The pages that follow contain

- Some excerpts (keep an eye on the page number!) from a tutorial/survey paper that I am writing about model based control of soft robots. The paper will be fully available through ArXiv in a few weeks
- My hand notes

In case you spot errors in these documents and/or if you have any suggestion about how to improve the lectures, I would be extremely grateful if you let me know by sending an email to c.dellasantina@tudelft.nl
Also, do not hesitate to use the same email if you have other questions/curiosities/… you wish to discuss.

You can also find a general introduction to articulated and soft robots here (Springer Encyclopedia of Robotics):

www.researchgate.net/profile/Cosimo-Della-Santina/publication/346266320_Soft_Robots/links/6037637592851c4ed595b7c3/Soft-Robots.pdf

Meeting you all in person has been quite nice. I look forward to reading your next papers, and to meeting again at the next in-person conference!

Cheers,
Cosimo

Finite dimensional approximations

The alternative to PDE formulations is to restrict the range of possible strains ξ to a finite dimensional functional space. Two classes of strategies exist to achieve this goal: piecewise constant strain models, and functional parametrizations. Both of them will be discussed in detail below. At the current stage, what is important to keep in mind is that using these techniques the strain ξ can be approximated as a function of the vector $q \in \mathbb{R}^n$ that serves as configuration of the soft robot. This key step enables the recasting concepts from classic discrete robotics to the new continuum context. For a start, the kinematics of a soft robot can now be defined as follows

$$\dot{x}(s, q(t)) = J(s, q(t))\, \dot{q}(t), \quad J(s, q) = \frac{\partial h(s, q)}{\partial q}, \tag{1}$$

where $h(s, q)$ is the map - called forward kinematics - connecting the configuration $q(t)$ to the posture $x(s, t)$ for each point s along the backbone. The matrix-valued function J is the Jacobian of h. The following set of ODEs can be directly derived from (1) sing standard Lagrangian mechanics machinery

$$\underbrace{M(q)\ddot{q} + C(q, \dot{q})\dot{q} + G(q)}_{\text{Multi-body dynamics}} + \underbrace{D(q)\dot{q} + K(q)}_{\text{Elastic and dissipative forces}} = \underbrace{A(q)\tau}_{\text{Model of underactuation}}, \tag{2}$$

where $(q, \dot{q})$ forms the robot state.

The inertia matrix $M(q) \in \mathbb{R}^{n \times n}$ is evaluated as follows

$$M(q) = \int_0^1 J^\top(q, s) \begin{bmatrix} m(s)I & 0 \\ 0 & \mathcal{J}(s) \end{bmatrix} J(q, s)\, \mathrm{d}s \succeq 0, \tag{3}$$

where $m(s)$ and $\mathcal{J}(s)$ are the mass and inertia distributions respectively. As for a rigid robot, this matrix verifies

$$||M(q)|| \leq c_\mathrm{m} + c'_\mathrm{m}||q||^2, \tag{4}$$

where $c_\mathrm{m}, c'_\mathrm{m}$ are two positive scalars. If the elongation is considered negligible, then $c'_\mathrm{m} = 0$. Coriolis and centrifugal forces $C(q, \dot{q})\dot{q} \in \mathbb{R}^n$ can be evaluated using the standard mathematical machinery (e.g. Christoffel symbols). Elastic $K(q)$ and gravitational $G(q)$ actions are defined as

$$K(q) = \frac{\partial U_\mathrm{K}}{\partial q}, \quad G(q) = \frac{\partial U_\mathrm{G}}{\partial q}, \tag{5}$$

where U_K and U_G are the associated potential energies, obtained as integration along the spacial coordinate of the energetic contributions of each infinitesimal elements. The elastic force field of the actuators are themselves mechanical (e.g. tendons actuated through electric motors, fluids pressurized through pistons)

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + D(q)\dot{q} + K(q) + G(q) + \frac{\partial U_{\mathrm{c}}}{\partial q}(q,\eta) = 0, \quad (12)$$

$$B(\eta)\ddot{\eta} + H(\eta,\dot{\eta})\dot{\eta} + \frac{\partial U_{\mathrm{c}}}{\partial \eta}(q,\eta) = \tau, \quad (13)$$

where we not include dissipation in the actuation system for simplicity. The configuration of all the actuators is collected in $\eta \in \mathbb{R}^m$, and $B, H \in \mathbb{R}^{m\times m}$ are the associated inertia and Coriolis matrices. The former is usually diagonal and configuration independent, and in turn $H = 0$. This is however not always the case, an exception being magnetically actuated soft robots with magnets moved by a rigid robot [89].

The coupling between the two dynamics (12) and (13) is purely mediated by the potential field U_{c}, which models elasticity of tendons, molecular interactions in compressible fluids, or electro-magnetic fields, just to cite a few. In case the dynamics of η is fast compared to q, as well as robustly globally stable, then (13) can be approximated with its steady state behavior $\eta \simeq \bar{\eta}(q,\tau)$. In this case $\partial U_{\mathrm{c}}(q,\bar{\eta}(q,\tau))/\partial q$ serves as a generalization of the the input field $A(q)\tau$ appearing in (2). Alternatively, singular perturbation theory can be used to separate the fast actuator dynamics from the slow soft robot one, without applying quasi-static approximations [90].

