11}$$

which is comprised of a feedback and a feedforward term, and derive *matching conditions* relating the vector of controller gains, $k = \mathrm{col}(k_1, k_2)$, with θ to ensure fulfillment of the control objectives. To this end, the dynamics of the model matching error $e := y - y_m$ is easily derived as

$$\dot{e} = -a_m e + (bk_1 + a_m - a)y + (bk_2 - b_m)r \tag{2.12}$$

Consequently, setting

$$k_1 = k_1^* := \frac{a - a_m}{b}\,, \qquad k_2 = k_2^* := \frac{b_m}{b} \tag{2.13}$$

yields the converging dynamics $\dot{e} = -a_m e$, hence the solution to the asymptotic model matching problem. The identities (2.13) are precisely the matching conditions mentioned above. The second step is to replace the fixed gains in the certainty equivalence controller (2.11) with *tunable gains*, $\hat{k} = \mathrm{col}(\hat{k}_1, \hat{k}_2)$ and propose, in place of (2.11), the dynamic controller

$$\begin{aligned} \dot{\hat{k}} &= \tau \\ u &= \hat{k}_1 y + \hat{k}_2 r \end{aligned} \tag{2.14}$$

where $\tau \in \mathbb{R}^2$ is an update law to be determined. This yields the formulation of the asymptotic model matching problem as an adaptive control problem, commonly known as the *Model Reference Adaptive Control (MRAC)* problem. Two strategies may be pursued: In the first one, *direct* adaptation of the tunable gain vector $\hat{k}$ is sought, on the basis of the minimization of a quadratic functional of the model matching error (or, as we will see, to enforce stability of the ensuing error system). This is referred to as *direct MRAC*. The second strategy consists in obtaining an estimate $\hat{\theta} = \mathrm{col}(\hat{a}, \hat{b})$ of the plant parameter vector θ through on-line system identification techniques, and then computing the tunable gains from the matching conditions, that is, by letting

$$\hat{k}_1(\hat{\theta}) := \frac{\hat{a} - a_m}{\hat{b}}\,, \qquad \hat{k}_2(\hat{\theta}) := \frac{b_m}{\hat{b}} \tag{2.15}$$

This approach is referred to as *indirect MRAC*. Note that in the indirect approach one needs to bound the estimate $\hat{b}(t)$ away from the singularity at $\hat{b} = 0$. This is usually accomplished by means of projection techniques, where the assumption made previously that $b \geq b_0 > 0$ becomes instrumental.

Direct Approach

Using the matching conditions, one readily obtains for the closed-loop system

$$\begin{aligned}\dot{e} &= -ay + b(\hat{k}_1 - k_1^* + k_1^*)y + b(\hat{k}_2 - k_2^* + k_2^*)r + a_m y_m - b_m r \\ &= -a_m e + b(\hat{k}_1 - k_1^*)y + b(\hat{k}_2 - k_2^*)r \\ &= -a_m e + b\phi^{\mathrm{T}}(t,e)\tilde{k} \end{aligned} \tag{2.16}$$

where $\tilde{k} := \hat{k} - k^*$ is the *parameter estimate error*, and $\phi^{\mathrm{T}}(t,e) := \begin{pmatrix} e + y_m(t) & r(t) \end{pmatrix}$ is a *known regressor*. Note that the dependence of the regressor on the reference signal, $r(\cdot)$, and the output of the reference model, $y_m(\cdot)$, has been regarded as an explicit dependence on time. Since $b > 0$, the function

$$V(e, \tilde{k}) := \tfrac{1}{2}e^2 + \tfrac{1}{2}b\gamma^{-1}\tilde{k}^{\mathrm{T}}\tilde{k}$$

where $\gamma > 0$ is a gain parameter, is a Lyapunov function candidate for the closed-loop system. Evaluation of the derivative of V along the vector field of the closed-loop system yields (recall that $\dot{\tilde{k}} = \dot{\hat{k}}$)

$$\dot{V} = -a_m e + \tfrac{b}{\gamma}\left[\tau + \gamma\phi(t,e)e\right]$$

leading to the obvious choice

$$\tau = -\gamma\phi(t,e)e$$

for the update law. Application of La Salle/ Yoshizawa Theorem (Theorem 1.3.18), yields global uniform stability of $(e, \tilde{k}) = (0,0)$, boundedness of all trajectories, and asympttotic convergence of $e(t)$ to zero.

Note that, at this point, we do not have enough tools yet to ascertain whether the origin $(e, \tilde{k}) = (0,0)$ is a uniformly asymptotically stable equilibrium, which is not ruled out by the conclusions of La Salle/ Yoshizawa Theorem. The following examples show that the possibility of achieving uniform asymptotic stability of the origin of the *error system*

$$\begin{aligned}\dot{e} &= -a_m e + b\phi^{\mathrm{T}}(t,e)\tilde{k} \\ \dot{\tilde{k}} &= -\gamma\phi(t,e)e \end{aligned} \tag{2.17}$$

depends indeed on the properties of the reference signal $r(\cdot)$.

Example 1: The case of constant reference signals. Consider the case $r(t) = r_0 = \text{const}$, and – for the sake of simplicity – let $a_m = 1$, $b_m = 1$ in the reference model (2.10). Letting $\tilde{y}_m := y_m - r_0$ one obtains the closed-loop error system in the form

$$\begin{aligned}\dot{\tilde{y}}_m &= -\tilde{y}_m \\ \dot{e} &= -a_m e + b\phi^{\mathrm{T}}(\tilde{y}_m, e, r_0)\tilde{k} \\ \dot{\tilde{k}} &= -\gamma\phi(\tilde{y}_m, e, r_0)e \end{aligned} \tag{2.18}$$

where

$$\phi^{\mathrm{T}}(\tilde{y}_m, e, r_0) := \begin{pmatrix} e + \tilde{y}_m + r_0 & r_0 \end{pmatrix}$$

is the regressor. Note that the overall system is autonomous, hence one can use La Salle's invariance principle instead of La Salle's / Yoshizawa theorem to assess the asymptotic properties of its solutions. Note also that r_0 is regarded as a constant parameter. It is easily seen that the origin $(\tilde{y}, e, \tilde{k}) = (0, 0, 0)$, albeit stable in the sense of Lyapunov, is not uniformly attractive, hence not uniformly asymptotically stable. This is a simple consequence of the fact that the origin is not an isolated equilibrium point. As a matter of fact, the system possess an equilibrium manifold (subspace) given by

$$\mathcal{M} = \left\{ (\tilde{y}, e, \tilde{k}) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^2 : \tilde{y} = 0,\ e = 0,\ \tilde{k}_1 = -\tilde{k}_2 \right\}$$

Example 2: The case of sinusoidal reference signals. Consider this time the reference signal $r(t) = \cos(\omega_0 t)$, $\omega > 0$. Let

$$y_m^\star(t) := \int_{-\infty}^{t} e^{\tau - t} \cos(\omega_0 \tau) \mathrm{d}\tau = \frac{\cos(\omega_0 t) + \omega_0 \sin(\omega_0 t)}{1 + \omega_0^2}$$

be the steady-state solution of the reference model, where, once again, it has been assumed that $a_m = 1$ and $b_m = 1$. The change of coordinates $\tilde{y}_m := y_m - y_m^\star$ yields the error system

$$\begin{aligned} \dot{\tilde{y}}_m &= -\tilde{y}_m \\ \dot{e} &= -a_m e + b\phi^{\mathrm{T}}(t, \tilde{y}_m, e)\tilde{k} \\ \dot{\tilde{k}} &= -\gamma\phi(t, \tilde{y}_m, e)e \end{aligned} \tag{2.19}$$

where

$$\phi^{\mathrm{T}}(t, \tilde{y}_m, e) := \begin{pmatrix} e + \tilde{y}_m + y_m^\star(t) & r(t) \end{pmatrix}$$

is the new regressor. Note that both the reference signal and the steady-state of the reference model can be generated by the autonomous linear system (a so-called *exosystem*)

$$\begin{pmatrix} \dot{w}_1 \\ \dot{w}_2 \end{pmatrix} = \begin{pmatrix} 0 & \omega_0 \\ -\omega_0 & 0 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}, \qquad \begin{pmatrix} r \\ y_m^\star \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \frac{\omega_0}{\omega_0^2+1} & \frac{1}{\omega_0^2+1} \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \tag{2.20}$$

with initial condition $w_1(0) = 0$, $w_2(0) = 1$. Note also that the equilibrium $(w_1, w_2) = (0, 0)$ of the exosystem is stable in the sense of Lyapunov, all trajectories of the exosystem are bounded, and that the state matrix is skew-symmetric. As a result, La Salle's invariance principle applies to the closed-loop error system augmented with the exosystem, with Lyapunov function candidate given by

$$V(w, \tilde{y}_m, e, \tilde{k}) := w^{\mathrm{T}} w + \tfrac{1}{2}\tilde{y}_m^2 + \tfrac{1}{2}e^2 + \tfrac{1}{2}b\gamma^{-1}\tilde{k}^{\mathrm{T}}\tilde{k}$$

where $w = (w_1, w_2)$. A simple analysis shows that the trajectories of the closed-loop system (2.19)–(2.20) converge to the largest invariant set $\mathcal{M} \subset \mathbb{R}^2 \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^2$ contained in the set

$$\mathcal{E} := \left\{ (w, \tilde{y}_m, e, \tilde{k}) \in \mathbb{R}^2 \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^2 : \tilde{y}_m = 0,\ e = 0 \right\}$$

This invariant set is obviously comprised of the trajectory $w(t) = (r(t), y_m^\star(t))$ and trajectories $\tilde{k}(t) = \tilde{k}^\star = \text{const}$ satisfying

$$\begin{pmatrix} \dfrac{\cos(\omega_0 t) + \omega_0 \sin(\omega_0 t)}{1 + \omega_0^2} & \cos(\omega_0 t) \end{pmatrix} \begin{pmatrix} \tilde{k}_1^\star \\ \tilde{k}_2^\star \end{pmatrix} = 0$$

for all $t \in \mathbb{R}$. Differentiation of both sided of the above identity yields the system of equations

$$Q(t)\tilde{k}^\star = 0 \qquad \forall\, t \in \mathbb{R}$$

where

$$Q(t) := \begin{pmatrix} \dfrac{\cos(\omega_0 t) + \omega_0 \sin(\omega_0 t)}{1+\omega_0^2} & \cos(\omega_0 t) \\ \dfrac{\omega_0^2 \cos(\omega_0 t) - \omega_0 \sin(\omega_0 t)}{1_+\omega_0^2} & -\omega_0 \sin(\omega_0 t) \end{pmatrix}$$

Since $\det Q(t) = -\omega_0^2/(1+\omega_0^2)$, it is concluded that, necessarily, $\tilde{k}^\star = 0$. As a result, the equilibrium $(\tilde{y}, e, \tilde{k}) = (0,0,0)$ of the time-varying system (2.19) is uniformly (globally) stable in the sense of Lyapunov and globally attractive. Unfortunately, we are not still in the position to conclude that the equilibrium is globally uniformly asymptotically stable, as *uniform* (global) attractivity has not been determined. However, it is noted that asymptotic convergence of the parameter estimates $\hat{k}(t)$ to the "true values" $k^\star$ has been established.

Indirect Approach

In the indirect approach, we use a *model estimator* of the form

$$\dot{\hat{y}} = -\hat{a}y + \hat{b}u + \ell(y - \hat{y}) \tag{2.21}$$

where $\ell > 0$ is the output injection gain. Next, define the *model estimation error* $\tilde{y} := \hat{y} - y$, with dynamics

$$\dot{\tilde{y}} = -\ell\tilde{y} - (\hat{a} - a)y + (\hat{b} - b)r \tag{2.22}$$

and the *estimated model mismatch error* $\hat{e} := \hat{y} - y_m$, with dynamics

$$\dot{\hat{e}} = -a_m\hat{e} - \ell\tilde{y} + (a_m - \hat{a})y - b_m r + \hat{b}u$$

Applying the certainty-equivalence control

$$u = \hat{k}_1(\hat{\theta})y + \hat{k}_2(\hat{\theta})r \tag{2.23}$$

where the tunable gains are given in (2.15), yields

$$\dot{\hat{e}} = -a_m\hat{e} - \ell\tilde{y} \tag{2.24}$$

Next, the equation of the $\tilde{y}$-dynamics (2.22) is written in the more compact form

$$\dot{\tilde{y}} = -\ell\tilde{y} + \psi^{\mathrm{T}}(t, \tilde{y}, \hat{e})\tilde{\theta} \tag{2.25}$$

where $\tilde{\theta} := \hat{\theta} - \theta$ is the *parameter estimate error*, and $\psi^{\mathrm{T}}(t, \tilde{y}, \hat{e}) := \begin{pmatrix} \tilde{y} - \hat{e} - y_m(t) & r(t) \end{pmatrix}$ is a *known regressor*. Following a similar reasoning as in the direct approach, consider the Lyapunov function candidate

$$W(\hat{e}, \tilde{y}, \tilde{\theta}) := \tfrac{1}{2}\hat{e}^2 + \tfrac{\lambda}{2}\left(\tilde{y}^2 + \gamma^{-1}\tilde{\theta}^{\mathrm{T}}\tilde{\theta}\right)$$

where $\lambda > 0$ is a scaling factor to be determined. Scaling the term in parenthesis in the Lyapunov function candiate above allows one to take into account the coupling between the

$\hat{e}$- and the $\tilde{y}$-subsystems in (2.24). Evaluation of the derivative of W along the vector field of the system one obtains

$$\dot{W} = -a_m\hat{e}^2 - \ell\hat{e}\tilde{y} + \lambda\left(-\ell\tilde{y}^2 + \tilde{y}\psi^{\mathrm{T}}(t,\tilde{y},\hat{e})\tilde{\theta} + \gamma^{-1}\tilde{\theta}\dot{\hat{\theta}}\right)$$

Choosing

$$\dot{\hat{\theta}} = -\gamma\psi(t,\tilde{y},\hat{e})\tilde{y}$$

for the update law yields

$$\dot{W} = -a_m\hat{e}^2 - \ell\hat{e}\tilde{y} - \lambda\ell\tilde{y}^2 \tag{2.26}$$

The selection $\lambda > \ell/(4a_m)$ ensures that the quadratic form on the right-hand side of (2.26) is negative definite. As a consequence, application of La Salle/ Yoshizawa Theorem yields boundedness of all trajectories and asymptotic regulation of both $\hat{e}(t)$ and $\tilde{y}(t)$, *if one can show that the control* (2.23) *is well-defined*, for instance, if one can ensure that $\hat{b}(t) \geq b_0$ for all $t \geq 0$. As we will see later in this chapter, this goal can be easily accomplished (at least for this simple example) by projecting the estimate $\hat{b}(t)$ onto the convex set $\mathcal{R} := \{\hat{b} \geq b_0\}$.

Comparing side-by-side the two controllers (and ignoring for the time being the issue of possible singularity of $\hat{b}$ in the indirect approach) yields

$$\text{direct:}\ \begin{cases} \dot{\hat{k}}_1 = -\gamma(y-y_m)y \\ \dot{\hat{k}}_2 = -\gamma(y-y_m)r \\ u = \hat{k}_1 y + \hat{k}_2 r \end{cases} \qquad \text{indirect:}\ \begin{cases} \dot{\hat{y}} = -\hat{a}y + \hat{b}u + \ell(y-\hat{y}) \\ \dot{\hat{a}} = \gamma(\hat{y}-y)y \\ \dot{\hat{b}} = -\gamma(\hat{y}-y)u \\ u = \dfrac{\hat{a}-a_m}{\hat{b}}y + \dfrac{b_m}{\hat{b}}r \end{cases} \tag{2.27}$$

where $\gamma > 0$ and $\ell > 0$ are the adaptation and the observer gains, respectively. It is clear that the indirect approach is more complex, as it involves a controller of higher dimesionality and requires an additional controller gain to be selected (the gain ℓ). This may be a disadvantage when the dimansion of the plant model is large, as the order of an indirect controller increases roughly by a factor of two with respect to its direct counterpart. Nonetheless, the indirect approach presents a clear advantage over the direct approach in the presence of bounded control inputs. Specifically, consider again the plant model (2.9), and assume that the control input is saturated, that is,

$$\dot{y} = -ay + b\,\mathrm{sat}\,(u)\ ,\ y(0) = y_0 \tag{2.28}$$

In this case, the direct design proceeds by ignoring the presence of the saturation function, essentially regaring the effect of input saturations as an unmeasurable disturbance. As a result, the ensuing direct controller is the same as the controller on the left in (2.27). Conversely, using the indirect approach one has the luxury of providing to the parameter estimator a model of the plant that incorporates the effect of the saturation. This task is achieved by replacing the controller on the right of (2.27) with the modified controller

$$\text{indirect (modified):}\ \begin{cases} \dot{\hat{y}} = -\hat{a}y + \hat{b}\,\mathrm{sat}\,(u) + \ell(y-\hat{y}) \\ \dot{\hat{a}} = \gamma(\hat{y}-y)y \\ \dot{\hat{b}} = -\gamma(\hat{y}-y)\mathrm{sat}\,(u) \\ u = \dfrac{\hat{a}-a_m}{\hat{b}}y + \dfrac{b_m}{\hat{b}}r \end{cases}$$

where, again, the issue of non-singularity of $\hat{b}$ has been set aside. It can be verified that this modification has a beneficial effect, similar to that of an anti-windup modification, as it prevents the adaptation law from reacting erroneously to the occurrence of input saturation (see the Matlab-Simulink example provided in the file repository.)

2.2 The Standard Adaptive Control Problem

It was shown in the previous sections that several adaptive control problems share a common formulation in which one needs to study the stability of the interconnection between a strictly passive and a passive system. We will refer to this particular setup as the *standard adaptive control problem*, or simply as the *standard problem*. Namely, we will analyze the stability of the equilibrium at the origin of a nonlinear time varying system of the form

$$\begin{aligned}\dot{x}_1 &= Ax_1 + B\phi^{\mathrm{T}}(t,x)x_2 \\ \dot{x}_2 &= -\gamma\phi(t,x)Cx_1\end{aligned} \tag{2.29}$$

where $x = \mathrm{col}(x_1, x_2) \in \mathbb{R}^{n_1+n_2}$ and γ is a positive constant. The vector field $\phi : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \to \mathbb{R}^{n_2}$, where $n = n_1 + n_2$, defined by the mapping $(t, x) \mapsto \phi(t, x)$ is piecewise continuous in t for any fixed x, and locally Lipschitz in x uniformly in t. In particular, we are interested in determining under which conditions system (2.29) possesses a UGAS (and LES) equilibrium a the origin. As we have already seen, the case in which the vector field ϕ depends only on x_1 and the triplet (A, B, C) is strictly passive or SPR can be easily dealt with using La Salle's invariance principle. A similar situation applies when the dependence on time is due to signals which can be generated as trajectories of autonomous exogenous systems, as in this case La Salle's invariance principle also applies.

A more interesting situation occurs obviously when ϕ depends explicitly on time, hence (2.29) is non-autonomous. We begin with considering the situation in which ϕ depends on t but not on the state x, and thus (2.29) takes the form of a time-varying linear system. Specifically, we consider first the linear time-varying system

$$\begin{aligned}\dot{x}_1 &= Ax_1 + B\phi^{\mathrm{T}}(t)x_2 \\ \dot{x}_2 &= -\gamma\phi(t)Cx_1\end{aligned} \tag{2.30}$$

with the following standing assumptions:

Assumption 2.2.1 *There exist $P = P^{\mathrm{T}} > 0$ and $Q = Q^{\mathrm{T}} > 0$ such that*

$$A^{\mathrm{T}}P + PA \leq -Q$$

$$PB = C^{\mathrm{T}}\,.$$

Assumption 2.2.2 *The function $\phi : \mathbb{R}_{\geq 0} \to \mathbb{R}^{n_1}$ is bounded and globally Lipschitz.*

Let $Q = Q_1^{\mathrm{T}}Q_1$, denote with $A^a(\cdot)$, C^a, and P^a respectively the mappings

$$A^a(t) = \begin{pmatrix} A & B\phi^{\mathrm{T}}(t) \\ -\gamma\phi(t)C & 0 \end{pmatrix}, \qquad C^a = \begin{pmatrix} Q_1 & 0 \end{pmatrix}, \qquad P^a = \begin{pmatrix} P & 0 \\ 0 & \gamma^{-1}I \end{pmatrix},$$

and endow system (2.30) with the output $y = C^a x$. Then, the following holds:

Proposition 2.2.3 *The system* (2.30) *is globally exponentially stable if the pair* $(C^a, A^a(\cdot))$ *is uniformly completely observable.*

Proof. The result follows directly from Proposition 1.3.20, using the Lyapunov function candidate $V(x) = x^{\mathrm{T}} P^a x$. □

The main problem in applying Proposition 2.2.3 to a given system (2.30) is to assess uniform complete observability of the pair $(C^a, A^a(\cdot))$. A direct evaluation of the observability gramian is a formidable task, as it requires the explicit computation of the transition matrix of $A^a(\cdot)$. A useful result is provided by the following lemma, which states that uniform complete observability is invariant under bounded output injection.