Simulators

A bottleneck to entering the field of soft robots control has been the need of implementing by yourself the simulator of the soft robot. This is especially troublesome when considering that the models used for simulation are typically way more sophisticated than the ones used for control. Luckily, there are now several open-source solutions available: SOFA uses velumetric FEM techniques [91], [92], while Elastica [93], TMTDyn [94], SimSOFT [52], and SoRoSim [95] implement discretizations of rod modes. More details on simulators for soft robots can be found here [96, Sec. VII]. Still, selecting the right model among all the available ones is a task with no clear solution. Experimental comparisons as the ones provided in [97], [98] can be a useful tool in this context.

Shape Control in the Fully Actuated Approximation

The primary task of control architectures in classic robotics is to accurately manage the posture of the robot - i.e. state space control. In the case of soft robots, this translates into devising strategies to control the whole shape of the system, that is controlling q. Depending on the model

Figure 7: Block schemes of controllers for task space control (position and impedance)

Posture regulation

Posture regulation is defined as follows: given a desired configuration $\bar{q} \in \mathbb{R}^n$ find a control action $\tau \in \mathbb{R}^m$ such that the configuration of the soft robot $q \in \mathbb{R}^n$ eventually converges to the desired one, i.e.

$$\lim_{t \to \infty} q(t) = \bar{q}. \tag{14}$$

It has been already discussed in the previous section that an equilibrium is always associated to any constant control input - as exemplified by (11). We show here that this equilibrium is also asymptotically stable under opportune conditions on the mechanical impedance of the robot. Consider the following purely feedforward controller

$$\tau(\bar{q}) = K(\bar{q}) + G(\bar{q}), \tag{15}$$

where K and G are the elastic and gravitational fields with potentials U_{K} and U_{G} respectively, as defined in (5). Plugging (15) in (2) and performing simple rearranging of the terms yield

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} = \underbrace{(K(\bar{q}) + G(\bar{q})) - (K(q) + G(q))}_{\text{Physical P-loop}} + \underbrace{D(q)(-\dot{q})}_{\text{Physical D-loop}}, \tag{16}$$

where we can recognize the same mathematical structure of a classic robot (left hand side) controlled through a nonlinear PD regulator (right hand side). Note indeed that $\dot{\bar{q}} = 0$ by hypothesis. The control community has devoted much attention to (nonlinear) PD controllers [99], which has produced a thriving literature which soft roboticist can borrow from [100]–

[102], by relying on (16), as for example done in the following theorem.

Theorem 1. *The state $(\bar{q}, 0) \in \mathbb{R}^{2n}$ is an asymptotically stable equilibrium of system* (2) *subject to the constant control action* (15) *if an open neighbourhood $\mathcal{N}(\bar{q}) \subseteq \mathbb{R}^n$ of $\bar{q}$ exists such that $\forall q \in \mathcal{N}(\bar{q})/\{\bar{q}\}$,*

$$(U_{\mathrm{G}}(q) + U_{\mathrm{K}}(q)) > (U_{\mathrm{G}}(\bar{q}) + U_{\mathrm{K}}(\bar{q})) + \left(\frac{\partial}{\partial q}(U_{\mathrm{G}}(q) + U_{\mathrm{K}}(q))\right)\Bigg|_{q=\bar{q}}^{\top}(q-\bar{q}), \tag{17}$$

and

$$G(q) + K(q) \neq G(\bar{q}) + K(\bar{q}). \tag{18}$$

These two conditions also imply that $\mathcal{N}(\bar{q})$ is fully included within the region of asymptotic stability of $\bar{q}$.

Proof. Consider as Lyapunov candidate the following generalization of the energy of the robot

$$V(q,\dot{q}) = \underbrace{\frac{1}{2}\dot{q}^{\top}M(q)\dot{q}}_{\text{Kinetic energy}} + \underbrace{U_{\mathrm{G}}(q) - U_{\mathrm{G}}(\bar{q}) + U_{\mathrm{K}}(q) - U_{\mathrm{K}}(\bar{q})}_{\text{Centered potential energy}} + \underbrace{(G(\bar{q}) + K(\bar{q}))^{\top}(\bar{q}-q)}_{\text{Correction term}}. \tag{19}$$

The kinetic energy is always strictly positive definite in $\dot{q}$ since $M \succ 0$. Thus, necessary and sufficient condition for V to be positive definite in $(q, \dot{q})$ is that $V - \dot{q}^{\top}M(q)\dot{q}/2$ is positive definite in q, which is equivalent to (17). The next step is to study the sign of the time derivative of (19), which is

$$\begin{aligned}
\dot{V}(q,\dot{q}) &= \dot{q}^{\top}M(q)\ddot{q} + \frac{1}{2}\dot{q}^{\top}\dot{M}(q)\dot{q} - \dot{q}^{\top}\left((K(\bar{q}) + G(\bar{q})) - (K(q) + G(q))\right) \\
&= -\dot{q}^{\top}C(q,\dot{q})\dot{q} + \dot{q}^{\top}D(q)(-\dot{q}) + \frac{1}{2}\dot{q}^{\top}\dot{M}(q)\dot{q} \\
&= -\dot{q}^{\top}D(q)\dot{q},
\end{aligned} \tag{20}$$

where the first step exploits (16) to express $M\ddot{q}$, and the second the passivity of the system $\dot{q}^{\top}(\dot{M}(q) - 2C(q,\dot{q}))\dot{q} = 0$. Eq. (20) is only semi-positive definite despite $D(q)$ being a strictly positive matrix, since $\dot{V}(q, 0) = 0$ for all q. Thanks to LaSalle's principle, the system converges to the set of $(q, 0)$ such that $\ddot{q} = 0$. To conclude the proof it is therefore sufficient to show that $\bar{q}$ is the only configuration in $\mathcal{N}(\bar{q})$ such that $\ddot{q} \neq 0$ for $\dot{q} = 0$, i.e.

$$G(q) + K(q) \neq \bar{\tau}, \tag{21}$$

which thanks to (15) is equivalent to the hypothesis (18), thus yielding the thesis. □

Eq. (17) is a convexity condition on the total potential energy $U_{\mathrm{G}}(q) + U_{\mathrm{K}}(q)$. As such, it can be locally checked by looking at the sign of the Hessian matrix. This results in the condition

$$\left(\frac{\partial K(q)}{\partial q}+\frac{\partial G(q)}{\partial q}\right)\bigg|_{q=\bar{q}}\succ 0, \tag{22}$$

which is to say that the force field linearized at the desired equilibrium is attractive. In turn, this also implies that the potential force field is locally not constant, therefore implying also hypothesis (18) at least in an infinitesimal neighborhood of $\bar{q}$. Additionally, (22) becomes a necessary condition for when $\succeq$ is used instead of $\succ$. The two terms in (22) are the stiffness matrices associated to elastic and potential fields. While the first is always positive definite - see (6) - the second is in general not definite in sign. Gravity may serve either as a destabilizing ($\partial G(q)/\partial q \preceq 0$) or as stabilizing force ($\partial G(q)/\partial q \succeq 0$). For the CC segment described in sidebar xxx, these two conditions corresponds to the robot pointing upwards ($\phi = \pi$) or downwards ($\phi = 0$) when in straight configuration ($q = 0$) respectively.

Thus, as already pointed out for the analysis of equilibria, the presence of an elastic field makes the control problem simpler to solve compared to the standard rigid case. This can be regarded as an instance of the so-called self stabilization property of soft robots, which has been recognized by several works in the literature [103], [104]. However, even if a feedforward action has proven to be sufficient for stiff enough systems, it is still interesting to consider what happens when a further feedback loop is introduced. This may serve several purposes, as for example enlarge the basin of attraction, shape the transient, and reject disturbances. Further following along with the analogy with nonlinear PDs, (15) can be extended as follows for the fully actuated case

$$\tau(\bar{q},q,\dot{q})=\underbrace{K(\bar{q})+G(\bar{q})}_{\text{Feedforward}}+\underbrace{\alpha(\bar{q}-q)-\beta\dot{q}}_{\text{PD}}. \tag{23}$$

Here, $\alpha, \beta \in \mathbb{R}^{n\times n}$ are two gain matrices weighting the proportional and derivative actions respectively.

Corollary 1. *The state $(\bar{q}, 0) \in \mathbb{R}^{2n}$ is an asymptotically stable equilibrium of the closed loop* (2)-(23) *if $D(q) \succ -\beta$, and an open neighbourhood $\mathcal{N}(\bar{q}) \subseteq \mathbb{R}^n$ of $\bar{q}$ exists such that $\forall q \in \mathcal{N}(\bar{q})/\{\bar{q}\}$,*

$$(U_{\mathrm{G}}(q)+U_{\mathrm{K}}(q))+\frac{1}{2}(q-\bar{q})^{\top}\alpha(q-\bar{q})>(U_{\mathrm{G}}(\bar{q})+U_{\mathrm{K}}(\bar{q}))+\left(\frac{\partial}{\partial q}\left(U_{\mathrm{G}}(q)+U_{\mathrm{K}}(q)\right)\right)\bigg|_{q=\bar{q}}^{\top}(q-\bar{q}), \tag{24}$$

and

$$G(q)+K(q)+\alpha(\bar{q}-q)\neq G(\bar{q})+K(\bar{q}). \tag{25}$$

These two conditions also imply that $\mathcal{N}(\bar{q})$ is fully included within the region of asymptotic stability of $\bar{q}$.