Lemma 2.2.4 *Given bounded matrix-valued functions* $A : \mathbb{R}_{\geq 0} \to \mathbb{R}^{n\times n}$, $C : \mathbb{R}_{\geq 0} \to \mathbb{R}^{p\times n}$ *and* $N : \mathbb{R}_{\geq 0} \to \mathbb{R}^{n\times p}$, *the pair* $(C(\cdot), A(\cdot))$ *is uniformly completely observable if and only if so is the pair* $(C(\cdot), A(\cdot) - N(\cdot)C(\cdot))$.

Proof. See [3, Lemma 4.8.1]. □

The above result can be used to replace the computation of the observability gramian of the original system with that of the system under output injection, provided that the latter takes a simpler form. For our purposes, it suffices to use

$$N^a(t) = \begin{pmatrix} AQ_1^{-1} \\ -\gamma\phi(t)CQ_1^{-1} \end{pmatrix}$$

to obtain

$$A^a(t) - N^a(t)C^a = \begin{pmatrix} 0 & B\phi^{\mathrm{T}}(t) \\ 0 & 0 \end{pmatrix}$$

for which the transition matrix can be easily computed as

$$\varPhi(t,\tau) = \begin{pmatrix} I & B\sigma(t,\tau) \\ 0 & I \end{pmatrix}, \qquad \sigma(t,\tau) \triangleq \int_\tau^t \phi^{\mathrm{T}}(s)ds\,.$$

It follows that the observability gramian of $(C^a(\cdot), A^a(\cdot) - N^a(\cdot)C^a(\cdot))$ reads as, after some manipulations,

$$W(t_1,t_2) = \int_{t_1}^{t_2} \begin{pmatrix} Q & QB\sigma(\tau,t_1) \\ \sigma^{\mathrm{T}}(\tau,t_1)B^{\mathrm{T}}Q & \sigma^{\mathrm{T}}(\tau,t_1)B^{\mathrm{T}}QB\sigma(\tau,t_1) \end{pmatrix} d\tau\,. \tag{2.31}$$

Proposition 2.2.5 *Assume that the function* $\phi(\cdot)$ *is bounded and globally Lipschitz, and that there exist constants* $\kappa > 0$, $\delta > 0$ *such that*

$$\int_t^{t+\delta} \phi(\tau)\phi^{\mathrm{T}}(\tau)d\tau \geq \kappa I\,, \qquad \forall\, t \geq 0\,. \tag{2.32}$$

Then, there exists $\mu > 0$ *such that*

$$W(t, t+\delta) \geq \mu I\,, \qquad \forall\, t \geq 0$$

where $W(\cdot,\cdot)$ *is the observability gramian in* (2.31).

Proof. See [3, Lemma 4.8.4]. □

The condition (2.32) is commonly referred to as a *persistence of excitation* (PE) condition. The PE condition plays a fundamental role in the analysis of the asymptotic properties of adaptive systems. In a nutshell, it guarantees that the time-varying signal $\phi(\cdot)$ yields enough couplings between the trajectories $x_1(\cdot)$ and $x_2(\cdot)$ of (2.30) to obtain uniform complete observability. The PE condition has been studied quite extensively in the adaptive control literature. For a comprehensive survey of the properties of PE signals and their role in control and system identification, the reader should consult [3], [8], [12], [15], and the recent paper [13], which provides a nice review of earlier results.

The PE property (2.32), used in conjunction with Assumption 2.2.1 and Assumption 2.2.2, yields a sufficient condition for global exponential stability of (2.30), established by means of Proposition 1.3.20. The result is summarized as follows:

Theorem 2.2.6 *Consider system* (2.30)*, and let assumptions 2.2.1 and 2.2.2 hold. Assume, in addition, that the function $\phi(\cdot)$ satisfies the PE condition* (2.32)*. Then, the origin is a globally exponentially stable equilibrium of* (2.30)*.*

Reverting back to the full nonlinear system (2.29), one may wonder to what extent the result of Theorem 2.2.6 can be used to find conditions for global uniform asymptotic stability of the origin, as opposed to the much weaker form of stability implied by La Salle/Yoshizawa theorem. For this purpose, assume that Assumption 2.2.1 holds for the triplet (A, B, C) in (2.29). As a result, by La Salle/Yoshizawa theorem, the origin is a uniformly globally stable equilibrium, and thus for any initial condition $(t_0, x_0) \in \mathbb{R} \times \mathbb{R}^n$ the corresponding trajectory $x(t; t_0, x_0)$ is bounded for all $t \geq t_0$. Let the parameterized family of functions

$$\tilde{\phi}_{(t_0,x_0)}(\cdot) \triangleq \phi(\cdot, x(\cdot\,; t_0, x_0))\,, \qquad (t_0, x_0) \in \mathbb{R} \times \mathbb{R}^n \tag{2.33}$$

be defined as the function $\phi(\cdot, \cdot)$ evaluated along the trajectories of (2.29), that is, as the mapping

$$t \mapsto \phi(t, x(t; t_0, x_0))\,, \qquad t \geq t_0$$

parameterized by the initial condition of (2.29). Note that since each *single trajectory* $x(t) \triangleq x(t; t_0, x_0)$ satisfies the differential equation

$$\begin{pmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{pmatrix} = \begin{pmatrix} A & B\tilde{\phi}^{\mathrm{T}}_{(t_0,x_0)}(t) \\ -\gamma\tilde{\phi}_{(t_0,x_0)}(t)C & 0 \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}\,, \tag{2.34}$$

to each trajectory of (2.29) one can associate a linear time-varying system, which can in principle be used to study the asymptotic properties of that particular trajectory. In particular, the standing assumptions on $\phi(\cdot, \cdot)$ and boundedness of $x(\cdot\,; t_0, x_0)$ imply that the system (2.34) is well defined for for each pair (t_0, x_0). Note, however, that it is not possible to replace (2.29) with (2.34), and that any conclusion about the asymptotic behavior of $x(t)$ drawn from (2.34) will be valid only for that particular trajectory, unless additional conditions hold.

Theorem 2.2.7 *Assume that, in addition to Assumption 2.2.1, the following conditions hold:*

i.) For any $(t_0, x_0) \in \mathbb{R} \times \mathbb{R}^n$*, the function* $\tilde{\phi}_{(t_0,x_0)}(\cdot)$ *is globally Lipschitz.*

ii.) The function $\tilde{\phi}_{(t_0,x_0)}(\cdot)$ *satisfies a PE condition, that is, for each pair* $(t_0, x_0) \in \mathbb{R} \times \mathbb{R}^n$ *there exist* $\kappa_0 > 0$ *and* $\delta_0 > 0$ *such that*

$$\int_t^{t+\delta_0} \tilde{\phi}_{(t_0,x_0)}(\tau)\tilde{\phi}^{\mathrm{T}}_{(t_0,x_0)}(\tau)d\tau \geq \kappa_0 I\,, \qquad \forall\, t \geq t_0\,.$$

Then, the system (2.29) *is globally exponentially convergent (see Definition 1.2.6).*

It is important to point out that Theorem 2.2.7 does not imply neither exponential stability nor uniform asymptotic stability of the origin of (2.29). As a matter of fact, Theorem 2.2.7 only improves on the results of La Salle/Yoshizawa theorem establishing convergence of $x(t)$ to the origin (as opposed to that of $x_1(t)$ alone,) but the convergence need not be uniform. Moreover, Theorem 2.2.7 may be difficult to apply, as in order to check the conditions i.) and ii.) above, knowledge of the solution $x(\cdot\,; t_0, x_0)$ may be required.

2.2.8 Uniform Asymptotic Stability of Adaptive Systems

From the above discussion, it is clear that for the prototype system (2.29) persistence of excitation of the parameterized family of functions (2.33) plays an important role in extending convergence to the origin of the trajectory $x_1(t)$, implied by LaSalle/Yoshizawa theorem, to the whole trajectory $x(t) = (x_1(t), x_2(t))$. The result, stated formally in Theorem 2.2.7, establishes "pointwise" convergence of each individual trajectory $x(t; t_0, x_0)$, interpreted as a parameterized family of functions indexed by the initial condition (t_0, x_0). A natural question to ask is whether such a convergence can be made uniform with respect to all (t_0, x_0) in any given set of the form $\mathbb{R}_{\geq 0} \times \bar{\mathcal{B}}_r$, so that the result of Theorem 2.2.7 can be extended to yield global uniform asymptotic stability of the origin, versus mere exponential converge. Not surprisingly, the key to achieving this goal is an enhanced persistence of excitation property for the family of functions $\tilde{\phi}_{(t_0,x_0)}$ in (2.33), which holds uniformly with respect to (t_0, x_0). In particular, the following definition is introduced in [13]:

Definition 2.2.9 *Assume that the system* (2.29) *is forward complete. The parameterized family of functions* $\tilde{\phi}_{(t_0,x_0)}$ *in* (2.33) *is said to be* uniformly persistently exciting *(u-PE) if for any* $r > 0$ *there exist* $\kappa > 0$ *and* $\delta > 0$ *such that for any* $(t_0, x_0) \in \mathbb{R}_{\geq 0} \times \bar{\mathcal{B}}_r$ *the corresponding trajectory* $x(t; t_0, x_0)$ *of* (2.29) *satisfies*

$$\int_t^{t+\delta} \tilde{\phi}_{(t_0,x_0)}(\tau)\tilde{\phi}^{\mathrm{T}}_{(t_0,x_0)}(\tau)d\tau \geq \kappa I\,, \qquad \forall\, t \geq t_0\,.$$

Applying the definition of u-PE directly to a system of the form (2.29) appears to be of limited use, as one needs to know *a priori* the solutions of (2.29) to be able to check that the given conditions are satisfied. However, it is possible to infer the u-PE property without solving explicitly the differential equation if appropriate conditions on the solutions of (2.29) and on the vector field $\phi(t, x)$ hold.

Proposition 2.2.10 *Let* $\phi(\cdot, x)$ *be piecewise continuous for each* $x \in \mathbb{R}^n$*, and let* $\phi(t, \cdot)$ *be locally Lipschitz uniformly in* t*. Consider a system of the form* (2.29)*, and assume that there exist:*

i. A number $\mu > 0$ *such that for any initial condition* $(t_0, x_0) \in \mathbb{R}_{\geq 0} \times \mathbb{R}^n$ *the corresponding solution* $x(\cdot; t_0, x_0)$ *satisfies* $\max\{\|x\|_\infty, \|x_1\|_2\} \leq \mu|x_0|$;

ii. A function $\bar{\phi} : \mathbb{R}_{\geq 0} \to \mathbb{R}^{n_2}$ *which is bounded and satisfies the PE condition* (2.32) *for some* $\kappa > 0$ *and some* $\lambda > 0$.

iii. A nondecreasing function $\psi : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ *and nonnegative constants* c_1 *and* c_2 *satisfying* $c_1 + c_2 > 0$ *such that for any unitary vector* $\xi \in \mathbb{R}^{n_2}$

$$|\phi_0^{\mathrm{T}}(t,x)\,\xi| \geq [\,c_1 + c_2\psi(|x_2|)|x_2|\,]\;|\bar{\phi}^{\mathrm{T}}(t)\xi| \tag{2.35}$$

where $\phi_0(t,x) = \phi(t,x)|_{x_1=0}$.

Then, the parameterized family of functions $\tilde{\phi}_{(t_0,x_0)}(\cdot)$ *in* (2.33) *is u-PE. Moreover, if* (2.35) *holds with* $c_1 > 0$*, the function* $\bar{\phi}$ *is not required to be bounded*

Proof. See [13, Prop.2]. □

A simplified version of the above result holds for the important case in which the vector field $\phi(t,x)$ does not depend on the component x_2, and the realization (A, B, C) is strictly passive.

Corollary 2.2.11 *For the given system* (2.29)*, let Assumption 2.2.1 hold. Assume that the vector field* $\phi(t,x)$ *does not depend on* x_2*, that is, let* $\phi(t,x) = \phi(t,x_1)$*. Then, if the function* $\phi_0 : \mathbb{R}_{\geq 0} \to \mathbb{R}^{n_2}$ *defined as* $\phi_0(t) = \phi(t,0)$ *is PE, then the parameterized family of functions* $\tilde{\phi}_{(t_0,x_0)}(\cdot) \triangleq \phi(\cdot, x_1(\cdot\,; t_0, x_0))$ *is u-PE.*

The concept of u-PE is instrumental in deriving a sufficient condition for global uniform asymptotic stability of system (2.29). Specifically, the following result can be proven using the arguments in [13, Theorem 1]:

Theorem 2.2.12 *Consider the system* (2.29)*, where the vector field* $\phi(t,x)$ *is such that* $\phi(\cdot,x)$ *is bounded for each fixed* $x \in \mathbb{R}^n$*, and* $\phi(t,\cdot)$ *is locally Lipschitz uniformly in* t*. Let Assumption 2.2.1 hold. If, in addition:*

i.) There exists a nondecreasing function $\rho : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ *such that*

$$\max\left\{ \left\| \frac{\partial \phi}{\partial x} \right\|, \left| \frac{\partial \phi}{\partial t} \right| \right\} \leq \rho(|x|)$$

for all $(t,x) \in \mathbb{R}_{\geq 0} \times \mathbb{R}^n$*;*

ii.) The parameterized family of functions $\tilde{\phi}_{(t_0,x_0)}(\cdot)$ *in* (2.33) *is u-PE.*

Then, the origin is a uniformly globally asymptotically and locally exponentially stable equilibrium of system (2.29).

2.3 The Issue of Robustness

Consider again the standard adaptive control system (2.29), endowed with Assumptions 2.2.1 and 2.2.2. Let us consider the presence of external disturbance signals $d = \mathrm{col}(d_1, d_2)$, with $d_1(\cdot) \in \mathcal{L}^\infty(\mathbb{R}^{n_1})$ and $d_2(\cdot) \in \mathcal{L}^\infty(\mathbb{R}^{n_2})$ as follows

$$\begin{aligned} \dot{x}_1 &= Ax_1 + B\phi^{\mathrm{T}}(t,x)x_2 + d_1(t) \\ \dot{x}_2 &= -\gamma\phi(t,x)Cx_1 + d_2(t) \end{aligned} \tag{2.36}$$

The aim of this section is to investigate the effect of bounded disturbances on the trajectories of system (2.36), in particular on the properties of boundedness and asymptotic regulation of $x_1(t)$, which are guaranteed by La Salle/Yoshizawa theoerm in absence of model perturbation. The first result is a direct consequence of the theorem of total stability (Theorem 1.3.11), which is behind the *raison d'être* for uniform global asymptotic stability:

Corollary 2.3.1 *Assume that the assumptions of Theorem 2.2.12 hold for system* (2.36). *Then, for any $\varepsilon > 0$ there exist $\delta_1 > 0$ and $\delta_2 > 0$ such that for all $t_0 \in \mathbb{R}$, all $x_0 \in \bar{\mathcal{B}}_{\delta_1} \subset \mathbb{R}^n$ and all $d(\cdot) \in \mathcal{L}^\infty(\mathbb{R}^n)$ such that $\|d\|_\infty \leq \delta_2$, the forward solution $x(t) := x(t; t_0, x_0, d_{[t_0,t)})$, $t \geq t_0$, of* (2.36) *satisfies $\|x(\cdot)\|_\infty \leq \varepsilon$.*

The above result establishes the property of *small-signal bounded-input bounded-state stabiliy* for the perturbed system (2.36), under the assumption of UGAS and LES of the equilibrium at the origin of the unforced system (2.29). It must be noted that the above result is local in nature (that is, it is only valid for "small" values of the $\mathcal{L}^\infty$-norm of the disturbance and the norm of the initial condition). The following example serves the purpose of clarifying this issue.

Example 2.3.2 Consider again the one-dimensional direct MRAC problem of Section 2.1.4, with the following assumptions:

1. The control input coefficient b is known (without loss of generality, let $b = 1$;)
2. The reference model is the identity operator, $y_m(t) = r(t)$, $t \geq 0$;
3. The reference signal is constant, $r(t) = r_0$, $t \geq 0$, where $r_0 \geq 0$.
4. The adaptation gain is selected as $\gamma = 1$;
5. The tracking error dynamics is affected by a constant disturbance, $d(t) = -d_0$, $t \geq 0$, where $d_0 \geq 0$.

Under these assumptions, the equations of the closed-loop system read as

$$\begin{aligned} \dot{e} &= -e + (e + r_0)\tilde{k} - d_0 \\ \dot{\tilde{k}} &= -(e + r_0)e \end{aligned} \tag{2.37}$$

where $\tilde{k} := \hat{k} - k^*$ is the estimation error, and $k^* = a - 1$ (refer to Section 2.1.4.)

Let us consider first the case in which $r_0 > 0$. It is readily seen that (2.37) has a unique equilibrium point at $(e, \tilde{k}) = (0, d_0/r_0)$. Changing coordinates as $\theta := \tilde{k} - d_0/r_0$, system (2.37) is written as

$$\begin{aligned}\dot{e} &= -\left(1 - \frac{d_0}{r_0}\right) e + (e + r_0)\theta \\ \dot{\theta} &= -(e + r_0)e\end{aligned} \tag{2.38}$$

where the equilibrium point has been shifted to the origin, $(e, \theta) = (0, 0)$. The jacobian matrix of the vector field of the system evaluated at the origin reads as

$$A = \begin{pmatrix} \dfrac{d_0}{r_0} - 1 & r_0 \\ -r_0 & 0 \end{pmatrix}$$

and its characteristic polynomial is $p_A(\lambda) = \lambda^2 + (1 - d_0/r_0)\lambda + r_0^2$. Clearly, for $r_0 > d_0$ the equilibrium at the origin of (2.38) is LES, whereas for $0 < r_0 < d_0$ the equilibrium is unstable. To determine the global portrait of the solutions, consider first the case $r_0 > d_0$. Using the Lyapunov function candidate $V(e, \theta) = e^2 + \theta^2$, one obtains

$$\dot{V}(e, \theta) = -2\left(1 - \frac{d_0}{r_0}\right) e^2 \leq 0$$

Application of La Salle's invariance principle (notice that the system is autonomous) yields that the only invariant set contained in the set $\{(e, \theta) \in \mathbb{R}^2 : \dot{V}(e, \theta) = 0\}$ is the origin, hence the origin is a globally asymptotically and locally exponentially stable equilibrium. Clearly, this situation also includes the case in which $d_0 = 0$, where in this case $\theta = \tilde{k}$.

For the case $0 < r_0 < d_0$, let us consider the *backward solutions* of (2.38), which are obtained as the forward solutions of system

$$\begin{aligned}\dot{e} &= \left(1 - \frac{d_0}{r_0}\right) e - (e + r_0)\theta \\ \dot{\theta} &= (e + r_0)e\end{aligned} \tag{2.39}$$

Once again, using the Lyapunov function candidate $V(e, \theta) = e^2 + \theta^2$, one obtains

$$\dot{V}(e, \theta) = 2\left(1 - \frac{d_0}{r_0}\right) e^2 \leq 0$$

hence, using the same reasoning as before, one concludes that the origin is a globally asymptotically stable equilibrium of (2.39). Consequently, reverting back to system (2.38), it is concluded that for any $\varepsilon > 0$ and any $R > 0$, there exists $T_{\varepsilon,R} > 0$ such that for all initial conditions $x(0) := \mathrm{col}(e(0), \theta(0)) \in \bar{\mathcal{B}}_R$ the corresponding backward trajectory satisfies $x(t) := \mathrm{col}(e(t), \theta(t)) \in \bar{\mathcal{B}}_\varepsilon$ for all $t \leq -T_{\varepsilon,R}$. This implies that all forward trajectories of (2.38), except the one originating at $x(0) = 0$, satisfy $\lim_{t\to+\infty} |x(t)| = +\infty$.

Finally, for the case $0 < r_0 = d_0$, it is readily seen that the function $V(e, \theta)$ is a first integral of motion for the system (that is, $\dot{V}(e, \theta) = 0$ for all $(e, \theta) \in \mathbb{R}^2$), hence the solutions generate a family of closed orbits given by the level curves $V(e, \theta) = c$, $c \geq 0$.

To summarize the behavior of the solutions of (2.38) when $r_0 > 0$:

- For $r_0 > d_0$, the origin is GAS and LES;
- For $0 < r_0 < d_0$, the origin is unstable, and solutions originating away from the origin diverge as $t \to \infty$;
- For $0 < r_0 = d_0$, solutions are bounded, as solutions originating away from the origin describe a closed orbit.

It is clear that $\mu := 1 - d_0/r_0 \in (-\infty, \infty)$ plays the role of a *bifurcation parameter* for the system, with $\mu = 0$ corresponding to the critical case. It is also clear that the *stability margin* of the system (in the sense of robustness of the stability of the equilibrium at the origin with respect to the constant disturbance d_0) depends on r_0: the larger the value of r_0, the larger the disturbance that can be accommodated by the system. As $r_0 \to 0$, the system loses robustness to constant disturbances. In particular, when $r_0 = 0$ and $d_0 = 0$, system (2.38) possesses an equilibrium manifold $\mathcal{A} := \{(e, \tilde{k}) \in \mathbb{R}^2 : e = 0\}$ that is globally attractive but not stable in the sense of Lyapunov (see the discussion in Example 2.1.3.) In this case, there is no robustness whatsoever, and even an infinitesimally small positive constant disturbance results in unbounded forward trajectories (note that when $r_0 = 0$ and $d_0 > 0$ the system does not have equilibrium points, hence no closed orbits either.)

The previous discussion has highlighted two important issues related to robustness of adaptive systems in the standard form (2.29) with respect to external disturbances:

- When the equilibrium $x = 0$ is UGAS and LES, there is robustness to "small enough" external disturbance signals, for solutions originating within a neighborhood of the origin, as provided by the theorem of total stability;
- In absence of a UGAS equilibrium at the origin (that is, when only the weaker properties provided by the La Salle/ Yoshizawa theorem hold) there is no guaranteed robustness to external disturbances.

Clearly, these issues make the application of adaptive control techniques less than ideal, especially in all those cases (which are indeed typical) when uniform persistence of excitation of the regressor can not be guaranteed. This lack of robustness to model perturbations has prompted the development of *robust update laws*, that is, modifications of the standard passivity-based update laws aiming at providing robustness to external disturbance of arbitrarily large magnitude. This will be the topic of the next section.

2.4 Robust Modifications of Passivity-based Update Laws

The aim of this section is to introduce three different strategies aimed at providing robustness of adaptive control systems to external bounded disturbances. For notational convenience, we write the standard adaptive control problem in the following form

$$\begin{aligned}
\dot{z} &= Az + B\phi^{\mathrm{T}}(t, z, \hat{\theta})\tilde{\theta} + d_1 \\
\dot{\hat{\theta}} &= \tau + d_2 \\
e &= Cz
\end{aligned} \tag{2.40}$$

where $z \in \mathbb{R}^{n_1}$ comprise the state of the plant model and that of the controller, $\hat{\theta} \in \mathbb{R}^{n_2}$ is the vector of parameter estimates, $\tilde{\theta} := \hat{\theta} - \theta^*$ is the parameter estimate error, $d = \text{col}(d_1, d_2) \in \mathbb{R}^{n_1+n_2}$ is an external disturbance, $e \in \mathbb{R}$ is the error to be regulated and $\tau \in \mathbb{R}^{n_2}$ is an update law to be designed. The regressor $\phi : \mathbb{R} \times \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \to \mathbb{R}^{n_2}$ defined by the mapping $(t, z, \hat{\theta}) \mapsto \phi(t, z, \hat{\theta})$ is continuous and bounded in t for any fixed z and $\hat{\theta}$, and locally Lipschitz in z and $\hat{\theta}$, uniformly in t. Furthermore, it is assumed that the triplet $\{C, A, B\}$ defines a strictly passive system with positive definite storage function $V_1(z) = z^{\mathrm{T}} P z$ and negative definite supply rate $W(z) = -z^{\mathrm{T}} Q z$. It has been shown in the previous sections that the passivity-based update law

$$\tau = -\gamma \phi(t, z, \hat{\theta}) C z \tag{2.41}$$

achieves boundedness of all trajectories and asymptotic regulation of $e(t)$ when $d = 0$, but does not ensure robustness (in the sense of bounded-input bounded-state behavior) to arbitrary disturbance signals $d \in \mathcal{L}^\infty(\mathbb{R}^n)$. To achieve the goal of ensuring bounded-input bounded-state behavior (and, possibly, preserving asymptotic regulation when $d = 0$), we will consider three modifications to the update law (2.41), namely *leakage*, *leakage with dead-zone*, and *parameter projection*.

2.4.1 Update Laws with Leakage

The first and simplest modification consists in adding a dissipation term (a so-called *leakage*) to the update law, namely to replace (2.41) with

$$\tau = -\gamma \phi(t, z, \hat{\theta}) C z - \sigma \gamma \hat{\theta} \tag{2.42}$$

where $\sigma > 0$ is a *small* gain parameter, resulting in the closed-loop system

$$\begin{aligned} \dot{z} &= Az + B\phi^{\mathrm{T}}(t, z, \hat{\theta})\tilde{\theta} + d_1 \\ \dot{\tilde{\theta}} &= -\gamma \phi(t, z, \hat{\theta}) C z - \sigma \gamma \tilde{\theta} - \sigma \gamma \theta^* + d_2 \end{aligned} \tag{2.43}$$

It is noted that the addition of the leakage term destroys the property of the closed-loop system possessing an equilibrium in $(z, \tilde{z}) = (0, 0)$ when $d = 0$, due to the presence of the constant term $-\sigma\gamma\theta^*$ on the equation of the $\tilde{\theta}$-dynamics. This is the reason why the gain of the leakage term should not be chosen too large in order to prevent an unduly deterioration of regulation performance.

Stability Analysis

As customary, consider the Lyapunov function candidate

$$V(z, \tilde{\theta}) = \tfrac{1}{2} z^{\mathrm{T}} P z + \tfrac{1}{2\gamma} \tilde{\theta}^{\mathrm{T}} \tilde{\theta} \tag{2.44}$$

and evaluate its derivative along the vector field of (2.43) to obtain

$$\begin{aligned} \dot{V}(z, \tilde{\theta}) &= -\tfrac{1}{2} z^{\mathrm{T}} Q z + z^{\mathrm{T}} P d_1 - \sigma \tilde{\theta}^{\mathrm{T}} \tilde{\theta} - \sigma \tilde{\theta}^{\mathrm{T}} \theta^* + \gamma^{-1} \tilde{\theta}^{\mathrm{T}} d_2 \\ &\le -\frac{\lambda_{\min}}{2} |z|^2 - \sigma |\tilde{\theta}|^2 + |z||P||d_1| + \gamma^{-1} |\tilde{\theta}||d_2| + \sigma |\tilde{\theta}||\theta^*| \end{aligned} \tag{2.45}$$

where $\lambda_{\min} > 0$ is the smallest eigenvalue of Q. Letting $x := \mathrm{col}(z, \tilde{\theta})$, one obtains (with a minor abuse of notation)

$$\dot{V}(x) \le -\lambda_0 |x|^2 + \mu_0 |x||d| + \sigma |x||\theta^*| \tag{2.46}$$

where $\lambda_0 := \min\{\lambda_{\min}/2, \sigma\}$ and $\mu_0 := |P| + \gamma^{-1}$. Using Young's inequality[4] in the expression

$$-\lambda_0 |x|^2 + \sqrt{\tfrac{\lambda_0}{2}}|x|\sqrt{\tfrac{2}{\lambda_0}}\mu_0|d| + \sqrt{\tfrac{\lambda_0}{2}}|x|\sqrt{\tfrac{2}{\lambda_0}}\sigma|\theta^*|$$

which is equivalent to the right-hand side of (2.46), one obtains

$$\dot{V}(x) \le -\tfrac{\lambda_0}{2}|x|^2 + \tfrac{\mu_0^2}{\lambda_0}|d|^2 + \tfrac{\sigma^2}{\lambda_0}|\theta^*|^2 \tag{2.47}$$

Defining the class-$\mathcal{N}$ function $\chi(\cdot)$ as

$$\chi(s) = \sqrt{\frac{2\mu_0^2}{\lambda_0^2}s^2 + \frac{2\sigma^2}{\lambda_0^2}|\theta^*|^2}$$

from (2.47) one obtains

$$|x| > \chi(|d|) \implies \dot{V}(x) < 0$$

therefore, by Theorem 1.3.13 the perturbed system (2.43) has the GUUB property when $d(\cdot) \in \mathcal{L}^\infty$.

2.4.2 Update Laws with Leakage and Dead-zone Modification

As mentioned, the leakage modification to the passivity-based update law has the undesired effect of destroying the equilibrium at the origin of the closed-loop system in the coordinates $(z, \tilde{\theta})$ in absence of the disturbance. To remedy the situation, a further modification is introduced via the use of a dead-zone function that "switches off" the leakage when the estimation error is inside a given compact set.

To begin, we need a preliminary assumption:

Assumption 2.4.3 *The parameter vector θ^* ranges over the interior a known compact and convex set, $\Theta \subset \mathbb{R}^{n_2}$, that is, $\theta^* \in \mathrm{int}\,\Theta$.*

Fix a number $\ell > 0$ such that

$$\ell > \max_{\theta \in \Theta}\{|\theta_1|, |\theta_2|, \dots, |\theta_{n_2}|\}$$

and consider the *decentralized multivariable dead-zone function* $\mathbf{dz}_\ell(\cdot) : \mathbb{R}^{n_2} \to \mathbb{R}^{n_2}$, defined as

$$\mathbf{dz}_\ell(\vartheta) = \begin{pmatrix} \mathrm{dz}_\ell(\vartheta_1) \\ \mathrm{dz}_\ell(\vartheta_2) \\ \vdots \\ \mathrm{dz}_\ell(\vartheta_{n_2}) \end{pmatrix}, \qquad \mathrm{dz}_\ell(\vartheta_i) = \vartheta_i - \ell\,\mathrm{sat}\left(\frac{\vartheta_i}{\ell}\right), \qquad \mathrm{sat}\,(\vartheta_i) = \begin{cases} -1 & \vartheta_i \le -1 \\ \vartheta_i & |\vartheta_i| < 1 \\ 1 & \vartheta_i \ge 1 \end{cases}$$

The decentralized dead-zone (hereby simply referred to as "dead-zone") with the given choice of the level ℓ has the following properties:

[4] Given $a \ge 0$ and $b \ge 0$, $ab \le a^2/2 + b^2/2$.

- For all $\vartheta \in \mathbb{R}^{n_2}$ and all $\theta \in \Theta$

$$\vartheta^{\mathrm{T}} \mathbf{dz}_\ell(\vartheta + \theta) \geq 0 \tag{2.48}$$

- There exist constants $c_1 > 0$ and $c_2 > 0$ such that for all $\vartheta \in \mathbb{R}^{n_2}$ satisfying $|\vartheta| \geq c_1$ and all $\theta \in \Theta$

$$\vartheta^{\mathrm{T}} \mathbf{dz}_\ell(\vartheta + \theta) \geq c_2 |\vartheta|^2 \tag{2.49}$$

Note also that $\mathbf{dz}_\ell(\theta) = 0$ for all $\theta \in \Theta$. The *leakage with dead-zone modification* of the passivity-based update law (2.41) is defined as

$$\tau = -\gamma \phi(t, z, \hat{\theta}) C z - \sigma\gamma\, \mathbf{dz}_\ell(\hat{\theta})\,, \qquad \sigma > 0 \tag{2.50}$$

resulting in the closed-loop system

$$\begin{aligned} \dot{z} &= Az + B\phi^{\mathrm{T}}(t, z, \hat{\theta})\tilde{\theta} + d_1 \\ \dot{\tilde{\theta}} &= -\gamma\phi(t, z, \hat{\theta}) C z - \sigma\gamma\, \mathbf{dz}_\ell(\tilde{\theta} + \theta^*) + d_2 \end{aligned} \tag{2.51}$$

Note that, as opposed to the standard leakage modification, when $d = 0$ the system preserves the equilibrium at $(z, \tilde{\theta}) = (0, 0)$, due to the fact that $\theta^* \in \Theta$ by assumption.

Stability Analysis

Consider the Lyapunov function candidate (2.44), and evaluate its derivative along the vector field of (2.51) to obtain

$$\begin{aligned} \dot{V}(z, \tilde{\theta}) &= -\tfrac{1}{2} z^{\mathrm{T}} Q z + z^{\mathrm{T}} P d_1 - \sigma\tilde{\theta}^{\mathrm{T}} \mathbf{dz}_\ell(\tilde{\theta} + \theta^*) + \gamma^{-1}\tilde{\theta}^{\mathrm{T}} d_2 && (2.52) \\ &\leq -\frac{\lambda_{\min}}{2}|z|^2 + |z||P||d_1| + \gamma^{-1}|\tilde{\theta}||d_2| - \sigma\tilde{\theta}^{\mathrm{T}} \mathbf{dz}_\ell(\tilde{\theta} + \theta^*) && (2.53) \end{aligned}$$

where $\lambda_{\min} > 0$ is the smallest eigenvalue of Q. As before, let $x := \mathrm{col}(z, \tilde{\theta})$. First, consider the case $|\tilde{\theta}| \leq c_1$, which together with (2.48) and (2.53) implies

$$\dot{V}(z, \tilde{\theta}) \leq -\frac{\lambda_{\min}}{2}|z|^2 + |z||P||d_1| + \frac{c_1}{\gamma}|d_2|$$

Applying Young's inequality to the right-hand side of the above inequality, and using the fact that $|d_i| \leq |d|$, $i = 1, 2$, one obtains

$$\dot{V}(z, \tilde{\theta}) \leq -\frac{\lambda_{\min}}{4}|z|^2 + \frac{|P|^2}{\lambda_{\min}}|d|^2 + \frac{c_1}{\gamma}|d| \tag{2.54}$$

Defining the class-$\mathcal{K}_\infty$ function $\chi_1(\cdot)$ as follows

$$\chi_1(s) = \sqrt{\frac{4|P|^2}{\lambda_{\min}^2}s^2 + \frac{4c_1}{\lambda_{\min}\gamma}s}$$

one obtains, from (2.54) and the assumption $|\tilde{\theta}| \leq c_1$,

$$|z| > \chi_1(|d|) \;\text{ and }\; |\tilde{\theta}| \leq c_1 \implies \dot{V}(z, \tilde{\theta}) < 0 \tag{2.55}$$

Assume now that $|\tilde{\theta}| > c_1$. Using (2.49), the right-hand side of (2.52) can be bounded as

$$\begin{aligned}\dot{V}(z,\tilde{\theta}) &\leq -\frac{\lambda_{\min}}{2}|z|^2 + |z||P||d_1| - c_2\sigma|\tilde{\theta}|^2 + \gamma^{-1}|\tilde{\theta}||d_2| \\ &\leq -\frac{\lambda_{\min}}{4}|z|^2 + \frac{|P|^2}{\lambda_{\min}}|d_1|^2 - \frac{c_2\sigma}{2}|\tilde{\theta}|^2 + \frac{1}{2c_2\sigma\gamma^2}|d_2|^2 \end{aligned} \tag{2.56}$$

where we have made again use of Young's inequality. Letting $x := \mathrm{col}(z, \tilde{\theta})$, one obtains

$$\dot{V}(x) \leq -\lambda_0|x|^2 + \mu_0|d|^2 \tag{2.57}$$

where $\lambda_0 := \min\{\lambda_{\min}/4, c_2\sigma/2\}$ and $\mu_0 := |P|^2/\lambda_{\min} + (2c_2\sigma\gamma^2)^{-1}$. As a result, defining the class-$\mathcal{K}_\infty$ function $\chi_2(\cdot)$ as

$$\chi_2(s) = \sqrt{\frac{\mu_0}{\lambda_0}}s$$

one obtains

$$|x| > \chi_2(|d|) \text{ and } |\tilde{\theta}| > c_1 \implies \dot{V}(x) < 0 \tag{2.58}$$

Next, we combine the two conditions (2.55) and (2.58) into a single one involving a class-$\mathcal{N}$ function. Let the class-$\mathcal{N}$ function $\chi(\cdot)$ be defined as

$$\chi(s) = \sqrt{c_1^2 + \chi_1^2(s) + \chi_2^2(s)}$$

and notice that $|x| > \chi(|d|)$ implies $|x| > \chi_2(|d|)$, and that $|x| > \chi(|d|)$ implies $|x|^2 > c_1^2 + \chi_1^2(|d|)$. In particular, when $|\tilde{\theta}| \leq c_1$ one obtains

$$c_1^2 + \chi_1^2(|d|) < |x|^2 \implies c_1^2 + \chi_1^2(|d|) < |z|^2 + |\tilde{\theta}|^2 \leq |z|^2 + c_1^2 \implies \chi_1^2(|d|) < |z|^2$$

hence

$$|\tilde{\theta}| \leq c_1 \text{ and } |x| > \chi(|d|) \implies |\tilde{\theta}| \leq c_1 \text{ and } |z| > \chi_1(|d|) \implies \dot{V}(x) < 0$$

Conversely,

$$|\tilde{\theta}| > c_1 \text{ and } |x| > \chi(|d|) \implies |\tilde{\theta}| > c_1 \text{ and } |z| > \chi_2(|d|) \implies \dot{V}(x) < 0$$

therefore, by Theorem 1.3.13 the perturbed system (2.51) has the GUUB property when $d(\cdot) \in \mathcal{L}^\infty$.

2.4.4 Update Laws with Parameter Projection

The last modification of the standard passivity-based update law presented in this section is applicable to those cases in which the disturbance affects only the z-dynamics of system (2.40), that is, when $d_2 = 0$. As in the previous section, it is assumed that Assumption 2.4.3 holds. Note that convexity of the parameter set Θ is a strict requirement, along with compactness. In this regard, we pose an additional requirement:

Assumption 2.4.5 *The set Θ is given by*

$$\Theta = \{\theta \in \mathbb{R}^{n_2} : \Pi(\theta) \leq 0\}$$

where $\Pi(\cdot) : \mathbb{R}^{n_2} \to \mathbb{R}$ is a convex and differentiable function.

Denote with $\nabla\Pi(\cdot)$ the gradient of $\Pi(\cdot)$, that is $\nabla\Pi(\theta) = \begin{pmatrix} \frac{\partial\Pi}{\partial\theta_1}(\theta) & \frac{\partial\Pi}{\partial\theta_2}(\theta) & \cdots & \frac{\partial\Pi}{\partial\theta_{n_2}}(\theta) \end{pmatrix}^{\mathrm{T}}$, and define the *projection operator onto* Θ as follows:

$$\underset{\hat\theta\in\Theta}{\mathrm{Proj}}\{\tau\} = \begin{cases} \tau & \text{if } \hat\theta \in \mathrm{int}\,\Theta \text{ or } \{\hat\theta \in \partial\Theta \text{ and } \nabla\Pi^{\mathrm{T}}(\hat\theta)\tau \leq 0\} \\ \left(I - \dfrac{\nabla\Pi(\hat\theta)\nabla\Pi^{\mathrm{T}}(\hat\theta)}{|\nabla\Pi(\hat\theta)|^2}\right)\tau & \text{if } \hat\theta \in \partial\Theta \text{ and } \nabla\Pi^{\mathrm{T}}(\hat\theta)\tau > 0 \end{cases}$$

The dynamics of the parameter vector estimate is selected as

$$\dot{\hat\theta} = \underset{\hat\theta\in\Theta}{\mathrm{Proj}}\{\tau\}\,, \quad \hat\theta(0) \in \mathrm{int}\,\Theta \tag{2.59}$$

where τ is the passivity-based update law (2.41) resulting in the closed-loop system[5]

$$\begin{aligned} \dot z &= Az + B\phi^{\mathrm{T}}(t,z,\hat\theta)\tilde\theta + d_1 \\ \dot{\hat\theta} &= \underset{\hat\theta\in\Theta}{\mathrm{Proj}}\left\{-\gamma\phi(t,z,\hat\theta)Cz\right\} \end{aligned} \tag{2.60}$$

The use of parameter projection ensures the following properties:

Proposition 2.4.6 *The set Θ is forward invariant under the flow of* (2.59).