Proof. The closed loop dynamics is $M(q)\ddot{q}+C(q,\dot{q})\dot{q} = (K(\bar{q}) - K(q))+(G(\bar{q})-G(q))+\alpha(\bar{q}-q)-(D(q)+\beta)\dot{q}$. The previously discussed proof generalizes to this case by adding $(\bar{q}-q)^\top\alpha(\bar{q}-q)/2$ to (19). The time derivative of this new Lyapunov candidate is $\dot{V} = -\dot{q}^\top (D(q) + \beta)\, \dot{q}$, which is semi-positive definite if $D(q) + \beta \succ 0$. So any $\beta \succeq 0$ implements a damping injection that does not destabilize the closed loop. The rest of the proof follows as in the feedforward case. □

The sufficient condition for local asymptotic stability is

$$\left. \left(\frac{\partial K(q)}{\partial q} + \frac{\partial G(q)}{\partial q} \right) \right|_{q=\bar{q}} + \alpha \succ 0, \tag{26}$$

which becomes necessary when only semi-positiveness is required. Note that (26) can always be fulfilled through a large enough proportional gain α. Yet, large gains may result in a stiffening of the soft robot [105], and in amplification of noise or excitation of neglected dynamics. Possibly nonlinear integral actions can also be added to (23) for compensating steady state errors and achieve global stabilization, as discussed in [102].

Trajectory tracking

In trajectory tracking the desired behavior is specified as an evolution of the full robot shape in time. Consider a twice differentiable function of time $\bar{q} : \mathbb{R} \to \mathbb{R}^n$, then the control goal is to find a control strategy τ such that

$$\lim_{t\to\infty} (q(t), \dot{q}(t)) - (\bar{q}(t), \dot{\bar{q}}(t)) = 0. \tag{27}$$

Usually the reference is considered bounded in norm $||(\bar{q}(t), \dot{\bar{q}}(t))|| < c_\mathrm{t}$, for some positive c_t. In theory, under the fully actuated approximation $n = m$, (2) can be completely feedback linearized with a computed torque scheme. However, such a strategy would be hardly applicable on a real system, as discussed in sidebar xxx. This section will focus on controllers achieving the trajectory tracking goal by relying minimally on direct model cancellations. For the sake of space, proof of convergence will not be provided. All of them can be obtained by adapting proofs from the nonlinear PD literature so to work for a system as (16), similarly as what it has been shown in Theorem 1.

If the reference trajectory is slow varying (i.e. $||\dot{\bar{q}}||$ small enough) then (15) and (23) can still be applied as they are, possibly with the inclusion of damping feedforward compensation terms - i.e., $D(\bar{q})\dot{\bar{q}}$ and $(D(\bar{q}) + \beta)\dot{\bar{q}}$ respectively. The state will not converge to $(\bar{q}, \dot{\bar{q}})$ at steady state, but to a neighborhood of it [106], [107]. Higher the gains and slower the reference, smaller is the neighborhood.

Dealing with undearctuation in shape control

Fully actuated approximations have proven to be effective in the practice, despite being a clear over-simplification of the control problem. By bringing under-actuation into the picture, the degrees of freedom not directly affected by the control action can be analyzed an potentially used in the design of the controller, towards solutions with improved performance and certifiable reliability. Thus, consider a non-square actuation matrix $A(q)$ with $m < n$. The first difficulty that arises is that the desired shape $\bar{q}$ may not be an attainable equilibrium of the system, i.e. $K(\bar{q}) + G(\bar{q}) \notin \mathrm{Span}(A(\bar{q}))$. In other terms, for a generic shape it will not exists a control action which makes it an equilibrium configuration. Similarly, in general it will not necessarily exist a control input evolution $\tau(t)$ such as a generic state $(\bar{q}, \dot{\bar{q}})$ can be reached from any initial condition. Authors of [121] discuss how different actuation patterns may affect the accessible set [122] of a soft robot.

Let us assume that the equilibrium $\bar{q}$ is attainable with the given under-actuation matrix $A(q)$. Under this assumption, then (15) can be generalized in

$$\tau = A^{\mathrm{L}}(\bar{q})(K(\bar{q}) + G(\bar{q})), \tag{29}$$

with A^{L} left-inverse of A, as for example the Moore-Penrose pseudoinverse $\left(A^\top A\right)^{-1} A^\top$. If A is configuration independent, this leads to the same closed loop equation (16). Thus the physical impedance acts as a stabilizing action not only on the collocated part, but also on the variables which are not directly reached by the actuation. If A is configuration dependent then its local changes may have destabilizing effects that must be considered in a modified Eq. (22), as discussed in the appendix of [123]. When dealing with slowly varying trajectories, similar considerations can be applied to the trajectory tracking problem. However, extending the results involving feedback actions - as for example (28) - is a substantially more complex challenge that is still to be addressed. Relying on linearized models can be a practically effective alternative, either when linearizing around the equilibrium [124] or around the desired trajectory [125].