Proof. At each point $\hat\theta \in \partial\Theta$

$$\nabla\Pi^{\mathrm{T}}(\hat\theta)\underset{\hat\theta\in\Theta}{\mathrm{Proj}}\{\tau\} = \begin{cases} \nabla\Pi^{\mathrm{T}}(\hat\theta)\tau & \text{if } \nabla\Pi^{\mathrm{T}}(\hat\theta)\tau \leq 0\} \\ \nabla\Pi^{\mathrm{T}}(\hat\theta)\left(I - \dfrac{\nabla\Pi(\hat\theta)\nabla\Pi^{\mathrm{T}}(\hat\theta)}{|\nabla\Pi(\hat\theta)|^2}\right)\tau & \text{if } \nabla\Pi^{\mathrm{T}}(\hat\theta)\tau > 0 \end{cases}$$

Clearly, if $\nabla\Pi^{\mathrm{T}}(\hat\theta)\tau \leq 0$, then $\nabla\Pi^{\mathrm{T}}(\hat\theta)\,\mathrm{Proj}_{\hat\theta\in\Theta}\{\tau\} \leq 0$ as well. Conversely, assume $\nabla\Pi^{\mathrm{T}}(\hat\theta)\tau > 0$, and decompose τ along the direction of the vector $\nabla\Pi(\hat\theta)$ and a given basis of the tangent plane to $\partial\Theta$ at $\hat\theta$, that is, let

$$\tau = \alpha\nabla\Pi(\hat\theta) + \psi$$

for some $\alpha > 0$ and $\psi \in \{\mathrm{span}\nabla\Pi(\hat\theta)\}^{\perp}$. Then

$$\begin{aligned} \nabla\Pi^{\mathrm{T}}(\hat\theta)\underset{\hat\theta\in\Theta}{\mathrm{Proj}}\{\tau\} &= \nabla\Pi^{\mathrm{T}}(\hat\theta)\left(\tau - \frac{\nabla\Pi^{\mathrm{T}}(\hat\theta)\tau}{|\nabla\Pi(\hat\theta)|^2}\nabla\Pi(\hat\theta)\right) \\ &= \nabla\Pi^{\mathrm{T}}(\hat\theta)\left(\alpha\nabla\Pi(\hat\theta) + \psi - \alpha\frac{\nabla\Pi^{\mathrm{T}}(\hat\theta)\nabla\Pi(\hat\theta)}{|\nabla\Pi(\hat\theta)|^2}\nabla\Pi(\hat\theta) - \frac{\nabla\Pi^{\mathrm{T}}(\hat\theta)\psi}{|\nabla\Pi(\hat\theta)|^2}\nabla\Pi(\hat\theta)\right) \\ &= \nabla\Pi^{\mathrm{T}}(\hat\theta)\left(\alpha\nabla\Pi(\hat\theta) + \psi - \alpha\nabla\Pi(\hat\theta)\right) \\ &= 0 \end{aligned}$$

As a consequence, $\nabla\Pi^{\mathrm{T}}(\hat\theta)\,\mathrm{Proj}_{\hat\theta\in\Theta}\{\tau\} \leq 0$ at each point $\hat\theta \in \partial\Theta$, hence the vector field of system (2.59) points inward along the boundary of Θ. □

[5] Recall that, by assumption, $d_2 = 0$.

Proposition 2.4.7 *Let* $\tilde{\theta} := \hat{\theta} - \theta^*$. *Then,* $\tilde{\theta}^{\mathrm{T}} \operatorname{Proj}_{\hat{\theta}\in\Theta}\{\tau\} \leq \tilde{\theta}^{\mathrm{T}}\tau$ *for all* $\tilde{\theta} \in \mathbb{R}^{n_2}$ *and all* $\theta^* \in \operatorname{int}\Theta$.

Proof. According to the definition of $\operatorname{Proj}_{\hat{\theta}\in\Theta}\{\tau\}$, we only need to prove the proposition in the case where $\hat{\theta} \in \partial\Theta$ and $\nabla\Pi^{\mathrm{T}}(\hat{\theta})\tau > 0$. From the convexity of the function $\Pi(\cdot)$ and the fact that $\theta^* \in \operatorname{int}\Theta$, it follows that

$$\tilde{\theta}^{\mathrm{T}}\nabla\Pi(\hat{\theta}) = (\hat{\theta} - \theta^*)^{\mathrm{T}}\nabla\Pi(\hat{\theta}) \geq 0 \quad \forall\, \hat{\theta} \in \partial\Theta$$

Consequently, if $\hat{\theta} \in \partial\Theta$ and $\nabla\Pi^{\mathrm{T}}(\hat{\theta})\tau > 0$

$$\tilde{\theta}^{\mathrm{T}} \operatorname*{Proj}_{\hat{\theta}\in\Theta}\{\tau\} \leq \tilde{\theta}^{\mathrm{T}}\tau = \tilde{\theta}^{\mathrm{T}}\tau - \frac{\tilde{\theta}^{\mathrm{T}}\nabla\Pi(\hat{\theta})\,\nabla\Pi^{\mathrm{T}}(\hat{\theta})\tau}{|\nabla\Pi(\hat{\theta})|^2} \leq \tilde{\theta}^{\mathrm{T}}\tau$$

□

Stability Analysis

Evaluation of the Lyapunov function candidate (2.44) along the vector field of the closed-loop system (2.60) yields

$$\begin{aligned}\dot{V}(z,\tilde{\theta}) &= -\tfrac{1}{2}z^{\mathrm{T}}Qz + z^{\mathrm{T}}PB\phi^{\mathrm{T}}(t,z,\hat{\theta}) + z^{\mathrm{T}}Pd_1 + \tilde{\theta}^{\mathrm{T}} \operatorname*{Proj}_{\hat{\theta}\in\Theta}\left\{-\gamma\phi(t,z,\hat{\theta})Cz\right\} \\ &\leq -\tfrac{1}{2}z^{\mathrm{T}}Qz + z^{\mathrm{T}}Pd_1 \\ &\leq -\frac{\lambda_{\min}}{2}|z|^2 + |z||P||d_1| \end{aligned} \tag{2.61}$$

where we have made use of Proposition 2.4.7. Adding and subtracting the term $\lambda_{\min}|\tilde{\theta}|^2/2$ to the right-hand side of the last inequality in (2.61), and recalling that the solution of (2.59) satisfies $\hat{\theta}(t) \in \Theta$ for all $t \geq 0$, one obtains

$$\begin{aligned}\dot{V}(z,\tilde{\theta}) &\leq -\frac{\lambda_{\min}}{2}|z|^2 - \frac{\lambda_{\min}}{2}|\tilde{\theta}|^2 + |z||P||d_1| + \frac{\lambda_{\min}}{2}|\tilde{\theta}|^2 \\ &\leq -\frac{\lambda_{\min}}{2}|x|^2 + |x||P||d_1| + \frac{\lambda_{\min}}{2}\mu^2 \end{aligned} \tag{2.62}$$

where $\mu = 2\max_{\theta\in\Theta}|\theta|$. Application of Young's inequality yields

$$\dot{V}(x) \leq -\frac{\lambda_{\min}}{4}|x|^2 + \frac{|P|^2}{\lambda_{\min}}|d_1|^2 + \frac{\lambda_{\min}}{2}\mu^2$$

Consequently, defining the class-$\mathcal{N}$ function

$$\chi(s) = \sqrt{\frac{4|P|^2}{\lambda_{\min}^2}s^2 + 2\mu^2}$$

one obtains

$$|x| > \chi(|d_1|) \implies \dot{V}(x) < 0$$

therefore, by Theorem 1.3.13 the perturbed system (2.60) has the GUUB property when $d_1(\cdot) \in \mathcal{L}^\infty$.

Chapter 3

Model-Reference Adaptive Control

3.1 Design for Linear SISO Systems

Consider the parameterized family of single-input single-output LTI systems

$$\begin{aligned}\dot{x} &= A(\mu)x + B(\mu)u \\ y &= C(\mu)x\end{aligned} \tag{3.1}$$

with state $x \in \mathbb{R}^n$, control input $u \in \mathbb{R}$, and regulated output $y \in \mathbb{R}$. It is assumed that $A(\cdot)$, $B(\cdot)$ and $C(\cdot)$ are continuous matrix-valued function of the parameter vector μ, which is assumed to range within a given compact set $\mathcal{P} \subset \mathbb{R}^p$. We begin with making the following assumptions on the model (3.1):

Assumption 3.1.1 *The pair $(A(\mu), B(\mu))$ is controllable and the pair $(C(\mu), A(\mu))$ is observable for any $\mu \in \mathcal{P}$.*

Assumption 3.1.2 *The model* (3.1) *has relative degree equal to one for any $\mu \in \mathcal{P}$. Moreover, the sign of the high-frequency gain $b(\mu) = C(\mu)B(\mu)$ is constant for any $\mu \in \mathcal{P}$, and known.*

It is worth noticing that Assumption 3.1.1 is stronger than what actually needed for the solvability of the problem, as stabilizability and detectability would suffice. The stronger assumption on the minimality of (3.1) is made with the only purpose of simplifying the problem at this stage, and can be easily removed. On the other hand, removal of the assumption on the relative degree of the model requires the development of additional tools, and will be pursued in the next chapters.

It is well known that Assumption 3.1.2 implies that, by means of a change of coordinates, system (3.1) can be written in the following form

$$\begin{aligned}\dot{z} &= A_{11}(\mu)z + A_{12}(\mu)y \\ \dot{y} &= A_{21}(\mu)z + a_{22}(\mu)y + b(\mu)u\end{aligned} \tag{3.2}$$

with $z \in \mathbb{R}^{n-1}$, where, without loss of generality, it is assumed that there exists a given constant $b_0 > 0$ such that $b(\mu) \geq b_0$ for all $\mu \in \mathcal{P}$. A *reference model* for the input/otput behavior of (3.1) is given as a stable and minimum-phase transfer function

$$\bar{y}_r(s) = W_m(s)\bar{u}_r(s)$$

where $\bar{y}_r(s)$ and $\bar{u}_r(s)$ denote the Laplace transform of the input and the output of the reference model, respectively. It is assumed that $u_r(\cdot)$ is a piecewise continuous and bounded signal. Since for obvious reasons the reference model is required to have relative degree at least equal to that of the plant, we consider the simplest such model, that is, a transfer function with realization

$$\dot{y}_r = -\alpha_m y_r + \beta_m u_r ,$$

where $\alpha_m > 0$ and $\beta_m > 0$. The reader will have no difficulty in extending the results of this section to more general reference models within the considered class.

The problem we want to address is that of finding a controller for (3.1) that, processing only information on the plant output y and the input and output pair (u_r, y_r), is capable to let $y(t)$ track $y_r(t)$ asymptotically, for any $\mu \in \mathcal{P}$. To approach the problem posed by parameter uncertainty, we look for a certainty-equivalence adaptive controller, that is, a parameterized family of controllers of the form

$$\begin{aligned} \dot{\xi} &= F_c(\hat{\theta})\xi + G_c(\hat{\theta})\mathbf{y} \\ u &= H_c(\hat{\theta})\xi + K_c(\hat{\theta})\mathbf{y} \end{aligned} \tag{3.3}$$

with state $\xi \in \mathbb{R}^\nu$ and input $\mathbf{y} = \mathrm{col}(y, u_r, y_r)$, endowed with an update law

$$\dot{\hat{\theta}} = \varphi(\hat{\theta}, \xi, \mathbf{y})$$

for the tunable parameter vector $\hat{\theta} \in \mathbb{R}^m$. A crucial assumption that ensures solvability of the problem is that the plant model is minimum phase, robustly with respect to $\mu \in \mathcal{P}$. More precisely, we assume the following:

Assumption 3.1.3 *There exist a continuous, symmetric and positive definite matrix-valued function $P : \mathbb{R}^p \to \mathbb{R}^{(n-1)\times(n-1)}$, and positive constants a_1, a_2 satisfying*

$$a_1 I \leq P(\mu) \leq a_2 I$$

$$A_{11}^{\mathrm{T}}(\mu)P(\mu) + P(\mu)A_{11}(\mu) \leq -I$$

for all $\mu \in \mathcal{P}$.

3.1.4 Design of the Certainty-Equivalence Controller

In what follows, the solution of the problem is derived under the assumption that the actual value of the parameter vector μ is known. The structure of the solution will be then exploited for the adaptive design. The starting point is to convert the tracking problem into a regulation problem, introducing the tracking error $e = y - y_r$ and deriving the error dynamics

$$\begin{aligned} \dot{e} &= A_{21}(\mu)z + a_{22}(\mu)y + b(\mu)u + \alpha_m y_r - \beta_m u_r \\ &= -\alpha_m e + A_{21}(\mu)z + (a_{22}(\mu) + \alpha_m)y - \beta_m u_r + b(\mu)u. \end{aligned}$$

From the equation above, it is clear that the control law that enforces convergence of the tracking error (with the same dynamics of the reference model) is

$$u = -\frac{1}{b(\mu)} \left[A_{21}(\mu)z + (a_{22}(\mu) + \alpha_m)y - \beta_m u_r \right] . \tag{3.4}$$

As a matter of fact, the resulting closed-loop system (in the coordinates (y_r, z, e)) reads as the stable cascade interconnection

$$\begin{aligned}\dot{y}_r &= -\alpha_m y_r + \beta_m u_r \\ \dot{z} &= A_{12}(\mu) y_r + A_{11}(\mu) z + A_{12}(\mu) e \\ \dot{e} &= -\alpha_m e \,,\end{aligned}$$

and thus the trajectory of the lower subsystem (the e-dynamics) converge asymptotically, while the trajectories of the remaining subsystems are bounded, as the signal $u_r(\cdot)$ is bounded.

Remark 3.1.5 The certainty-equivalence control law (3.4) has an interesting interpretation as the superposition $u = u_{\text{zd}} + u_{\text{st}}$ of two distinct control actions, namely

$$u_{\text{zd}} = -\frac{1}{b(\mu)} \left[A_{21}(\mu) z + (a_{22}(\mu) + \alpha_m) y_r - \beta_m u_r \right]$$

and

$$u_{\text{st}} = -\frac{1}{b(\mu)} (a_{22}(\mu) + \alpha_m) e \,.$$

The control u_{zd} is the unique control law that renders invariant the zero dynamics of the augmented system

$$\begin{aligned}\dot{y}_r &= -\alpha_m y_r + \beta_m u_r \\ \dot{z} &= A_{11}(\mu) z + A_{12}(\mu) y \\ \dot{y} &= A_{21}(\mu) z + a_{22}(\mu) y + b(\mu) u \\ e &= y - y_r\end{aligned}$$

with respect to the input u and the output e. In the given set of coordinates, the zero dynamics of the augmented system is given precisely by the forced trajectories of the system

$$\begin{aligned}\dot{y}_r &= -\alpha_m y_r + \beta_m u_r \\ \dot{z} &= A_{12}(\mu) y_r + A_{11}(\mu) z\end{aligned}$$

evolving on an n-dimensional submanifold of the state space $\mathcal{X} = \mathbb{R} \times \mathbb{R}^{n-1} \times \mathbb{R}$ of the augmented system. The control u_{st}, on the other hand, renders the zero dynamics submanifold globally attractive, with assigned transversal dynamics. The fact that the control law is designed to force the trajectories on the system onto its zero dynamics, explains why the minimum-phase assumption is required. ◇

It is worth noting that the memoryless feedback law (3.4) is not yet in the desired form for the certainty-equivalent design, since the state z is not available for feedback. We proceed by modifying the control law (3.4) with the introduction of a reduced-order observer for z. Using (3.2), the candidate observer is chosen a system of the form

$$\dot{\xi} = F(\mu) \xi + G_1(\mu) y + G_2(\mu) u \tag{3.5}$$

with state $\xi \in \mathbb{R}^{n-1}$, where the matrices F and $G \triangleq (G_1 \; G_2)$ are allowed – at this stage – to depend explicitly on μ. As in every reduced-order observer design, the observation error is defined as

$$\chi = z - \xi - L(\mu)y$$

where $L(\mu) \in \mathbb{R}^{(n-1)\times 1}$ is an output-injection gain to be chosen. The dynamics of the observation error yields (omitting the parameter vector μ to simplify the notation)

$$\begin{aligned}\dot{\chi} &= A_{11}[\chi + \xi + Ly] + A_{12}y - F\xi - G_1 y - G_2 u - LA_{21}[\chi + \xi + Ly] \\ &\quad -a_{22}Ly - bLu \\ &= (A_{11} - LA_{21})\chi + (A_{11} - LA_{21} - F)\xi + (A_{12} - G_1 - LA_{21}L + A_{11}L - a_{22}L)y \\ &\quad -(G_2 + bL)u\,.\end{aligned}$$

Since the pair $(C(\mu), A(\mu))$ is assumed to be observable for any $\mu \in \mathcal{P}$, a simple application of the PHB test reveals that the pair $(A_{21}(\mu), A_{11}(\mu))$ is also observable for any $\mu \in \mathcal{P}$. As a result, given any Hurwitz polynomial

$$p_d(\lambda) = \lambda^{n-1} + d_{n-2}\lambda^{n-2} + \cdots + d_1\lambda + d_0$$

there exists $L(\mu)$ such that

$$\det\left(A_{11}(\mu) - L(\mu)A_{21}(\mu) - \lambda I\right) = p_d(\lambda)\,.$$

Fix $\mu \in \mathcal{P}$, and determine the corresponding output-injection gain such that the characteristic polynomial of $A_{11}(\mu) - L(\mu)A_{21}(\mu)$ coincides with a given Hurwitz polynomial $p_d(\lambda)$. Selecting the matrices of the reduced-order observer as

$$\begin{aligned}F(\mu) &= A_{11}(\mu) - L(\mu)A_{21}(\mu) \\ G_1(\mu) &= A_{12}(\mu) - L(\mu)A_{21}(\mu)L(\mu) + A_{11}(\mu)L(\mu) - a_{22}(\mu)L(\mu) \\ G_2(\mu) &= -b(\mu)L(\mu)\,,\end{aligned} \tag{3.6}$$

the observer error is assigned the dynamics of the autonomous asymptotically stable system

$$\dot{\chi} = (A_{11}(\mu) - L(\mu)A_{21}(\mu))\chi\,.$$

Using the available signal $\xi + L(\mu)y$ to replace z in the certainty-equivalence control law, one obtains

$$\begin{aligned}u &= -\frac{1}{b(\mu)}\left[A_{21}(\mu)\xi + (a_{22}(\mu) + \alpha_m + A_{21}(\mu)L(\mu))y - \beta_m u_r\right] \qquad (3.7)\\ &= -\frac{1}{b(\mu)}\left[A_{21}(\mu)z - A_{21}(\mu)\chi + (a_{22}(\mu) + \alpha_m)y - \beta_m u_r\right]\,.\end{aligned}$$

As a result, the closed-loop system, using for the observer the coordinates χ in place of ξ, reads as

$$\begin{aligned}\dot{y}_r &= -\alpha_m y_r + \beta_m u_r \\ \dot{z} &= A_{12}(\mu)y_r + A_{11}(\mu)z + A_{12}(\mu)e \\ \dot{\chi} &= (A_{11}(\mu) - L(\mu)A_{21}(\mu))\chi \\ \dot{e} &= A_{21}(\mu)\chi - \alpha_m e\,,\end{aligned}$$

and thus, by inspection, the trajectory $(\chi(t), e(t))$ converges to the origin, while the trajectory $(y_r(t), z(t))$ remains bounded.

Finally, the expression of the controller (3.5)-(3.7) in the desired form (3.3) is obtained replacing in the observer equation the expression of u, yielding the system

$$\begin{aligned}\dot{\xi} &= A_{11}(\mu)\xi + [A_{11}(\mu)L(\mu) + A_{12}(\mu) + \alpha_m L(\mu)]\, y - \beta_m L(\mu) u_r \\ u &= -\tfrac{1}{b(\mu)}\left[A_{21}(\mu)\xi + (a_{22}(\mu) + \alpha_m + A_{21}(\mu)L(\mu))y - \beta_m u_r\right] .\end{aligned} \tag{3.8}$$

It is worth noting the controller (3.8) is internally stable, as the matrix $A_{11}(\mu)$ is Hurwitz by assumption.