Control design and analysis get substantially more complex when it comes to stabilizing unstable equilibria of underactuated models. In this case, (22) is not verified, and feedback actions must be necessarily involved. A discussion and experimental validation on combining local linear control, an accurate FEM model, and a Luenberger Observer, for designing a damping injection loop is provided in [126], [127]. A FEM-Based Gain-Scheduling Controller is used in [128] to cover the state space of the robot with linear set-point regulators including integral actions. Moving a step towards the nonlinear domain, the simple controller (23) can be extended to

$$\tau(\bar{q}, q, \dot{q}) = A^{\mathrm{L}}(K(\bar{q}) + G(\bar{q})) + \alpha A^\top(\bar{q} - q) - \beta A^\top \dot{q}, \tag{30}$$

which is a generalization of (23) to the underactuated domain. Note that the two gains α, β are still elements of $\mathbb{R}^{m\times m}$, and thus they weight the involvement of the actuators into the control loop.

Corollary 2. *The thesis of Corollary* (1) *is verified for the closed loop* (2)-(30)*, with constant A, if the same set of hypotheses obtained is verified when formally switching α and β with $A\alpha A^\top$ and $A\beta A^\top$ respectively, and if*

$$(I - AA^{\mathrm{L}})(K(\bar{q}) + G(\bar{q})) = 0. \tag{31}$$

Proof. Under hypothesis (31), the following holds $AA^{\mathrm{L}}(K(\bar{q})+G(\bar{q})) = K(\bar{q})+G(\bar{q})$. The closed loop dynamics is thus structurally equivalent to the one in Corollary 1, i.e. $M(q)\ddot{q} + C(q,\dot{q})\dot{q} = (K(\bar{q}) - K(q)) + (G(\bar{q}) - G(q)) + A\alpha A^\top(\bar{q} - q) - (D(q) + A\beta A^\top)\dot{q}$. Thus, the rest of the proof follows as in the fully actuated case. □

The sufficient convergence condition becomes

$$\left.\left(\frac{\partial K(q)}{\partial q} + \frac{\partial G(q)}{\partial q} + A\alpha A^\top\right)\right|_{q=\bar{q}} \succ 0, \tag{32}$$

where if $\alpha \succ 0$ then $A\alpha A^\top \succeq 0$ but $\mathrm{Rank}\left(A\alpha A^\top\right) \leq m < n$. Thus, the equilibrium $\bar{q}$ can be stabilized using (30) only if the actuation is collocated on the directions in which the effective stiffness loses rank. Other recent works deal with the regulation of equilibria under similar collocated conditions. In [129] energy shaping controller is proposed for set-point posture regulation one planar segment modeled as a sequence of rigid links, with the same torque applied to all links. Moving to more general systems, [71] tests in simulation the use of computed torque plus zero-dynamics damping injection in a geometrical exact discrete Cosserat model. This technique was already used for controlling a eel-like hyper-redundant robot in [130]. No proof of convergence is provided, but simulations show good performance.

If also (32) cannot be verified, then the problem must be analyzed using less local strategies. For example, if the left hand of (32) is only semi-positive definite, then the extended version of the Lyapunov function (19) may still be positive definite. If however this term is not definite in sign for all α, then there are directions on which the actuators is not acting directly, and for which the potential field $K(q) + G(q)$ is repulsive. In this case, stabilization must occur by relying on dynamic couplings. This is largely an unexplored ground in soft robotics. A very first step in this direction is discussed in [63], where a soft inverted pendulum is introduced as an a soft extension of the acrobot [131]. The stabilization of an unstable equilibrium is discussed analytically, and it is shown that there is a range of low stiffnesses for which the robot can be stabilized only by means of non-collocated feedback.

Actuators dynamics and constraints

Actuators dynamics plays a much important role in shaping the soft robot behavior, especially if compared to classic rigid robots. Nonetheless, few are the works so far that have explicitly taken into account a dynamics formulation as (13) in the design of the controller. Some actuation technologies require already to accurately consider the control problem for a single isolated actuator. This is the case of electro-thermally-active materials [132]–[135], and of magnetic actuation of micro and nano robots [136], [137]. If a clear separation exists between the response times of actuators (13) and the robot (12), then singular perturbation approach [138] could be used to improve the performance of the model based controllers introduced above. Alternatively, backstepping design achieves the same goal without any assumption on the relative time scales [139], but at the cost of a more complex control architecture. Both techniques have been extensively used to control flexible robots actuated with similar modalities as typically found in soft robotics, as tendon driven [140], pistons [141], and artificial muscles [142], [143]. Nonetheless, the only example of application in soft robotics that we are aware of is a backstepping controller for a single segment approximated with a linear model of the robot and of the air flow [144].