3.1.6 Controller Parametrization

The dynamic controller derived in the previous section, although it can expressed in the form (3.3), does not lend itself easily to an adaptive redesign. The reason is that the controller dynamics depend on the unknown parameter, which complicates the design of the update law. A much preferable situation would occur if the dependence on the unknown parameters was confined to the output map of the controller (that is, equation (3.7),) while the dynamic equations were independent of μ. The approach we take in this section is to find a different parametrization of the controller (3.8) for which this is indeed the case, and thus an update law can be easily derived. We begin with reverting back to the controller given by (3.5)-(3.7), and writing it in the form

$$\begin{aligned}\dot{\xi} &= F(\mu)\xi + G_1(\mu)y + G_2(\mu)u \\ u &= H(\mu)\xi + K_1(\mu)y + K_2(\mu)u_r\end{aligned} \tag{3.9}$$

where F, G_1 and G_2 are given in (3.6), and

$$H(\mu) = -\frac{1}{b(\mu)}A_{21}(\mu)\,,\ K_1(\mu) = -\frac{1}{b(\mu)}[a_{22}(\mu) + \alpha_m + A_{21}(\mu)L(\mu)]\,,\ K_2(\mu) = \frac{\beta_m}{b(\mu)} \tag{3.10}$$

The controller is viewed as the interconnection, depicted in Figure 3.1, between the memoryless system given by the feed-through terms $K_1 y$ and $K_2 u_r$, and the dynamic system given by the triplet (F, G, H), with $G = (G_1\ G_2)$. Let $W(s) = (W_1(s)\ W_2(s))$ denote the transfer function matrix of the dynamic system (F, G, H), that is, let

$$\bar{u}_1(s) = W_1(s)\bar{y}(s) + W_2(s)\bar{u}(s)$$

denote the output (in the Laplace domain) of the dynamic part of the controller, where

$$W_1(s) = H(\mu)(sI - F(\mu))^{-1}G_1(\mu)\,, \quad W_2(s) = H(\mu)(sI - F(\mu))^{-1}G_2(\mu)\,.$$

It is obvious that, in general, neither (F, G_1, H) nor (F, G_2, H) need to be minimal realizations of their respective transfer functions $W_1(s)$ and $W_2(s)$. However, since the matrix F has been obtained by means of an output injection from the pair (A_{21}, A_{11}), it follows that the pair (H, F) is observable. This implies that the triplets (F, G, H), (F, G_1, H), and

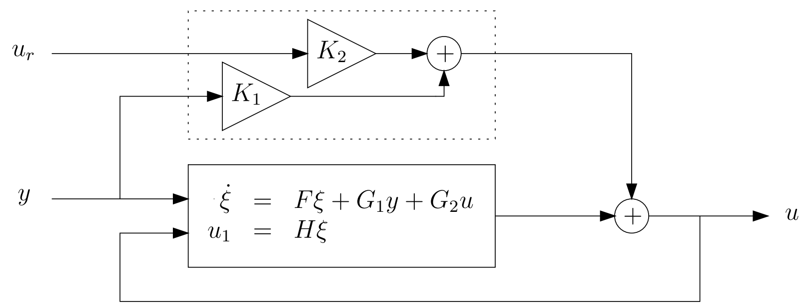

Figure 3.1: Certainty-equivalence controller

(F, G_2, H) are all observable realizations of $W(s)$, $W_1(s)$, and $W_2(s)$, respectively. Moreover, if one assumes that $p_d(\lambda)$ and the characteristic polynomial of A_{11} are coprime (that is, if the output injection $A_{11} - LA_{21}$ does not leave any eigenvalue of A_{11} unchanged), then the pair (F, G_2) is controllable as well, and thus (F, G_2, H) is a minimal realization of $W_2(s)$. This is stated formally in the following assumption:

Assumption 3.1.7 *The triplet (F, G_2, H) is a minimal realization of $W_2(s)$.*

Note that controllability of (F, G_2) implies controllability of (F, G), and thus Assumption 3.1.7 guarantees that (F, G, H) is a *minimal realization* of $W(s)$.

The key for a successful adaptive redesign of the certainty-equivalence control law is to find a realization of $W(s)$ that avoids the dependence of the state equation on the unknown parameter. This can accomplished simply by realizing $W_1(s)$ and $W_2(s)$ in controller canonical form, an then obtaining the required realization of $W(s)$ from their parallel interconnection. More precisely, define the matrices in companion form

$$\Phi = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & \cdots & 1 \\ -d_0 & -d_1 & -d_2 & \cdots & -d_{n-2} \end{pmatrix}, \qquad \Gamma = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$$

and let the vectors $\theta_1 \in \mathbb{R}^{n-1}$, $\theta_2 \in \mathbb{R}^{n-1}$ be such that

$$W_1(s) = \theta_1^{\mathrm{T}}(sI - \Phi)^{-1}\Gamma\,, \qquad W_2(s) = \theta_2^{\mathrm{T}}(sI - \Phi)^{-1}\Gamma\,, \tag{3.11}$$

which is possible, because (Φ, Γ) is controllable and Φ and F have the same characteristic polynomial. The required non-minimal realization of $W(s)$ is given by the $2(n-1)$–dimensional system

$$\begin{aligned} \dot{\zeta}_1 &= \Phi\zeta_1 + \Gamma y \\ \dot{\zeta}_2 &= \Phi\zeta_2 + \Gamma u \\ u_1 &= \theta_1^{\mathrm{T}}\zeta_1 + \theta_2^{\mathrm{T}}\zeta_2\,. \end{aligned} \tag{3.12}$$

Finally, letting

$$\theta_3 = K_1(\mu)\,, \qquad \theta_4 = K_2(\mu) \tag{3.13}$$

we write the new parametrization of the certainty equivalence controller as

$$\begin{aligned} \dot{\zeta}_1 &= \varPhi\,\zeta_1 + \varGamma y \\ \dot{\zeta}_2 &= \varPhi\,\zeta_2 + \varGamma u \\ u &= \theta_1^{\mathrm{T}}\zeta_1 + \theta_2^{\mathrm{T}}\zeta_2 + \theta_3 y + \theta_4 u_r\,, \end{aligned} \tag{3.14}$$

In the above system, the matrices $\varPhi$ and $\varGamma$ are known, while the *controller parameter vector* $\theta \in \mathbb{R}^{2n}$, given by

$$\theta = \begin{pmatrix} \theta_1^{\mathrm{T}} & \theta_2^{\mathrm{T}} & \theta_3 & \theta_4 \end{pmatrix}^{\mathrm{T}}\,,$$

depends on the plant parameter vector μ. The system (3.12) employs twice as many states as needed to realize $W(s)$. However, it is internally stable, and equivalent to (F, G, H) in terms of the I/O response. Since the pair

$$\begin{pmatrix} \varPhi & 0 \\ 0 & \varPhi \end{pmatrix}, \quad \begin{pmatrix} \varGamma & 0 \\ 0 & \varGamma \end{pmatrix}$$

is controllable, it follows that the pair

$$\begin{pmatrix} \theta_1^{\mathrm{T}} & \theta_2^{\mathrm{T}} \end{pmatrix}, \quad \begin{pmatrix} \varPhi & 0 \\ 0 & \varPhi \end{pmatrix}$$

is necessarily unobservable, and that the subspace of unobservable states, denoted by $\mathcal{V}_{\mathrm{no}}$, has dimension equal to the number of excess states, that is $\dim(\mathcal{V}_{\mathrm{no}}) = n - 1$. As a result, system (3.12) can be decomposed with respect to observability by means of a change of basis adapted to the subspace of unobservable states. To find the required decomposition, we proceed first by noticing that, by virtue of Assumption 3.1.7, the triplet $(\varPhi, \varGamma, \theta_2^{\mathrm{T}})$ is a minimal realization of $W_2(s)$. Since all modes of the ζ_2-dynamics are observable at the output u_1 of (3.12) (and reachable from the input u), the modal subspace

$$\mathcal{V} = \mathrm{im}\begin{pmatrix} 0 \\ I \end{pmatrix}$$

is an invariant subspace for (3.12) satisfying

$$\mathcal{V} \cap \mathcal{V}_{\mathrm{no}} = \{0\}\,,$$

and thus $\mathcal{V}_{\mathrm{no}}$ is necessarily complementary to $\mathcal{V}$. Recalling that $\dim \mathcal{V}_{\mathrm{no}} = n - 1$, the subspace in question can be expressed as

$$\mathcal{V}_{\mathrm{no}} = \mathrm{im}\begin{pmatrix} I \\ X \end{pmatrix}$$

where the matrix $X \in \mathbb{R}^{(n-1)\times(n-1)}$ satisfies

$$\begin{pmatrix} \Phi & 0 \\ 0 & \Phi \end{pmatrix} \begin{pmatrix} I \\ X \end{pmatrix} = \begin{pmatrix} I \\ X \end{pmatrix} R\,, \qquad \begin{pmatrix} \theta_1^{\mathrm{T}} & \theta_1^{\mathrm{T}} \end{pmatrix} \begin{pmatrix} I \\ X \end{pmatrix} = 0$$

for some $R \in \mathbb{R}^{(n-1)\times(n-1)}$. The above conditions merely state the property that $\mathcal{V}_{\mathrm{no}}$ is the largest invariant subspace for (3.12) contained in $\ker(\theta_1^{\mathrm{T}}\ \theta_2^{\mathrm{T}})$. It is easy to see that necessarily $R = \Phi$, and thus X is computed as the solution of the linear equation

$$\begin{aligned} \Phi X &= X\Phi \\ \theta_2^{\mathrm{T}} X &= -\theta_1^{\mathrm{T}}\,. \end{aligned}$$

The change of coordinates $\bar{\zeta}_2 = \zeta_2 - X\zeta_1$ puts system (3.12) in the desired Kalman canonical form

$$\begin{aligned} \begin{pmatrix} \dot{\zeta}_1 \\ \dot{\bar{\zeta}}_2 \end{pmatrix} &= \begin{pmatrix} \Phi & 0 \\ 0 & \Phi \end{pmatrix} \begin{pmatrix} \zeta_1 \\ \bar{\zeta}_2 \end{pmatrix} + \begin{pmatrix} \Gamma & 0 \\ \bar{\Gamma} & \Gamma \end{pmatrix} \begin{pmatrix} y \\ u \end{pmatrix} \\ u_1 &= \begin{pmatrix} 0 & \theta_2^{\mathrm{T}} \end{pmatrix} \begin{pmatrix} \zeta_1 \\ \bar{\zeta}_2 \end{pmatrix}, \end{aligned} \tag{3.15}$$

where $\bar{\Gamma} = -X\Gamma$. The observable and controllable system

$$\begin{aligned} \dot{\bar{\zeta}}_2 &= \Phi\bar{\zeta}_2 + \bar{\Gamma} y + \Gamma u \\ u_1 &= \theta_2^{\mathrm{T}} \bar{\zeta}_2\,, \end{aligned}$$

being a minimal realization of $W(s)$, is related to (F, G, H) by means of a nonsingular transformation (see [16]). Specifically, there exists an invertible matrix $Y \in \mathbb{R}^{(n-1)\times(n-1)}$ such that

$$Y^{-1}\Phi Y = F\,, \quad Y^{-1}\bar{\Gamma} = G_1\,, \quad Y^{-1}\Gamma = G_2\,, \quad \theta_2^{\mathrm{T}} Y = H\,,$$

from which the following important result is concluded.

Proposition 3.1.8 *The change of coordinates*

$$\eta_1 = \zeta_1 \qquad \eta_2 = Y^{-1}\zeta_2 - Y^{-1}X\zeta_1$$

puts (3.14) *into the form*

$$\begin{aligned} \dot{\eta}_1 &= \Phi\,\eta_1 + \Gamma y \\ \dot{\eta}_2 &= F(\mu)\,\eta_2 + G_1(\mu) y + G_2(\mu) u \\ u &= H(\mu)\eta_2 + K_1(\mu) y + K_2(\mu) u_r\,, \end{aligned} \tag{3.16}$$

from which it is seen that the new controller (3.14) *embeds a diffeomorphic copy of the original controller* (3.9).

In the terminology of output regulation theory, one would say that (3.9) is *immersed* into (3.14). The additional dynamics introduced in (3.12) is unobservable from the output, and the unobservable trajectory $\eta_1(t)$ remains bounded as long as $y(t)$ is bounded. The introduction of the unobservable dynamics has the remarkable effect of allowing a representation of the controller for which the dependence on the uncertain parameters is shifted to the output map, and collected into the parameter vector θ.

Consider now the plant model (3.2), augmented with the reference model, in closed-loop with the certainty-equivalence controller (3.14)

$$\begin{aligned}
\dot{y}_r &= -\alpha_m y_r + \beta_m u_r \\
\dot{\zeta}_1 &= \Phi \zeta_1 + \Gamma y \\
\dot{\zeta}_2 &= \Phi \zeta_2 + \Gamma[\theta_1^{\mathrm{T}}\zeta_1 + \theta_2^{\mathrm{T}}\zeta_2 + \theta_3 y + \theta_4 u_r] \\
\dot{z} &= A_{11}(\mu)z + A_{12}(\mu)y \\
\dot{y} &= A_{21}(\mu)z + a_{22}(\mu)y + b(\mu)[\theta_1^{\mathrm{T}}\zeta_1 + \theta_2^{\mathrm{T}}\zeta_2 + \theta_3 y + \theta_4 u_r] \\
e &= y - y_r
\end{aligned} \tag{3.17}$$

The following result establishes the stability properties of the closed-loop system under the non-minimal certainty-equivalence controller:

Proposition 3.1.9 *Consider the closed-loop system* (3.17)*, where the controller parameter vector θ is such that equations* (3.11) *and* (3.13) *hold. Let Assumption 3.1.7 hold. Then, for any $\mu \in \mathcal{P}$, the trajectory of* (3.17) *originating for any initial condition is bounded, and satisfies* $\lim_{t\to\infty} e(t) = 0$.

Proof. Apply the preliminary change of coordinates

$$\eta_1 = \zeta_1\,, \qquad \eta_2 = Y^{-1}\zeta_2 - Y^{-1}X\zeta_1$$

and (3.11), (3.13) to obtain the equivalent expression of the closed-loop system

$$\begin{aligned}
\dot{y}_r &= -\alpha_m y_r + \beta_m u_r \\
\dot{\eta}_1 &= \Phi \eta_1 + \Gamma y \\
\dot{\eta}_2 &= F(\mu)\,\eta_2 + G_1(\mu)y + G_2(\mu)[H(\mu)\eta_2 + K_1(\mu)y + K_2(\mu)u_r] \\
\dot{z} &= A_{11}(\mu)z + A_{12}(\mu)y \\
\dot{y} &= A_{21}(\mu)z + a_{22}(\mu)y + b(\mu)[H(\mu)\eta_2 + K_1(\mu)y + K_2(\mu)u_r]\,.
\end{aligned}$$

Changing again coordinates as

$$\chi_1 = \eta_1\,, \qquad \chi_2 = z - \eta_2 - Ly\,, \qquad e = y - y_r\,,$$

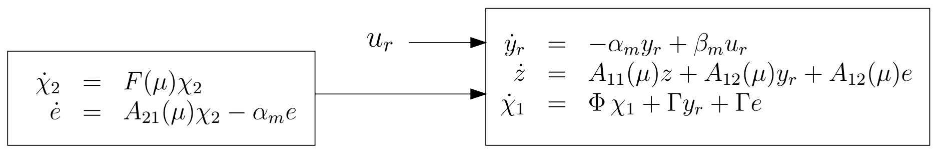

Figure 3.2: Closed-loop system (3.18).

and recalling (3.6) and (3.10), one obtains

$$\begin{aligned} \dot{y}_r &= -\alpha_m y_r + \beta_m u_r \\ \dot{z} &= A_{11}(\mu)z + A_{12}(\mu)y_r + A_{12}(\mu)e \\ \dot{\chi}_1 &= \Phi\,\chi_1 + \Gamma y_r + \Gamma e \\ \dot{\chi}_2 &= F(\mu)\chi_2 \\ \dot{e} &= A_{21}(\mu)\chi_2 - \alpha_m e\,. \end{aligned} \tag{3.18}$$

The above system is the cascade of two asymptotically stable systems, with the autonomous (χ_2, e)-dynamics driving the (y_r, χ_1, z)-dynamics, which is also forced by the bounded reference input u_r (see Figure 3.2.) Consequently, the trajectories of the driving system converge to zero, while the trajectories of the driven subsystem remain bounded, for any initial condition. □

It is interesting to investigate whether stronger conclusions on the stability properties of the closed-loop system than boundedness of all trajectories and regulation of the tracking error can be inferred. It is clear that the trajectories of the upper subsystem in (3.18) converge to a unique trajectory, determined solely by the reference input u_r, which represents the steady-state behavior of the system. The steady-state in question for the trajectory $y_r(t)$ is easily computed as

$$\bar{y}_r(t) = \int_{-\infty}^{t} \beta_m \mathrm{e}^{-\alpha_m(t-\tau)} u_r(\tau)d\tau\,,$$

while the steady-state for the trajectory $(z(t), \chi_1(t))$ is given by

$$\bar{z}(t) = \int_{-\infty}^{t} \mathrm{e}^{A_{11}(\mu)(t-\tau)} A_{12}(\mu)\bar{y}_r(\tau)d\tau\,, \qquad \bar{\chi}_1(t) = \int_{-\infty}^{t} \mathrm{e}^{\Phi(t-\tau)}\Gamma\bar{y}_r(\tau)d\tau\,.$$

The integrals above are well-defined, as the signal u_r is assumed bounded. Introducing the deviations from steady-state as new state variables

$$\tilde{y}_r = y_r - \bar{y}_r(t)\,, \qquad \tilde{z} = z - \bar{z}(t)\,, \qquad \tilde{\chi}_1 = \chi_1 - \bar{\chi}_1(t)$$

one obtains from (3.18)

$$
\begin{array}{rcl}
\dot{\tilde{y}}_r &=& -\alpha_m \tilde{y}_r \\
\dot{\tilde{z}} &=& A_{11}(\mu)\tilde{z} + A_{12}(\mu)\tilde{y}_r + A_{12}(\mu)e \\
\dot{\tilde{\chi}}_1 &=& \Phi\,\tilde{\chi}_1 + \Gamma \tilde{y}_r + \Gamma e \\
\dot{\chi}_2 &=& F(\mu)\chi_2 \\
\dot{e} &=& A_{21}(\mu)\chi_2 - \alpha_m e\,.
\end{array}
\tag{3.19}
$$

Note that in the new coordinates the closed-loop system system is autonomous, and that the matrix

$$
\mathcal{A}(\mu) = \begin{pmatrix} -\alpha_m & 0 & 0 & 0 & 0 \\ A_{12}(\mu) & A_{11}(\mu) & 0 & 0 & A_{12}(\mu) \\ \Gamma & 0 & \Phi & 0 & \Gamma \\ 0 & 0 & 0 & F(\mu) & 0 \\ 0 & 0 & 0 & A_{21}(\mu) & -\alpha_m \end{pmatrix}
\tag{3.20}
$$

is Hurwitz for all $\mu \in \mathcal{P}$. As a result, it can be concluded that the origin is a globally exponentially stable equilibrium of (3.19).

3.1.10 Adaptive Design

The controller (3.14) implicitly depends on the plant parameters μ by way of the parameter vector θ. Obviously, if the actual value of μ is not available, so is the actual value of θ. To implement the controller, we resort to the principle of certainty equivalence, and replace θ with an estimate $\hat{\theta}$, and look for a suitable update law. To simplify the notation, denote with $\phi(t, \zeta, y)$ the *regressor*

$$
\phi(t, \zeta, y) = \begin{pmatrix} \zeta_1 \\ \zeta_2 \\ y \\ u_r(t) \end{pmatrix}
$$

and write the controller (3.14) as

$$
\begin{array}{rcl}
\dot{\zeta}_1 &=& \Phi\,\zeta_1 + \Gamma y \\
\dot{\zeta}_2 &=& \Phi\,\zeta_2 + \Gamma u \\
u &=& \phi^{\mathrm{T}}(t, \zeta, y)\hat{\theta}\,.
\end{array}
$$

Note that in $\phi(t, \zeta, y)$ the explicit dependence on time is due to the exogenous signal $u_r(\cdot)$, which is not a state variable of the closed-loop system. Using in place of $(y_r,\ z,\ \zeta_1,\ \zeta_2,\ y)$ the coordinates $(\tilde{y}_r,\ \tilde{z},\ \tilde{\chi}_1,\ \chi_2,\ e)$ defined in the previous section, and introducing the "estimation error" $\tilde{\theta} = \hat{\theta} - \theta$, one obtains the following expression for the closed-loop system:

$$\begin{array}{rcl}
\dot{\tilde{y}}_r &=& -\alpha_m \tilde{y}_r \\
\dot{\tilde{z}} &=& A_{11}(\mu)\tilde{z} + A_{12}(\mu)\tilde{y}_r + A_{12}(\mu)e \\
\dot{\tilde{\chi}}_1 &=& \Phi\,\tilde{\chi}_1 + \Gamma\tilde{y}_r + \Gamma e \\
\dot{\chi}_2 &=& F(\mu)\chi_2 \\
\dot{e} &=& A_{21}(\mu)\chi_2 - \alpha_m e + b(\mu)\phi^{\mathrm{T}}(t,\zeta,y)\tilde{\theta}\,.
\end{array} \tag{3.21}$$

In writing the expression of the regressor $\phi(\cdot)$ in the above system, we have kept the original coordinates ζ and y, since they are directly available for feedback, and can be used to derive the required update law for $\hat{\theta}$. As a matter of fact, note that the state χ_2 can not be used in the update law, as it depends on the unknown matrices X and Y. Finally, collecting all state variables in the vector

$$\mathbf{x} = \begin{pmatrix} \tilde{y}_r & \tilde{z} & \tilde{\chi}_1 & \chi_2 & e \end{pmatrix}^{\mathrm{T}}$$

and introducing the matrices

$$\mathcal{B}(\mu) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ b(\mu) \end{pmatrix}, \qquad \mathcal{C} = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

the closed-loop system is written in the simpler form

$$\begin{array}{rcl}
\dot{\mathbf{x}} &=& \mathcal{A}(\mu)\mathbf{x} + \mathcal{B}(\mu)\phi^{\mathrm{T}}(t,\zeta,y)\tilde{\theta} \\
e &=& \mathcal{C}\mathbf{x}\,,
\end{array}$$

where $\mathcal{A}(\mu)$ is given by (3.20).

Proposition 3.1.11 *The triplet $(\mathcal{A}(\mu), \mathcal{B}(\mu), \mathcal{C})$ is strictly passive for all $\mu \in \mathcal{P}$. In particular, there exist a continuous, symmetric and positive definite matrix-valued function $\mathcal{Q}(\mu)$ and positive constants c_1, c_2 satisfying*

$$c_1 I \leq \mathcal{Q}(\mu) \leq c_2 I$$

and

$$\begin{array}{l}
\mathcal{A}^{\mathrm{T}}(\mu)\mathcal{Q}(\mu) + \mathcal{Q}(\mu)\mathcal{A}(\mu) \leq -I \\
\mathcal{B}^{\mathrm{T}}(\mu)\mathcal{Q}(\mu) = \mathcal{C}
\end{array}$$

for all $\mu \in \mathcal{P}$.

Proof. ... □

The above result suggests the use of the passivity-based update law

$$\dot{\tilde{\theta}} = -\gamma\phi(t,\zeta,y)e\,, \qquad \gamma > 0\,,$$

which yields the interconnection

$$\begin{aligned} \dot{\mathbf{x}} &= \mathcal{A}(\mu)\mathbf{x} + \mathcal{B}(\mu)\phi^{\mathrm{T}}(t,\zeta,y)\tilde{\theta} \\ \dot{\tilde{\theta}} &= -\gamma\phi(t,\zeta,y)\mathcal{C}\mathbf{x}\,. \end{aligned}$$

To analyze the behavior of the closed loop system, we begin with expressing the regressor in terms of the new coordinates $\mathbf{x}$ and the exogenous signals as

$$\phi(t,\zeta,y) = \begin{pmatrix} \zeta_1 \\ \zeta_2 \\ y \\ u_r(t) \end{pmatrix} = \begin{pmatrix} \tilde{\chi}_1 \\ Y[\tilde{z} - \chi_2 - Le] + X\tilde{\chi}_1 \\ e + \tilde{y}_r \\ 0 \end{pmatrix} + \begin{pmatrix} \bar{\chi}_1(t) \\ Y[\bar{z}(t) - L\bar{y}_r(t)] + X\bar{\chi}_1(t) \\ \bar{y}_r(t) \\ u_r(t) \end{pmatrix}.$$

As a result, the regressor can be expressed as a linear combination of the state variables $\mathbf{x}$ and the components of the vector of exogenous signals $\mathbf{w} = (u_r(t)\ \bar{y}_r(t)\ \bar{z}(t)\ \bar{\chi}_1(t))^{\mathrm{T}}$

$$\phi(t,\zeta,y) = M_1\mathbf{x} + M_2\mathbf{w}(t)\,,$$

where

$$M_1 = \begin{pmatrix} 0 & 0 & I & 0 & 0 \\ 0 & Y & X & -Y & -YL \\ 1 & 0 & 0 & 0 & 1 \end{pmatrix}, \qquad M_2 = \begin{pmatrix} 0 & 0 & I & 0 \\ -YL & Y & X & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

With a minor abuse of notation, we write the regressor as $\phi(t,\mathbf{x})$, with the understanding that $\phi(t,0) = M_2\mathbf{w}(t)$. This allows us to write the closed-loop system as the standard adaptive control problem

$$\begin{aligned} \dot{\mathbf{x}} &= \mathcal{A}(\mu)\mathbf{x} + \mathcal{B}(\mu)\phi^{\mathrm{T}}(t,\mathbf{x})\tilde{\theta} \\ \dot{\tilde{\theta}} &= -\gamma\phi(t,\mathbf{x})\mathcal{C}\mathbf{x}\,. \end{aligned} \tag{3.22}$$

Theorem 3.1.12 *For system* (3.22)*, the following results hold:*

(i) If $u_r \in \mathcal{L}_\infty$, for any $\mu \in \mathcal{P}$ the origin is a globally uniformly stable equilibrium of (3.22)*, and thus the trajectory $(\mathbf{x}(t),\tilde{\theta}(t))$ originating from any initial condition in $\mathbb{R}^{3n-1} \times \mathbb{R}^{2n}$ is bounded. Moreover, the trajectory $\mathbf{x}(t)$ satisfies $\lim_{t\to\infty}\mathbf{x}(t) = 0$*

(ii) If $u_r \in \mathcal{L}_\infty$, $\dot{u}_r \in \mathcal{L}_\infty$, and $\mathbf{w}(\cdot)$ is PE, then for any given $\mu \in \mathcal{P}$ the origin is a globally uniformly asymptotically stable and locally exponentially stable equilibrium of (3.22)*.*

Proof. The proof is left as an exercise / exam problem. □

Chapter 4

Adaptive Observers

4.1 Observers for Linear Systems

Consider the strictly proper transfer function

$$G(s) = \frac{n(s)}{d(s)} = \frac{b_{n-1}s^{s-1} + b_{n-2}s^{n-2} + \ldots + b_1 s + b_0}{s^n + a_{n-1}s^{s-1} + a_{n-2}s^{n-2} + \ldots + a_1 s + a_0}$$

in which $n(s)$ and $d(s)$ are relatively coprime, i.e., they do not possess common factors. Note that in case the relative degree of $G(s)$ is greater than one, say $1 < r \leq n$, the numerator polynomial satisfies $b_{n-1} = b_{n-2} = \ldots b_{r-2} = 0$. A minimal realization of $G(s)$ in *observer canonical form* is given by the system

$$\begin{aligned} \dot{x} &= A_o x + B_o u \\ y &= C_o^{\mathrm{T}} x \end{aligned} \tag{4.1}$$

where

$$A_o = \begin{pmatrix} -a_{n-1} & 1 & 0 & \cdots & 0 \\ -a_{n-2} & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -a_1 & 0 & 0 & \cdots & 1 \\ -a_0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \quad B_o = \begin{pmatrix} b_{n-1} \\ b_{n-2} \\ \vdots \\ b_1 \\ b_0 \end{pmatrix}, \quad C_o^{\mathrm{T}} = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

The problem we address in this section is the design of an *asymptotic observer* for the state of (4.1) under the assumption that the coefficients of the polynomials $n(s)$ and $d(s)$ (equivalently, the location of the zeros and the poles of $G(s)$) are not known precisely. At this regard, let us collect the unknown coefficients $b_0, \ldots, b_{n-1}$ and $a_0, \ldots, a_{n-1}$ into the $2n$-dimensional parameter vector θ

$$\theta = \begin{pmatrix} b_{n-1} & b_{n-2} & \cdots & b_0 & a_{n-1} & a_{n-2} & \cdots & a_0 \end{pmatrix}^{\mathrm{T}}$$

and write (4.1) as

$$\begin{aligned} \dot{x} &= A_b x + \Psi(u, y)\theta \\ y &= C_b x \end{aligned} \tag{4.2}$$

where A_b, C_b are in Brunowsky form

$$A_b = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \quad C_b = C_o^{\mathrm{T}}$$

and $\Psi(u, y) \in \mathbb{R}^{n \times 2n}$ is given by

$$\Psi(u, y) = \begin{pmatrix} u & 0 & \cdots & 0 & -y & 0 & \cdots & 0 \\ 0 & u & \cdots & 0 & 0 & -y & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & u & 0 & 0 & \cdots & -y \end{pmatrix}.$$

It is quite clear that, since the pair (A_b, C_b) is observable, if the parameter vector θ is known exactly, an asymptotic observer for (4.2) is given by a system of the form

$$\dot{\hat{x}} = A_b \hat{x} + \Psi(u, y)\theta + K(y - C_b \hat{x})$$

with $K = \begin{pmatrix} k_{n-1} & k_{n-2} & \cdots & k_0 \end{pmatrix}^{\mathrm{T}}$ chosen in such a way that the characteristic polynomial of $A_b - KC_b$, i.e., the polynomial

$$s^n + k_{n-1}s^{s-1} + k_{n-2}s^{n-2} + \ldots + k_1 s + k_0 \tag{4.3}$$

has all roots in $\mathbb{C}^-$. If this is indeed the case, the dynamics of the *observation error* $e = x - \hat{x}$ is governed by

$$\dot{e} = (A_b - KC_b)e\,,$$

therefore satisfying $\lim_{t\to\infty} |e(t)| = 0$. However, it is easy to see that the above condition requires exact cancellation of the term $\Psi(u, y)\theta$, as otherwise the error dynamics would not possess an equilibrium at $e = 0$. In case the vector θ is not known exactly, we resort to the principle of *certainty equivalence*, and look for an observer of the kind

$$\dot{\hat{x}} = A_b \hat{x} + \Psi(u, y)\hat{\theta}(t) + K(y - C_b \hat{x})$$

in which the unknown parameter vector θ has been replaced by an "estimate" $\hat{\theta}(t)$, whose dynamics is governed by a suitable update law that must guarantee boundedness of $(e(t), \hat{\theta}(t))$ and convergence of $e(t)$ to the origin. We expect the design of the update law to be particularly challenging, as only the I/O pair $(u(t), y(t))$ of (4.2) is available for measurement.

4.1.1 Adaptive observer form

In this section, we will analyze a particular case for which the design of an adaptive observer is almost straightforward, and convergence analysis follows directly from the results of the previous chapter. Then, in the next section we will show how to extend the design to the general case (4.2).

Consider, in place of (4.2), the system

$$\begin{aligned} \dot{z} &= A_b z + d\,\beta(t)^{\mathrm{T}}\vartheta \\ y &= C_b z \end{aligned} \tag{4.4}$$

where

4.4.a) $\vartheta \in \mathbb{R}^p$ is a vector of unknown constant parameters,

4.4.b) $\beta : \mathbb{R} \to \mathbb{R}^p$ is a known bounded vector-valued function of time,

4.4.c) $d^{\mathrm{T}} = \begin{pmatrix} d_{n-1} & d_{n-2} & \cdots & d_0 \end{pmatrix}^{\mathrm{T}}$ is a *known vector* such that the polynomial

$$d_{n-1}s^{s-1} + d_{n-2}s^{n-2} + \ldots + d_1 s + d_0$$

has all roots in $\mathbb{C}^-$, and $d_{n-1} > 0$.

Systems having the structure in (4.4) and satisfying 4.4.a-c are said to be in *adaptive observer form.* Note that the transfer function

$$H(s) = C_b(sI - A_b)^{-1}d = \frac{d_{n-1}s^{s-1} + d_{n-2}s^{n-2} + \ldots + d_1 s + d_0}{s^n}$$

is minimum phase by assumption, and its relative degree is equal to 1. Then, the following result holds:

Proposition 4.1.2 *Fix $\lambda > 0$ arbitrarily, and choose the vector $K \in \mathbb{R}^n$ as*

$$\begin{aligned} &\begin{pmatrix} k_{n-1} & k_{n-2} & \cdots & k_1 & k_0 \end{pmatrix}^{\mathrm{T}} = \\ &\qquad = \frac{1}{d_{n-1}} \begin{pmatrix} d_{n-2} + \lambda\, d_{n-1} & d_{n-3} + \lambda\, d_{n-2} & \cdots & d_0 + \lambda\, d_1 & \lambda\, d_0 \end{pmatrix}^{\mathrm{T}}. \end{aligned} \tag{4.5}$$

The system

$$\begin{aligned} \dot{\hat{z}} &= A_b\hat{z} + d\,\beta(t)^{\mathrm{T}}\hat{\vartheta} + K(y - C_b\hat{z}) \\ \dot{\hat{\vartheta}} &= \gamma\beta(t)(y - C_b\hat{z}) \end{aligned} \tag{4.6}$$

where K is chosen as in (4.5)*, and $\gamma > 0$ is an arbitrary design parameter, is an adaptive observer for* (4.4)*.*

Proof. Changing coordinates as $e = z - \hat{z}$ and $\tilde{\vartheta} = \vartheta - \hat{\vartheta}$ we obtain the error system

$$\begin{aligned} \dot{e} &= (A_b - KC_b)e + d\,\beta(t)^{\mathrm{T}}\tilde{\vartheta} \\ \dot{\tilde{\vartheta}} &= -\gamma\beta(t)C_b e\,. \end{aligned} \tag{4.7}$$

It is easy to see that, since

$$d_{n-1}s^n + (d_{n-2} + \lambda\, d_{n-1})s^{s-1} + \ldots + \lambda\, d_0 \;=\; (s+\lambda)(d_{n-1}s^{s-1} + d_{n-2}s^{n-2} + \ldots + d_0)$$

and the characteristic polynomial of the matrix $A_b - KC_b$ is exactly (4.3), we obtain

$$C_b(sI - A_b + KC_b)^{-1}d = \frac{1}{s+\lambda}\,.$$

Therefore, the triplet $\{(A_b - KC_b),\, d,\, C_b\}$ is SPR, and system (4.7) is precisely in the form that we have studied in the previous chapters, that is, the negative feedback interconnection of a strictly passive and a passive system. Since $\beta(t)$ is assumed bounded, we conclude that all trajectories $(e(t), \tilde{\vartheta}(t))$ of (4.7) are bounded, and satisfy $\lim_{t\to\infty} e(t) = 0$. □

It is clear that the role of the output injection vector K is that of inducing $n-1$ stable pole-zero cancellations, and assigning the remaining pole to $s = -\lambda$, to render the transfer function $C_b(sI - A_b + KC_b)^{-1}d$ strictly positive real. It is also worth noticing that, apart from the obvious robustness issues, persistency of excitation of $\beta(t)$ is not needed to achieve asymptotic observation of $z(t)$.

4.1.3 The general case

The result of the previous section shows how to choose the output injection gain and the update law to build an adaptive observer for systems in the particular form (4.4). Clearly, this result applies for the original system (4.1) only if the term $\Psi(u(t), y(t))\theta$ can be factorized as $\Psi(u(t), y(t))\theta = d\,\beta(t)^{\mathrm{T}}\vartheta$, with β, ϑ and d satisfying 4.4.a-c, which does not hold in general. As a matter of fact, it is not difficult to see that if the transfer function $G(s)$ has relative degree $r > 1$, or is non-minimum phase, a factorization like the one suggested above can not be found at all. To overcome this difficulty, we will introduce a system augmentation and a parameter-dependent change of coordinates that transform the augmented system into a form compatible with the construction of the adaptive observer derived in Proposition 4.1.2.

Consider again system (4.1), and perform a time-varying change of coordinates of the form

$$z = x - M(t)\theta \tag{4.8}$$

where the matrix-valued function of time $M : \mathbb{R} \to \mathbb{R}^{n\times 2n}$, yet to be determined, is assumed bounded. In the new coordinates z, system (4.1) reads as

$$\begin{aligned} \dot{z} &= A_b\, z + \Big[A_b M(t) + \Psi(u, y) - \dot{M}(t)\Big]\,\theta \\ y &= C_b x\,. \end{aligned}$$

Choose arbitrarily a vector $d^{\mathrm{T}} = \begin{pmatrix} 1 & d_{n-2} & \cdots & d_0 \end{pmatrix}^{\mathrm{T}}$ such that the polynomial $s^{s-1} + d_{n-2}s^{n-2} + \ldots + d_1 s + d_0$ has all roots in $\mathbb{C}^-$. Clearly, d defined in this way satisfies 4.4.c. Also, note that $C_b\, d = 1$. Next, we look for a vector-valued function $\beta : \mathbb{R}^{2n} \to \mathbb{R}$ such that

$$d\,\beta(t)^{\mathrm{T}} = A_b M(t) + \Psi(u(t), y(t)) - \dot{M}(t). \tag{4.9}$$

If β satisfying (4.9) exists, pre-multiplying each side of (4.9) by C_b, we obtain necessarily

$$\beta(t)^{\mathrm{T}} = C_b A_b M(t) + C_b \Psi(u(t), y(t)) + C_b \dot{M}(t)\,. \tag{4.10}$$

In order to solve (4.9) and (4.10) for $\beta(t)$, we remove the appearance of $\dot{M}(t)$ in (4.10) choosing

$$M(t) = \begin{pmatrix} 0 \\ N(t) \end{pmatrix} \tag{4.11}$$

with $N(t) \in \mathbb{R}^{n-1 \times 2n}$. This choice for $M(t)$ implies that $C_b \dot{M}(t) = 0$, and thus (4.10) yields

$$\beta(t)^{\mathrm{T}} = C_b A_b M(t) + C_b \Psi(u(t), y(t))\,. \tag{4.12}$$

Substituting the expression of $\beta(t)$ into (4.9), we obtain

$$\dot{M}(t) = [A_b - d\, C_b A_b]\, M(t) + [I - d\, C_b]\, \Psi(u(t), y(t))$$

which, in turn, gives the following expression for the matrix differential equation for $N(t)$

$$\dot{N}(t) = A_d N(t) + B_d \Psi(u(t), y(t)) \tag{4.13}$$

where

$$A_d = \begin{pmatrix} -d_{n-2} & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -d_1 & 0 & \cdots & 0 & 1 \\ -d_0 & 0 & \cdots & 0 & 0 \end{pmatrix}, \qquad B_d = \begin{pmatrix} -d_{n-2} & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -d_1 & 0 & \cdots & 1 & 0 \\ -d_0 & 0 & \cdots & 0 & 1 \end{pmatrix}.$$

The matrix differential equation (4.13) can be thought as a set of $2n$ linear systems of the form

$$\dot{n}_i = A_d n_i + B_d \psi_i(u(t), y(t))\,, \qquad i = 1, \ldots, 2n$$

where $n_i \in \mathbb{R}^{n-1}$ and $\psi_i(u, y) \in \mathbb{R}^{n-1}$ are the i-th columns of N and $\Psi(u, y)$, respectively. Note that the matrix A_d is Hurwitz by assumption, as its characteristic polynomial coincides with $s^{s-1} + d_{n-2}s^{n-2} + \ldots + d_1 s + d_0$. The change of variables (4.8), where $M(t)$ satisfies (4.11) and (4.13), is known as a *filtered transformation*. By virtue of (4.9), system (4.1) in the coordinates z reads as

$$\begin{aligned} \dot{z} &= A_b z + d\, \beta(t)^{\mathrm{T}} \theta \\ y &= C_b z \end{aligned} \tag{4.14}$$

and is therefore in adaptive observer form. However, since this time the state z of (4.14) is related to the original state x by means of a transformation involving the unknown parameter vector, convergence of the estimates $\hat{\theta}(t)$ to θ is indeed necessary to reconstruct the state in the original coordinates.

Proposition 4.1.4 *The output $\hat{x}(t)$ of the system*

$$\begin{aligned} \dot{N} &= A_d N + B_d \Psi(u(t), y(t)) \\ \dot{\hat{z}} &= A_b \hat{z} + d\, \beta(t)^{\mathrm{T}} \hat{\theta} + K(y - C_b \hat{z}) \\ \dot{\hat{\theta}} &= \gamma \beta(t)(y - C_b \hat{z}) \\ \hat{x} &= \hat{z} + \begin{pmatrix} 0 \\ N \end{pmatrix} \hat{\theta} \end{aligned} \tag{4.15}$$

where K is chosen as in (4.5)[1], $\gamma > 0$ *is an arbitrary design parameter, and* $\beta(t)$ *satisfies* (4.12) *yields an asymptotic estimate of the state* $x(t)$ *of system* (4.1), *provided that:*

a. $\beta(t)$ *is bounded for all* $t \geq 0$*,*

b. $\beta(t)$ *satisfies the persistence of excitation condition.*

The proof follows along the same lines of Proposition 4.1.2, and is therefore omitted. However, it must be stressed that, in order for the result to hold, the function $\beta(t)$ must be bounded. Since $\beta(t)$ is obtaining filtering the input $u(t)$ and the output $y(t)$ of system (4.1) by means of the stable filter (4.13) (see equation (4.10)), a sufficient condition for $\beta(t)$ to be a bounded function of time is that the system (4.1) is $\mathcal{L}_\infty$-stable and $u(\cdot) \in \mathcal{L}_\infty$. This, in turn, is equivalent to internal stability, since (4.1) is a minimal realization of $G(s)$. If, in addition to internal stability, the plant model is minimum phase, persistency of excitation of $\beta(t)$ is guaranteed if $u(t)$ is *sufficiently rich* of order at least $2n$, that is, the spectrum of $u(t)$ contains at least n sinusoids of distinct frequencies.

[1]Note that in this case $d_{n-1} = 1$.

Chapter 5

Design Example: Control of a 6-DOF Model of a Fighter Jet

In this chapter, we describe the design of an adaptive controller for a 6-DOF model of a mock fighter jet. The design of the controller is given as a design project to be carried out by the reader. In the following sections, we describe the nonlinear aircraft model, the linearized models and their Matlab-Simulink implementations, provided in the file repository. Then, the various steps of the design process are explained, to be completed and implemented in Matlab-Simulink by the reader.