In soft robotic actuation, it is often the case that the input space can only take values in a subset of $\mathbb{R}^m$. This may be due to upper bounds to the maximum force, and to unilateral constraints induced by tendons that can only pull, or pressure chambers that can only push. These constraints are usually dealt with heuristics which mask their existence to controllers carefully tuned to not exceed the limits of actuation. As an alternative to heuristics, the masking can also be devised through model based techniques as closed form solution of optimal control allocation problems [145]. Alternatively, Model Predictive Controllers (MPC) can generate control actions that inherently verify the constraints. In [146] linear MPC is used to control a pneumatically actuated humanoid robot, with joint-like localized bending and under a decentralized approximation. In [147] the strategy is extended to nonlinear MPC, and Evolutionary algorithms are used to solve the nonlinear optimization. In [148] nonlinear order reduction techniques are used to generate accurate relaxations of a nonlinear finite horizon optimal control problem, including state and input constraints, and formulated on nonlinear FEM models.

Task space regulation and tracking

The task space of a robot is usually identified with the configuration of its end effector. In soft robots this corresponds to the configuration of the tip $x(1,t) = h(1,q(t))$. For simplicity of notation we will drop the s coordinate in this section. This also allows to stress that the results that we will discuss below are general for any s and even for any smooth function h of the configuration q. Examples are the potential energy, or the distance of the soft robot from an obstacle. Thus, we say that a task is fulfilled if

$$\lim_{t\to\infty} h(q(t)) - \bar{x}(t) = 0, \tag{33}$$

where the desired task coordinates $\bar{x}$ can be either a constant value (regulation) or a function of time (tracking).

A substantial body of literature [32], [149]–[154] deals with the problem under the kinematic approximation. For a fully actuated model, this means assuming that the robot evolution is described by (1), with $\dot{q}$ being the control input. This is a well known problem in robotics [155]–[157], which can be solved with the control loop

$$\dot{q} = J^{+}(q)\left(K_{\mathrm{e}}\left(\bar{x} - h(q)\right) + \dot{\bar{x}}\right). \tag{34}$$

Indeed, combining (1) and (34) yields the closed loop dynamics $\mathrm{d}(x-\bar{x})/\mathrm{d}t = K_{\mathrm{e}}(x-\bar{x})$ that fulfills (33) exponentially fast for all $K_{\mathrm{e}} \succ 0$. Note that for $\dot{\bar{x}} = 0$, the time discretization [158] of (34) is equivalent of apply gradient descent to solving the following quadratic programming problem

$$\min_{q\in\mathbb{R}^n} \quad ||h(q) - \bar{x}||_2^2. \tag{35}$$

Soft and hard constrains can be explicitly included in (35), and possibly reflected in (34) using multi-task prioritization. In the practice, (34) is integrated numerically and the result serves as reference $\bar{q}$ for a low level controller which regulates q. This can happen completely in feedforward or as an high level feedback loop. In the latter case q and $h(q)$ are directly measured. Alternatively, the kinematic behavior can be forced on the system by means of model-based cancellations [159]. Therefore, the use of a kinematic controller implicitly lies on the assumption that all configurations q are attainable through low level controller as the ones discussed in previous sections is available.

In order to extend (34) to the underactuated case, one has to introduce some extra assumptions. First, it must be assumed that a low level feedback loop $\tau(\bar{\eta}, \eta, \dot{\eta}, q, \dot{q})$ is available such that if applied to (13) then η converges to $\bar{\eta}$ in a short time. Under this assumption η and $\bar{\eta}$ can be used interchangeably. This is a strong assumption in general. However, if the robot dynamics is negligible compared to the actuators one - e.g. lightweight robot with strongly reduced actuation - standard actuator-side regulation $\tau(\bar{\eta}, \eta, \dot{\eta})$ is sufficient. This is for example the case of lightweight continuum medical devices [160], [161]. Second, it has to be assumed that the robot is drawn to a stable equilibrium $\bar{q}$, whenever a constant η is imposed. This is equivalent to say that the feedforward action (29) generates a stable equilibrium. See the previous subsection for more discussions on the topic. Third, a one-to-one map must exist from $\bar{\eta}$ to $\bar{q}$, which we refer to as $q(\eta)$. We call $J_\eta(\eta)$ the Jacobian of this map.

If these three hypotheses are simultaneously verified, then the following differential kinematic description can be constructed that goes directly from actuators space η to task space x

$$\dot{x} = \overbrace{J(q(\eta))\,\underbrace{J_\eta(\eta)\dot{\eta}}_{\dot{q}}}^{\text{Ent-to-end Jacobian}}. \tag{36}$$

This is formally equivalent to (1) from a mathematical standpoint. Thus, a kinematic controller can be constructed by following the same line of reasoning of (34), resulting in the control action

$$\dot{\eta} = (J(q(\eta))J_\eta(\eta))^{+}\left(K_{\mathrm{e}}\left(\bar{x} - h(q(\eta))\right) + \dot{\bar{x}}\right). \tag{37}$$

This formulation is quite powerful since $J(\eta)J_\eta(q(\eta))$ is in general full rank rows even if $J_\eta(q(\eta))$ is a strongly higher rectangular matrix (strong underactuaiton of the state), as soon as the dimension of x is smaller or equal than m. This is for example the case of a long soft tentacle being actuated with three tendons, and controlled to reach a goal location with the tip. Finally, it is worth underlying that similar steps can be followed by bypassing the actuators models, and directly reasoning on (2). This can be achieved by focusing on τ rather than on η. In this case J_{a} can be derived from (11). A similar loop as (37) can thus be used to evaluate the control action in (29).