5.1 Nonlinear Aircraft Model

The nonlinear aircraft model implemented in the file `Jet_model.slx` is the following [18, Chapter 2]

$$\begin{aligned}
\dot{h} &= -\boldsymbol{e}_3^{\mathrm{T}} \boldsymbol{R}_{eb}(\boldsymbol{\eta})\boldsymbol{\nu} && (5.1)\\
\dot{\boldsymbol{\nu}} &= -\boldsymbol{\omega} \times \boldsymbol{\nu} + \tfrac{1}{m}\boldsymbol{F}_{\mathrm{grav}} + \tfrac{1}{m}\boldsymbol{F}_{A,\mathrm{base}} + \tfrac{1}{m}\boldsymbol{F}_{A,\delta} + \tfrac{1}{m}\boldsymbol{F}_T && (5.2)\\
\dot{\boldsymbol{\eta}} &= \boldsymbol{E}(\boldsymbol{\eta})\boldsymbol{\omega} && (5.3)\\
\boldsymbol{J}\dot{\boldsymbol{\omega}} &= -\boldsymbol{\omega} \times \boldsymbol{J}\boldsymbol{\omega} + \boldsymbol{M}_{A,\mathrm{base}} + \boldsymbol{M}_{A,\delta} && (5.4)
\end{aligned}$$

The relevant nomenclature is given in Table 5.1. The model parameters are contained in the file `model_paramters.mat`, whereas the initial conditions corresponding to the equilibrium at Mach 1 must be loaded from the file `initial_conditions.mat`. The altitude dynamics (5.1) is used only to derive the equilibrium point of the vehicle at a given Mach number, and is not used in the control design. It is therefore assumed that the air density and the local speed of sound are constant during the simulation time. The control input is given by

$$(\delta_T, \boldsymbol{\delta}) \in \mathbb{R}^4 \qquad (5.5)$$

corresponding respectively to the throttle setting δ_T (controlling $\boldsymbol{F}_T$) and the deflection of the three main aerodynamic control surfaces $\boldsymbol{\delta}$ (controlling $\boldsymbol{M}_{A,\delta}$). Limits are imposed to the magnitude of the control inputs, as follows:

$$|\delta_T| \leq 5\,, \quad |\delta_i| \leq 0.75 \text{ [rad]}\,,\ i = a, e, r$$

$\mathscr{F}_e, \mathscr{F}_b$	Earth-fixed and Body-fixed coordinate frames
$\boldsymbol{R}_{eb} \in SO(3)$	Rotation matrix from $\mathscr{F}_b$ to $\mathscr{F}_e$
$\boldsymbol{\nu} = [u\ v\ w]^{\mathrm{T}}$	Translational velocity of the vehicle in $\mathscr{F}_b$
$\boldsymbol{\omega} = [p\ q\ r]^{\mathrm{T}}$	Angular velocity of the vehicle in $\mathscr{F}_b$
$\boldsymbol{\eta} = [\phi\ \theta\ \psi]^{\mathrm{T}}$	Euler-angle parameterization of $\boldsymbol{R}_{eb}$
$\boldsymbol{E}(\boldsymbol{\eta}) \in \mathbb{R}^{3\times 3}$	Jacobian of the Euler-angle kinematic transformation
$\boldsymbol{F}_{\mathrm{grav}} \in \mathbb{R}^3$	Gravity force
$\boldsymbol{F}_{A,\mathrm{base}},\ \boldsymbol{M}_{A,\mathrm{base}} \in \mathbb{R}^3$	Baseline aerodynamic force and moment
$\boldsymbol{F}_{A,\delta},\ \boldsymbol{M}_{A,\delta} \in \mathbb{R}^3$	Control aerodynamic force and moment
$\boldsymbol{F}_T \in \mathbb{R}^3$	Force due to engine thrust
$V_T = \sqrt{u^2+v^2+w^2}$	Airspeed
$\alpha = \arctan(w/u)$	Angle-of-attack
$\beta = \arcsin(v/V_T)$	Sideslip angle
$\gamma = \theta - \alpha$	Flight-path angle (FPA), assuming $\phi = 0$
h	Altitude
$\rho(h)$	Air density
$a(h)$	Local speed of sound
$M_\infty = V_T/a(h)$	Mach number
m	Mass
$\boldsymbol{J} \in \mathbb{R}^{3\times 3}$	Inertia matrix, $\boldsymbol{J} = \boldsymbol{J}^{\mathrm{T}} > \boldsymbol{0}$
δ_T	Throttle
$\boldsymbol{\delta} = [\delta_a\ \delta_e\ \delta_r]^{\mathrm{T}}$	Aerodynamic control surface deflections (aileron, elevator, rudder)

Table 5.1: Nomenclature for the vehicle models

The force $\boldsymbol{F}_{A,\delta}$ generated by the aerosurfaces is regarded as a disturbance. Aerodynamic forces and moments, as well as the propulsion forces, are nonlinear functions of Mach number, angle-of-attack, sideslip angle and the control inputs; these functions are provided in the form of look-up tables in the nonlinear model. The output to be regulated is given by

$$(M_\infty, \gamma, \phi, \psi) \in \mathbb{R}^4 \tag{5.6}$$

corresponding respectively to Mach number, flight-path angle (FPA), roll angle and yaw angle. Specifically, we aim at tracking reference trajectories $M_{\infty\mathrm{ref}}(t)$, $\gamma_{\mathrm{ref}}(t)$, $\psi_{\mathrm{ref}}(t)$ generated by second-order reference models, while keeping $\phi(t)$ as small as possible. It is stressed that all state variables are available for feedback.

5.2 Linear Aircraft Models

The equations of the nonlinear model (5.1)–(5.4) are linearized around an equilibrium point corresponding to a given set point, which is referred to as a *trim condition.* The trim condition is selected as the vehicle flying in wing-level flight, at constant airspeed and constant altitude, and zero turn rate, corresponding to the constant setpoint

$$M_\infty = M_\infty^0\,, \quad h = h^0\,, \quad \phi = 0\,, \quad \dot{\psi} = 0$$

yielding the trim values

$$\begin{array}{llll} \boldsymbol{\nu}^0 = [u^0\ 0\ w^0]^{\mathrm{T}}\,, & \boldsymbol{\eta}^0 = [0\ \alpha^0\ 0]^{\mathrm{T}}\,, & \boldsymbol{\omega}^0 = [0\ 0\ 0]^{\mathrm{T}} & \\ M_\infty^0 = \dfrac{\sqrt{(u^0)^2 + (w^0)^2}}{a(h^0)}\,, & \alpha^0 = \arctan\left(\dfrac{w^0}{u^0}\right)\,, & \beta^0 = 0\,, & \gamma = 0 \\ \delta_T = \delta_T^0\,, & \delta_a = 0\,, & \delta_e = \delta_e^0 & \delta_r = 0 \end{array}$$

Note that, without loss of generality, the trim value for ψ has been selected to be $\psi = 0$ (any constant value would do.) Also, note that the trim value for θ is the same as the trim value for α, since the level flight condition entails $\gamma = 0$. The linearized equations of motion of the full model (minus the altitude dynamics) read as

$$\begin{aligned} \dot{\boldsymbol{x}} &= \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}\boldsymbol{u} \\ \boldsymbol{y} &= \boldsymbol{C}\boldsymbol{x} \end{aligned} \tag{5.7}$$

where the state vector $\boldsymbol{x} \in \mathbb{R}^9$, the control input $\boldsymbol{u} \in \mathbb{R}^4$ and the regulated output $\boldsymbol{y} \in \mathbb{R}^4$ are given by

$$\boldsymbol{x} = \begin{pmatrix} M_\infty - M_\infty^0 \\ \alpha - \alpha^0 \\ \beta \\ \phi \\ \theta - \alpha^0 \\ \psi \\ p \\ q \\ r \end{pmatrix}\,, \qquad \boldsymbol{u} = \begin{pmatrix} \delta_t - \delta_T^0 \\ \delta_a \\ \delta_e - \delta_e^0 \\ \delta_r \end{pmatrix}\,, \qquad \boldsymbol{y} = \begin{pmatrix} M_\infty - M_\infty^0 \\ \gamma \\ \phi \\ \psi \end{pmatrix}$$

It should be noted that a coordinate transformation has been applied to the nonlinear model prior to linearization to express the translational velocity directly in the coordinates $(M_\infty, \alpha, \beta)$. Two separate subsystems are extracted from (5.7), namely the *longitudinal* and the *lateral* linearized dynamics:

$$\text{Longitudinal}: \begin{cases} \dot{\boldsymbol{x}}_{\text{long}} = \boldsymbol{A}_{\text{long}}\boldsymbol{x}_{\text{long}} + \boldsymbol{B}_{\text{long}}\boldsymbol{u}_{\text{long}} \\ \boldsymbol{y}_{\text{long}} = \boldsymbol{C}_{\text{long}}\boldsymbol{x}_{\text{long}} \end{cases}, \quad \text{Lateral}: \begin{cases} \dot{\boldsymbol{x}}_{\text{lat}} = \boldsymbol{A}_{\text{lat}}\boldsymbol{x}_{\text{lat}} + \boldsymbol{B}_{\text{lat}}\boldsymbol{u}_{\text{lat}} \\ \boldsymbol{y}_{\text{lat}} = \boldsymbol{C}_{\text{lat}}\boldsymbol{x}_{\text{lat}} \end{cases}$$

File name	Mach	Altitude (ft)	Full model	Long. model	Lat. model
`Jet_model_mach1.mat`	1	25×10^3	`Jet1`	`Jet_long1`	`Jet_lat1`
`Jet_model_mach05.mat`	0.5	25×10^3	`Jet05`	`Jet_long05`	`Jet_lat05`
`Jet_model_mach15.mat`	1.5	25×10^3	`Jet15`	`Jet_long15`	`Jet_lat15`

Table 5.2: Linearized models

where (recall that $\gamma = \theta - \alpha$)

$$\boldsymbol{x}_{\text{long}} = \begin{pmatrix} M_\infty - M_\infty^0 \\ \alpha - \alpha^0 \\ \theta - \alpha^0 \\ q \end{pmatrix}, \qquad \boldsymbol{u}_{\text{long}} = \begin{pmatrix} \delta_T - \delta_T^0 \\ \delta_e - \delta_e^0 \end{pmatrix}, \qquad \boldsymbol{y}_{\text{long}} = \begin{pmatrix} M_\infty - M_\infty^0 \\ \gamma \end{pmatrix}$$

and

$$\boldsymbol{x}_{\text{lat}} = \begin{pmatrix} \beta \\ \phi \\ \psi \\ p \\ r \end{pmatrix}, \qquad \boldsymbol{u}_{\text{lat}} = \begin{pmatrix} \delta_a \\ \delta_r \end{pmatrix}, \qquad \boldsymbol{y}_{\text{lat}} = \begin{pmatrix} \phi \\ \psi \end{pmatrix}$$

Three different sets of linearized models have been provided, each corresponding to a different trim condition, as seen in Table 5.2. The files listed in the first column of Table 5.2 contain also the trim values $(M_\infty^0, \alpha^0, \delta_T^0, \delta_e^0)$, as this information is needed by the controller. The linearized model at Mach 1 is the main design model. The other linearized model are used to test the capability of the control algorithm to adapt to different model coefficients.

5.3 Control Design for the Lateral Dynamics

Let us start from the design for the lateral dynamics, as it is less involved than the one for the longitudinal dynamics. The equations of the linearized lateral model read as

$$\begin{aligned} \dot{\beta} &= a_{11}\beta + a_{12}\phi + a_{14}p - r + b_{11}\delta_a + b_{12}\delta_r \\ \dot{\phi} &= p + a_{25}r \\ \dot{\psi} &= r \\ \dot{p} &= a_{41}\beta + b_{41}\delta_a + b_{42}\delta_r \\ \dot{r} &= a_{51}\beta + b_{51}\delta_a + b_{52}\delta_r \end{aligned} \tag{5.8}$$

It is noted that $b_{41} > 0$ and $b_{52} > 0$. This property is also preserved throughout the different linear models, as it is a structural property of the aircraft. Since $|a_{25}| << 1$ and $|b_{42}| << |b_{41}|$, the terms $a_{25}r$ and $b_{42}\delta_r$ will be neglected in the design, but preserved in the model. Also, the presence of saturations on δ_a and δ_r are neglected in the design, but should be implemented in the controller.

5.3.1 Adaptive Stabilization of the Roll Dynamics

Owing to the model simplifications listed above, the control design model for the roll dynamics reads as

$$\begin{aligned}\dot{\phi} &= p \\ \dot{p} &= a_{41}\beta + b_{41}\delta_a\end{aligned} \tag{5.9}$$

Recall that the goal is to stabilize the roll dynamics and decouple it from the dynamics of the sideslip angle. We proceed with a recursive design, starting from the roll angle. Define a *reference trajectory* for the roll rate and the corresponding *tracking error* as

$$p_{\text{ref}} := -k_\phi \phi\,, \qquad e_p := p - p_{\text{ref}}$$

where $k_\phi > 0$ is a controller gain to be selected. Using the coordinates (ϕ, e_p) in place of (ϕ, p), one obtains

$$\begin{aligned}\dot{\phi} &= -k_\phi \phi + e_p \\ \dot{e}_p &= a_{41}\beta + b_{41}\delta_a + k_\phi p \\ &= b_{41}\left[\mu_{11}\beta + \mu_{12}p + \delta_a\right]\end{aligned} \tag{5.10}$$

where

$$\mu_{11} := \frac{a_{41}}{b_{41}}\,, \qquad \mu_{12} := \frac{k_\phi}{b_{41}}$$

are unknown parameters with $b_{41} > 0$. The aileron input is used to adaptively cancel the unknown terms and stabilize the dynamics. To this end, let the control δ_a be defined as

$$\delta_a = -k_p e_p - k_{p,\phi}\phi - \hat{\mu}_{11}\beta - \hat{\mu}_{12}p$$

where $k_p, k_{p,\phi} > 0$ are controller gains to be selected, and $\hat{\mu}_1 := [\hat{\mu}_{11}\ \hat{\mu}_{12}]^{\mathrm{T}}$ is the vector of parameter estimates. The closed-loop system reads as

$$\begin{aligned}\dot{\phi} &= -k_\phi \phi + e_p \\ \dot{e}_p &= -b_{41}k_p e_p - b_{41}k_{p,\phi}\phi - b_{41}\left[\tilde{\mu}_{11}\beta + \tilde{\mu}_{12}p\right] \\ &= -b_{41}(k_p e_p + k_{p,\phi}\phi) - b_{41}\Phi_1^{\mathrm{T}}(\beta, p)\tilde{\mu}_1\end{aligned} \tag{5.11}$$

where $\tilde{\mu}_{1i} := \hat{\mu}_{1i} - \mu_{1i}$, $i = 1, 2$, and $\tilde{\mu}_1 := [\tilde{\mu}_{11}\ \tilde{\mu}_{12}]^{\mathrm{T}}$. Using the positive definite and radially unbounded Lyapunov function candidate

$$V(\phi, e_p, \tilde{\mu}_1) := \frac{1}{2}\phi^2 + \frac{1}{2b_{41}}e_p^2 + \frac{1}{2\Gamma_1}\tilde{\mu}_1^{\mathrm{T}}\tilde{\mu}_1$$

where $\Gamma_1 > 0$ is a scalar gain, one obtains

$$\dot{V}(\phi, e_p, \tilde{\mu}_1) = -k_\phi\phi^2 - k_p e_p^2 + (1 - k_{p,\phi})\phi\, e_p + \frac{1}{\Gamma_1}\tilde{\mu}_1^{\mathrm{T}}\left[\dot{\hat{\mu}} - \Gamma_1\Phi_1(\beta, e_p)e_p\right]$$

The obvious choice

$$k_{p,\phi} = 1\,, \qquad \dot{\hat{\mu}}_1 = \Gamma_1\Phi_1(\beta, e_p)e_p$$

yields

$$\dot{V}(\phi, e_p, \tilde{\mu}_1) = -k_\phi\phi^2 - k_p e_p^2$$

hence asymptotic regulation of $\phi(t)$ and $e_p(t)$ to zero. The selection of the remaining gain parameters (k_ϕ, k_p, Γ_1) should be made on the basis of simulation studies.

5.3.2 Direct Adaptive Control of the Yaw Dynamics

The control design model for the yaw dynamics reads as

$$\begin{aligned} \dot{\psi} &= r \\ \dot{r} &= a_{51}\beta + b_{51}\delta_a + b_{52}\delta_r \end{aligned} \tag{5.12}$$

The goal is to track a reference trajectory for the yaw angle and decouple the yaw dynamics from the dynamics of the sideslip angle and the roll angle. We proceed with a recursive design, starting from the tracking error for the yaw angle. Let the *reference model* for the yaw angle be given by the second-order linear system

$$\begin{aligned} \dot{\xi}_1 &= \xi_2 \\ \dot{\xi}_2 &= \omega_n^2\xi_1 + 2\zeta\omega_n\xi_2 + \omega_n^2\psi_{\text{cmd}} \\ \psi_{\text{ref}} &= \xi_1 \\ \dot{\psi}_{\text{ref}} &= \xi_2 \\ \ddot{\psi}_{\text{ref}} &= -\omega_n^2\xi_1 - 2\zeta\omega_n\xi_2 + \omega_n^2\psi_{\text{cmd}} \end{aligned} \tag{5.13}$$

where $\psi_{\text{ref}}(t)$ is the reference to be tracked, $\psi_{\text{cmd}}(t)$ is the command input to the reference model, $\omega_n > 0$ is the natural frequency of the second-order system, and $\zeta \in (0, 1)$ is the damping ratio. It is well known that the reference model is stable if $\omega_n > 0$ and $\zeta \in (0, 1)$. Define the *tracking error* for the yaw angle as $e_\psi := \psi - \psi_{\text{ref}}$, with dynamics

$$\dot{e}_\psi = r - \dot{\psi}_{\text{ref}}$$

The *reference trajectory* for the yaw rate and the corresponding *tracking error* are defined as

$$r_{\text{ref}} := -k_\psi e_\psi + \dot{\psi}_{\text{ref}}\,, \qquad e_r := r - r_{\text{ref}}$$

where $k_\psi > 0$ is a controller gain to be selected. Using the coordinates (e_ψ, e_r) in place of (ψ, r), one obtains

$$\begin{aligned} \dot{e}_\psi &= -k_\psi e_\psi + e_r \\ \dot{e}_r &= a_{51}\beta + b_{51}\delta_a + b_{52}\delta_r + k_\psi(r - \dot{\psi}_{\text{ref}}) - \ddot{\psi}_{\text{ref}} \\ &= b_{52}\left[\mu_{21}\beta + \mu_{22}(k_\psi r - k_\psi\dot{\psi}_{\text{ref}} - \ddot{\psi}_{\text{ref}}) + \mu_{23}\delta_a + \delta_r\right] \\ &= b_{52}\left[\Phi_2^{\mathrm{T}}(t, \beta, r, \delta_a)\mu_2 + \delta_r\right] \end{aligned} \tag{5.14}$$

where $\mu_2 = [\mu_{21}\ \mu_{22}\ \mu_{23}]^{\mathrm{T}}$, and

$$\mu_{21} := \frac{a_{51}}{b_{52}}\,, \qquad \mu_{22} := \frac{1}{b_{52}}\,, \qquad \mu_{23} := \frac{b_{51}}{b_{52}}$$

are unknown parameters with $b_{52} > 0$. The rudder input is used to adaptively cancel the unknown terms and stabilize the tracking error dynamics. To this end, the control δ_r is defined as

$$\delta_a = -k_r e_r - e_\psi - \Phi_2^{\mathrm{T}}(t, \beta, r, \delta_a)\hat{\mu}_2$$

where $k_r > 0$ is a controller gain to be selected, and $\hat{\mu}_2$ is the estimate of the unknown parameter vector μ_2. The closed-loop system reads as

$$\begin{aligned}
\dot{e}_\psi &= -k_\psi e_\psi + e_r \\
\dot{e}_r &= -b_{41} k_p e_p - b_{41} k_{p,\phi} \phi - b_{41} \left[\tilde{\mu}_1 \beta + \tilde{\mu}_2 p\right] \\
&= -b_{52}(k_r e_r + e_\psi) - b_{52} \Phi_2^{\mathrm{T}}(t, \beta, r, \delta_a) \tilde{\mu}_2
\end{aligned} \tag{5.15}$$

where $\tilde{\mu}_2 := \hat{\mu}_2 - \mu_2$. Using the positive definite and radially unbounded Lyapunov function candidate

$$V(e_\psi, e_r, \tilde{\mu}_2) := \frac{1}{2} e_\psi^2 + \frac{1}{2b_{52}} e_r^2 + \frac{1}{2\Gamma_2} \tilde{\mu}_2^{\mathrm{T}} \tilde{\mu}_2$$

where $\Gamma_2 > 0$ is a scalar gain, and selecting the passivity-based update law

$$\dot{\hat{\mu}}_2 = \Gamma_2 \Phi_2(t, \beta, r, \delta_a) e_r$$

yields

$$\dot{V}(e_\psi, e_r, \tilde{\mu}_2) = -k_\psi e_\psi^2 - k_r e_r^2$$

hence asymptotic regulation of $e_\psi(t)$ and $e_r(t)$ to zero. The selection of the gain parameters (k_ψ, k_r, Γ_2) should be made on the basis of simulation studies.