Several variations on the kinematic inversion strategies has been proposed in the literature. The Cosserat kinematic model is combined with linearized task space control in [162], and with sliding mode control in [163], [164]. Visual servoing based kinematic PCC model, where the camera looks the robot, is used to devise the closed loop [165]. The inverse kinematics problem is tackled for parallel soft robots by relying on rigid link discretization in [48], on FEM models in [166], and on Cosserat parallel kinematics in [25], [167].

As an alternative to the many assumptions required by the kinematic approximation, task control of under-actuated dynamic models can be directly embedded in the dynamic controller by relying on the operational space formulation [123], [168]. As for classic rigid robots, this can be done by differentiating one more time (34), and combining the result with (16). Algebraic manipulations yield the operational or task space dynamics

$$\underbrace{\Lambda(q)\,\ddot{x} + \eta(q,\dot{q}) + J_M^{+\top}(q)\,(G(q)}_{\text{Terms commonly found in rigid robots}} + K(q) + D(q)\dot{q}) = J_M^{+\top}(q)A(q)\tau, \tag{38}$$

where the inertia matrix in the task space is $\Lambda = (JM^{-1}J^{\top})^{-1} \in \mathbb{R}^{m \times m}$, Coriolis and centrifugal

Sidebar: Dynamics of a Constant Curvature Segment

The goal of this sidebar is to help a novice in soft robotics to familiarize with the topic by concisely presenting the derivation of the main ingredients of what is arguably the simplest soft robot: a constant curvature (CC) segment. Regardless its simplicity, this case already allows to build many intuitions that can directly generalized to more complex and general cases. Consider a single planar segment as in Fig. 8, which is an arc with fixed length L but curvature possibly varying in time. The scalar curvature $q \in \mathbb{R}$ is sufficient to describe its full configuration. Note that since the curvature is defined here w.r.t. the normalized arc length, then q is the angle subtended by the arc - also called bending angle. The two concepts have been used interchangeably in this paper. As a comparison, Fig. 8 reports also the non-continuum element of which a CC segment can be consider the direct extension of: a revolute joint connecting two rigid links of length $L/2$. This can be seen as a rigid-link lumped approximation of the CC segment.

The shape $x(s,t)$ of the soft robot can be expressed by collecting the position and orientation of all the reference frames S_s connected to the coordinate $s \in [0,1]$. These quantities can be retrieved via simple geometrical arguments, as visually illustrated by Fig. 8. The result is

$$x(s,t) = h(s, q(t)) = L \begin{bmatrix} \frac{\sin s\, q(t)}{q(t)} & \frac{1-\cos s\, q(t)}{q(t)} & \frac{s}{L} q(t) \end{bmatrix}^{\top}. \tag{44}$$

Thus, q can be defined also as the angle between base frame and tip frame. Note that $x(s,t)$ has no singularity point, since its limit in the straight configuration ($q = 0$) is well defined and equal to $[L\ 0\ 0]^{\top}$. However, the division by 0 can generate numerical instabilities in the practice. Fig. (9) compares how the shape of a CC segment changes compared to the one of its lumped discrete approximation. The two models gets progressively more different with the increase of $|q|$, one reason being that the length arc to which both links of the rigid model are tangent shrinks of a factor $(q/2)\cot{(q/2)}$.

According to (1), the Jacobian matrix mapping the time derivative of the curvature $\dot{q}(t) \in \mathbb{R}$ to $\dot{x}(s,t) \in \mathbb{R}^3$ is

$$J(s,q) = L \begin{bmatrix} \frac{sq\cos(sq) - \sin(sq)}{q^2} & \frac{(\cos(sq)-1) + sq\sin(sq)}{q^2} & \frac{s}{L} \end{bmatrix}^{\top}. \tag{45}$$

This kinematic description is sufficient to express the inertia according to (3). If an uniform distribution of mass ($m(s) = m$) and a very thin rod ($\mathcal{J} \simeq 0$) are assumed, then the configuration dependent inertia is

$$M(q) = \frac{mL^2}{20} \underbrace{\left(\frac{20}{3} \frac{q^3 + 6q - 12\sin(q) + 6q\cos(q)}{q^5} \right)}_{\lim_{q\to 0} * = 1} > 0. \tag{46}$$

Note that similar closed form solutions for M can be found for different mass distributions and non null inertia. These assumptions are introduced here only for the sake of conciseness. Fig. 10 shows a plot of $M(q)$ for all the curvatures in $[-2\pi, 2\pi]$. The inertia decreases with the increase of $|q|$ following a bell curve that goes to 0 when $|q| \to \infty$. This is because changes in q are reflected in progressively smaller changes in the shape of the soft robot when the curvature is larger - i.e., $||x(s, q + \delta_q) - x(s, q)||_2^2$ decreases with the increase of $|q|$ for all fixed $\delta_q > 0$. It is also worth noticing that the inertia of the lumped model with homogeneous distribution of mass is $(m/2)(L/2)^2/3 = mL^2/24$, which is smaller than $M(0)$, despite the two system being perfectly superimposed in the straight configuration. This can be explained by considering that the rigid model neglects the motion of the lower half of the robot, and so an actuation torque sees only the inertia produced by half of the robot's body.