5.3.3 Internal Dynamics: Sideslip Angle

The internal dynamics of the system is given by the dynamics of the sideslip angle, which can be written as the perturbed time-varying system

$$\dot{\beta} = \big(a_{11} - b_{11}\hat{\mu}_{11}(t) - b_{21}\hat{\mu}_{21}(t)\big)\beta + \Delta(t, \phi, e_p, e_\psi, e_r)$$

Since $a_{11} < 0$, $0 < b_{11} << 1$ and $0 < b_{12} << 1$, it can be shown that the time-varying system above is bounded-input, bounded-state stable if the estimation errors $\tilde{\mu}_{11}(t)$ and $\tilde{\mu}_{21}(t)$ remain sufficiently small. As a matter of fact, if the true values of these estimates are employed, then it is verified that $a_{11} - b_{11}\mu_{11} - b_{21}\mu_{21} < 0$ for all considered linearized models. In a practical implementation, the key is to choose the initial conditions of the estimated parameters at the nominal values for the design model (the one trimmed at Mach 1) and avoid selecting the gain of the update law too large to prevent the estimates to drift too rapidly away from the nominal values.

5.4 Control of the Longitudinal Model

Before proceedings with the design of the controller for the longitudinal dynamics, it is convenient to change coordinates in order to substitute the angle-of-attack with the flight-path angle, as this latter is a variable to be regulated[1]. Specifically, let

$$m_\infty := M_\infty - M_\infty^0\,, \qquad \vartheta := \theta - \alpha^0\,, \qquad \bar{\delta}_T := \delta_T - \delta_T^0\,, \qquad \bar{\delta}_e := \delta_e - \delta_e^0$$

[1]The LTI longitudinal model provided in the Matlab file uses the *original coordinates*, $(m_\infty, \alpha - \alpha^0, \vartheta, q)$.

denote deviations from the trim conditions, and recall that $\gamma = \theta - \alpha = \vartheta - (\alpha - \alpha^0)$. The equations of the longitudinal dynamics in the coordinates $(m_\infty, \gamma, \vartheta, q)$ read as follows

$$\begin{aligned}
\dot{m}_\infty &= a_{11} m_\infty + a_{12}\gamma + a_{13}\vartheta + b_{11}\text{sat}\left(\bar{\delta}_T\right) + b_{12}\bar{\delta}_e \\
\dot{\gamma} &= a_{21} m_\infty - a_{22}\gamma + a_{22}\vartheta + b_{21}\text{sat}\left(\bar{\delta}_T\right) + b_{22}\bar{\delta}_e \\
\dot{\vartheta} &= q \\
\dot{q} &= a_{41} m_\infty + a_{42}\gamma - a_{42}\vartheta + b_{42}\bar{\delta}_e
\end{aligned} \tag{5.16}$$

It is noted that $a_{22} > 0$, $b_{11} > 0$ and $b_{42} > 0$. This property is also preserved throughout the different linear models, as it is a structural property of the aircraft. Since $|b_{12}| << 1$, $|b_{21}| << 1$ and $|b_{22}| << 1$, the corresponding entries in the above model will be neglected in the design, but preserved in the model for simulation. Also, the presence of saturations on $\bar{\delta}_e$ is neglected in the design, but should be implemented in the controller. However, the controller design will take into account the saturation on the throttle input, $\bar{\delta}_T$. Note that since the original limits for the throttle were $|\delta_T| \leq 5$, the limits for $\bar{\delta}_T$ are

$$\delta_T^0 - 5 \leq \bar{\delta}_T \leq 5 + \delta_T^0$$

5.4.1 Indirect Adaptive Control of the Airspeed Dynamics

The goal is to track a reference $m_{\infty,\text{ref}}$ for m_∞, generated by a reference model of the same form of (5.13), this time driven by a bounded Mach number command signal, $m_{\infty,\text{cmd}}(t)$. We use an indirect approach to incorporate explicitly the presence of the input saturation in the design. Owing to the fact that $|b_{12}| << 1$, the Mach number dynamics is written as

$$\begin{aligned}
\dot{m}_\infty &= a_{11} m_\infty + a_{12}\gamma + a_{13}\vartheta + b_{11}\text{sat}\left(\bar{\delta}_T\right) \\
&= \Psi_1^{\text{T}}(m_\infty, \gamma, \vartheta, \bar{\delta}_T)\chi_1
\end{aligned}$$

where $\chi_1 = [a_{11}\ a_{12}\ a_{13}\ b_{11}]^{\text{T}}$ is the vector of unknown model parameters. For this system, a *model estimator* is built in the form

$$\begin{aligned}
\dot{\hat{m}}_\infty &= \hat{a}_{11} m_\infty + \hat{a}_{12}\gamma + \hat{a}_{13}\vartheta + \hat{b}_{11}\text{sat}\left(\bar{\delta}_T\right) - \ell(\hat{m}_\infty - m_\infty) \\
&= \Psi_1^{\text{T}}(m_\infty, \gamma, \vartheta, \bar{\delta}_T)\hat{\chi}_1 - \ell\tilde{m}_\infty
\end{aligned} \tag{5.17}$$

where $\ell > 0$ is the gain of the model estimator, $\hat{\chi}_1 = [\hat{a}_{11}\ \hat{a}_{12}\ \hat{a}_{13}\ \hat{b}_{11}]^{\text{T}}$ is the vector of parameter estimates, and $\tilde{m}_\infty := \hat{m}_\infty - m_\infty$ is the model estimation error. The dynamics of the model estimation error read as

$$\dot{\tilde{m}}_\infty = \Psi_1^{\text{T}}(m_\infty, \gamma, \vartheta, \bar{\delta}_T)\tilde{\chi}_1 - \ell\tilde{m}_\infty$$

where $\tilde{\chi}_1 := \hat{\chi}_1 - \chi_1$ is the parameter estimation error. Using the Lyapunov function candidate

$$V(\tilde{m}_\infty, \tilde{\chi}_1) := \frac{1}{2}\tilde{m}_\infty^2 + \frac{1}{2\Upsilon_1}\tilde{\chi}_1^{\text{T}}\tilde{\chi}_1$$

where $\Upsilon_1 > 0$ is a scalar gain, and selecting the passivity-based update law

$$\dot{\hat{\chi}}_1 = -\Upsilon_1 \Psi_1(m_\infty, \gamma, \vartheta, \bar{\delta}_T)\tilde{m}_\infty \tag{5.18}$$

yields

$$\dot{V}(\tilde{m}_\infty, \tilde{\chi}_1) = -\ell \tilde{m}_\infty^2$$

hence asymptotic regulation of $\tilde{m}_\infty(t)$ to zero. The selection of the gain parameters (ℓ, Υ_1) should be made on the basis of simulation studies. The control law for $\bar{\delta}_T$ is designed on the basis of the *tracking error dynamics* for the model estimator. Specifically, let $\hat{e}_m := \hat{m}_\infty - m_{\infty,\mathrm{ref}}$ and write the corresponding dynamics as

$$\dot{\hat{e}}_m = \hat{a}_{11} m_\infty + \hat{a}_{12}\gamma + \hat{a}_{13}\vartheta + \hat{b}_{11}\mathrm{sat}\left(\bar{\delta}_T\right) - \ell\tilde{m}_\infty - \dot{m}_{\infty,\mathrm{ref}} \tag{5.19}$$

The control input is selected as

$$\bar{\delta}_T = \frac{1}{\hat{b}_{11}}\left[-\hat{a}_{11} m_\infty - \hat{a}_{12}\gamma - \hat{a}_{13}\vartheta + \ell\tilde{m}_\infty + \dot{m}_{\infty,\mathrm{ref}} - k_m \hat{e}_m\right] \tag{5.20}$$

where $k_m > 0$ is a gain to be selected, yielding (when $\delta_T^0 - 5 < \bar{\delta}_T < 5 + \delta_T^0$) the converging dynamics

$$\dot{\hat{e}}_m = -k_m \hat{e}_m$$

It must be noted that, since it is required that $\hat{b}_{11}(t)$ be bounded away from zero to avoid singularity of the control law, a parameter projection should be implemented in the update law (5.18), to ensure that $\hat{b}_{11}(t) \geq \varepsilon$ for all $t \geq 0$, where $\varepsilon > 0$ is a small positive constant. Specifically, given $\varepsilon > 0$ such that the coefficient b_{11} satisfies $b_{11} \geq \varepsilon$ for all considered linearized models, the update law for $\hat{b}_{11}$ is modified by the introduction of parameter projection onto the convex set $\{\varepsilon - \hat{b}_{11} \leq 0\}$ as follows:

$$\dot{\hat{b}}_{11} = \begin{cases} -\Upsilon_1 \mathrm{sat}\left(\bar{\delta}_T\right)\tilde{m}_\infty & \text{if } \hat{b}_{11} > \varepsilon \text{ or } \left(\hat{b}_{11} = \varepsilon \text{ and } -\Upsilon_1 \mathrm{sat}\left(\bar{\delta}_T\right)\tilde{m}_\infty \geq 0\right) \\ 0 & \text{if } \hat{b}_{11} = \varepsilon \text{ and } -\Upsilon_1 \mathrm{sat}\left(\bar{\delta}_T\right)\tilde{m}_\infty < 0 \end{cases}$$

5.4.2 Adaptive Control with Integral Augmentation for the Flight-path Angle Dynamics

The design goal is to let γ track a reference $\gamma_{\mathrm{ref}}(t)$ generated by a reference model of the same form of (5.13), this time driven by a bounded flight-path angle command signal, $\gamma_{\mathrm{cmd}}(t)$. Since direct compensation of the uncertainty and the disturbances in the FPA dynamics would lead to an unduly complex control law, we employ integral error augmentation to ensure zero steady-state error for constant set points. It is worth nothing that $\gamma_{\mathrm{cmd}}(t) = $ const corresponds to a *steady climb* (or *descent*), whereas $\gamma_{\mathrm{cmd}}(t) = 0$ corresponds to a *level flight*.

Integral Control of the FPA Dynamics

On the basis of the model simplifications introduced at the beginning of the section, the dynamics of the FPA and the pitch angle read as

$$\begin{aligned} \dot{\gamma} &= a_{21} m_\infty - a_{22}\gamma + a_{22}\vartheta \\ \dot{\vartheta} &= q \\ \dot{q} &= a_{41} m_\infty + a_{42}\gamma - a_{42}\vartheta + b_{42}\bar{\delta}_e \end{aligned} \tag{5.21}$$

where the coupling between the FPA dynamics and the control effectors has been neglected. Define the *tracking error* for the FPA as $e_\gamma := \gamma - \gamma_{\text{ref}}$, with dynamics

$$\begin{aligned} \dot{e}_\gamma &= a_{21} m_\infty + a_{22}(\vartheta - \gamma) - \dot{\gamma}_{\text{ref}} \\ &= a_{21} m_\infty + a_{22}(-e_\gamma - \gamma_{\text{ref}} + \vartheta) - \dot{\gamma}_{\text{ref}} \end{aligned} \tag{5.22}$$

It is important to recall that $a_{22} > 0$. The FPA error dynamics (5.22) is augmented with the integral of the error, as follows

$$\begin{aligned} \dot{\zeta} &= e_\gamma \\ \dot{e}_\gamma &= a_{21} m_\infty + a_{22}(-e_\gamma - \gamma_{\text{ref}} + \vartheta) - \dot{\gamma}_{\text{ref}} \end{aligned} \tag{5.23}$$

where $\zeta \in \mathbb{R}$ is the state of the integrator. In system (5.23), the pitch angle, ϑ, is regarded as a *virtual control*, whereas m_∞ and $\dot{\gamma}_{\text{ref}}$ are regarded as *external disturbances* (hence the need for an integral action.) The *reference trajectory* for the pitch angle and the corresponding *tracking error* are defined as

$$\vartheta_{\text{ref}} := -k_\zeta \zeta + \dot{\gamma}_{\text{ref}}\,, \qquad e_\vartheta := \vartheta - \vartheta_{\text{ref}}$$

where $k_\zeta > 0$ is a controller gain to be selected. Accordingly, system (5.23) is written as

$$\begin{aligned} \dot{\zeta} &= e_\gamma \\ \dot{e}_\gamma &= -a_{22} e_\gamma - k_\zeta a_{22} \zeta + a_{22} e_\vartheta + a_{21} m_\infty - \dot{\gamma}_{\text{ref}} \end{aligned} \tag{5.24}$$

Assume that the command signals for Mach number and FPA provided to the reference models are constant, that is, assume that

$$m_{\infty,\text{cmd}}(t) = m^\star_{\infty,\text{cmd}}\,, \qquad \gamma_{\text{cmd}}(t) = \gamma^\star_{\text{cmd}}$$

This assumption implies that the reference trajectories $m_{\infty,\text{ref}}(t)$, $\gamma_{\text{ref}}(t)$ generated by reference models of the form (5.13) converge to the constant values $m^\star_{\infty,\text{cmd}}$, $\gamma^\star_{\text{cmd}}$ as $t \to \infty$. As a result, the reference trajectories satisfy

$$\begin{aligned} &\lim_{t\to\infty} \Delta m_{\infty,\text{ref}}(t) = 0\,, && \Delta m_{\infty,\text{ref}}(t) := m_{\infty,\text{ref}}(t) - m^\star_{\infty,\text{cmd}} \\ &\lim_{t\to\infty} \Delta\gamma_{\text{ref}}(t) = 0\,, && \Delta\gamma_{\text{ref}}(t) := \gamma_{\text{ref}}(t) - \gamma^\star_{\text{cmd}} \end{aligned}$$

and

$$\lim_{t\to\infty} \dot{\gamma}_{\text{ref}}(t) = 0$$

This assumption allows one to rewrite system (5.24) in the form

$$\begin{aligned} \dot{\zeta} &= e_\gamma \\ \dot{e}_\gamma &= -a_{22} e_\gamma - k_\zeta a_{22} \zeta + a_{22} e_\vartheta + a_{21} m^\star_{\infty,\text{cmd}} + \Delta_\gamma(t) \end{aligned} \tag{5.25}$$

where

$$\Delta_\gamma(t) := a_{21} e_m(t) + a_{21} \Delta m_{\infty,\text{ref}}(t) - \dot{\gamma}_{\text{ref}}(t)$$

is a disturbance that vanishes asymptotically. Finally, letting

$$e_\zeta := \zeta - \frac{a_{21} m^\star_{\infty,\text{cmd}}}{k_\zeta a_{22}}$$

one obtains

$$\begin{aligned}\dot{e}_\zeta &= e_\gamma \\ \dot{e}_\gamma &= -a_{22}e_\gamma - k_\zeta a_{22}e_\zeta + a_{22}e_\vartheta + \Delta_\gamma(t)\end{aligned} \tag{5.26}$$

Note that the integral term has "absorbed" the constant term $a_{21}m^\star_{\infty,\mathrm{cmd}}$. It is easy to see that, since $a_{22} > 0$, system (5.26) has a globally exponentially stable equilibrium at the origin $(e_\zeta, e_\gamma) = (0, 0)$ for all $k_\zeta > 0$ when $e_\vartheta = 0$ and $\Delta_\gamma = 0$. The perturbed system is also *input-to-state stable* with respect to the inputs e_ϑ and Δ_γ. The ISS property can be easily verified with the aid of a quadratic Lyapunov function for the unperturbed system, which is also an ISS-Lyapunov function for the perturbed one. Since $\Delta_\gamma(t)$ is a vanishing perturbation, input-to-state stability of (5.26) yields

$$\lim_{t\to\infty} |e_\vartheta(t)| = 0 \implies \lim_{t\to\infty} |e_\zeta(t)| = 0 \text{ and } \lim_{t\to\infty} |e_\gamma(t)| = 0$$

As a result, asymptotic regulation of e_γ is implied by asymptotic regulation of e_ϑ.

Direct Adaptive Control of the Pitch Dynamics

The goal of this section is to design an adaptive controller to achieve asymptotic regulation of the tracking error for the pitch angle $e_\vartheta(t)$, whose dynamics reads as

$$\dot{e}_\vartheta = q - \dot{\vartheta}_{\mathrm{ref}} = q - \dot{\gamma}_{\mathrm{ref}} + k_\zeta e_\gamma$$

To achieve this goal, define the *reference trajectory* for the pitch rate and the corresponding *tracking error* as

$$q_{\mathrm{ref}} := -k_\vartheta e_\vartheta + \dot{\gamma}_{\mathrm{ref}} - k_\zeta e_\gamma\,, \qquad e_q := q - q_{\mathrm{ref}}$$

where $k_\vartheta > 0$ is a controller gain to be selected. Using the coordinates (e_ϑ, e_q) in place of (ϑ, q), one obtains from (5.21)

$$\begin{aligned}\dot{e}_\vartheta &= -k_\vartheta e_\vartheta + e_q \\ \dot{e}_q &= a_{41}m_\infty + a_{42}\gamma - a_{42}\vartheta + b_{42}\bar{\delta}_e + k_\vartheta\big(q - \dot{\gamma}_{\mathrm{ref}} + k_\zeta e_\gamma\big) - \ddot{\gamma}_{\mathrm{ref}} \\ &\qquad + k_\gamma\big(a_{21}m_\infty + a_{22}(\vartheta - \gamma) - \dot{\gamma}_{\mathrm{ref}}\big) \\ &= (a_{41} + k_\gamma a_{21})m_\infty + (k_\gamma a_{22} - a_{42})(\vartheta - \gamma) + \varphi + b_{42}\bar{\delta}_e\end{aligned} \tag{5.27}$$

where the known signal $\varphi(t)$ is defined as

$$\varphi := k_\vartheta\big(q - \dot{\gamma}_{\mathrm{ref}} + k_\zeta e_\gamma\big) - \ddot{\gamma}_{\mathrm{ref}} - k_\gamma \dot{\gamma}_{\mathrm{ref}}$$

Introducing the unknown parameter vector $\chi_2 = [\chi_{21}\ \chi_{22}\ \chi_{23}]^{\mathrm{T}}$, where

$$\chi_{21} := \frac{a_{41} + k_\gamma a_{21}}{b_{42}}\,, \qquad \chi_{22} := \frac{k_\gamma a_{22} - a_{42}}{b_{42}}\,, \qquad \chi_{23} := \frac{1}{b_{42}}$$

and recalling that $b_{42} > 0$, one obtains from (5.27)

$$\begin{aligned}\dot{e}_\vartheta &= -k_\vartheta e_\vartheta + e_q \\ \dot{e}_q &= b_{42}\left[\chi_{21}m_\infty + \chi_{22}(\vartheta - \gamma) + \chi_{23}\varphi + \bar{\delta}_e\right] \\ &= b_{42}\left[\Psi_2^{\mathrm{T}}(m_\infty, \gamma, \vartheta, \varphi)\chi_2 + \bar{\delta}_e\right]\end{aligned} \tag{5.28}$$

Following the same procedure detailed in section 5.3.2, the control law for $\bar{\delta}_e$ is selected as the certainty-equivalence adaptive control

$$\bar{\delta}_e = -k_q e_q - e_\vartheta - \Psi_2^{\mathrm{T}}(m_\infty, \gamma, \vartheta, \varphi)\hat{\chi}_2$$

where $k_q > 0$ is a controller gain to be selected, and $\hat{\chi}_2$ is the estimate of the unknown parameter vector χ_2. The closed-loop system reads as

$$\begin{aligned} \dot{e}_\vartheta &= -k_\vartheta e_\vartheta + e_q \\ \dot{e}_q &= -b_{42}(k_q e_q + e_\vartheta) - b_{42}\Psi_2^{\mathrm{T}}(m_\infty, \gamma, \vartheta, \varphi)\tilde{\chi}_2 \end{aligned} \tag{5.29}$$

where $\tilde{\chi}_2 := \hat{\chi}_2 - \chi_2$. Using the positive definite and radially unbounded Lyapunov function candidate

$$V(e_\vartheta, e_q, \tilde{\chi}_2) := \frac{1}{2}e_\vartheta^2 + \frac{1}{2b_{42}}e_q^2 + \frac{1}{2\Upsilon_2}\tilde{\chi}_2^{\mathrm{T}}\tilde{\chi}_2$$

where $\Upsilon_2 > 0$ is a scalar gain, and selecting the passivity-based update law

$$\dot{\hat{\chi}}_2 = \Upsilon_2 \Psi_2(m_\infty, \gamma, \vartheta, \varphi)e_q$$

yields

$$\dot{V}(e_\vartheta, e_q, \tilde{\chi}_2) = -k_\vartheta e_\vartheta^2 - k_q e_q^2$$

hence asymptotic regulation of $e_\vartheta(t)$ and $e_q(t)$ to zero.

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