Since M is not constant, this formulation of the CC segment dynamics is affected by the following centrifugal force

$$\begin{aligned} C(q,\dot{q})\dot{q} &= \frac{1}{2}\frac{\mathrm{d}M}{\mathrm{d}t}\dot{q} \\ &= -\frac{mL^2}{3}\frac{12\,q - 30\,\sin(q) + 3\,q^2\,\sin(q) + 18\,q\,\cos(q) + q^3}{q^6}\dot{q}^2. \end{aligned} \tag{47}$$

Note that we could evaluate C by direct differentiation of M since both are scalar. Fig. 10 reports the evolution of this force when q changes. As expected from a centrifugal action $-C(q,\dot{q})\dot{q}$ tends to increase $|q|$ for all $\dot{q} \neq 0$.

Consider the base of the robot being oriented with a generic angle ϕ w.r.t. a gravity acceleration of intensity g. The gravity potential can be calculated by summing up the contributions of each infinitesimal element

$$U_{\mathrm{G}}(q,\phi) = \int_0^1 \underbrace{m\,g\,\left(x(s,0) - x(s,q)\right)^\top \begin{bmatrix}\cos(\phi)\\ \sin(\phi)\\ 0\end{bmatrix}}_{\text{Infinitesimal contribution of element } s} \mathrm{d}s, \tag{48}$$

which is the variation of the center of mass location with respect to the straight configuration, projected to the direction of the gravity acceleration, and multiplied for mg. According to (5), direct differentiation of the associated potential yields the gravitational torque

$$
\begin{aligned}
G(q,\phi) &= -\, m\, g \left(\int_0^1 J(s,q)\,\mathrm{d}s \right)^{\top} \begin{bmatrix} \cos(\phi) \\ \sin(\phi) \\ 0 \end{bmatrix} \\
&= -\, m\, g\, L \left(2\, \frac{\cos(q-\phi) - \cos(\phi)}{q^3} + \frac{\sin(q-\phi) - \sin(\phi)}{q^2} \right).
\end{aligned}
\tag{49}
$$

Fig. 10 depicts the case of $\phi = 0$, corresponding to a gravity field aligned with the straight configuration of the robot (pointing downward in Fig. 9). Two relevant symmetries that may help thinking about how G changes with ϕ are $G(q,\phi) = -G(q,\phi+\pi)$ and $G(q,\phi) = -G(-q,-\phi)$. The flexural rigidity can be modeled as a torque proportional to the local bending of the robot, which is the curvature q. Thus, the elastic force is

$$
K(q) = \frac{\partial}{\partial q} \overbrace{\int_0^1 \underbrace{\frac{1}{2}\, k(s)\, q^2}_{\text{Infinitesimal contribution}} \, \mathrm{d}s}^{U_{\mathrm{K}}(q)} = \underbrace{\left(\int_0^1 k(s)\,\mathrm{d}s \right)}_{\text{Average stiffness}} q,
\tag{50}
$$

where $k(s) \in \mathbb{R}$ is the local stiffness in s, which is assumed to be almost constant in order for the CC assumption to hold. Similarly, the damping torque can be evaluated by assuming local dissipation proportional to the variation of curvature. The torque needs then to be mapped in q leveraging the kinetostatic duality

$$
D(q)\dot{q} = \int_0^1 \underbrace{J(s,q)^{\top} \begin{bmatrix} 0 \\ 0 \\ d(s)\,\dot{q} \end{bmatrix}}_{\text{Infinitesimal contribution}} \mathrm{d}s = \underbrace{\left(\int_0^1 s\, d(s)\,\mathrm{d}s \right)}_{\text{Equivalent damping}} q.
\tag{51}
$$

where $d(s) \in \mathbb{R}$ is the local damping in s. Thus, both elastic and damping forces are linear under the discussed assumptions. Equivalent results are obtained also when infinitesimal springs and dampers proportional to the elongation are assumed distributed along the thickness of the robot [43]. Finally, consider the robot to be actuated with a pure torque applied at the tip, resulting in

$$
A(q)\tau = J(1,q)^{\top} \begin{bmatrix} 0 \\ 0 \\ \tau \end{bmatrix} = \tau.
\tag{52}
$$

Eqs. (46)-(52) can be combined by using (2), yielding a scalar second order dynamics for q which has the same structure and structural properties of a lumped joint model with parallel impedance, but with different and more complex expressions. Examples of the resulting evolutions are shown in Fig. 11

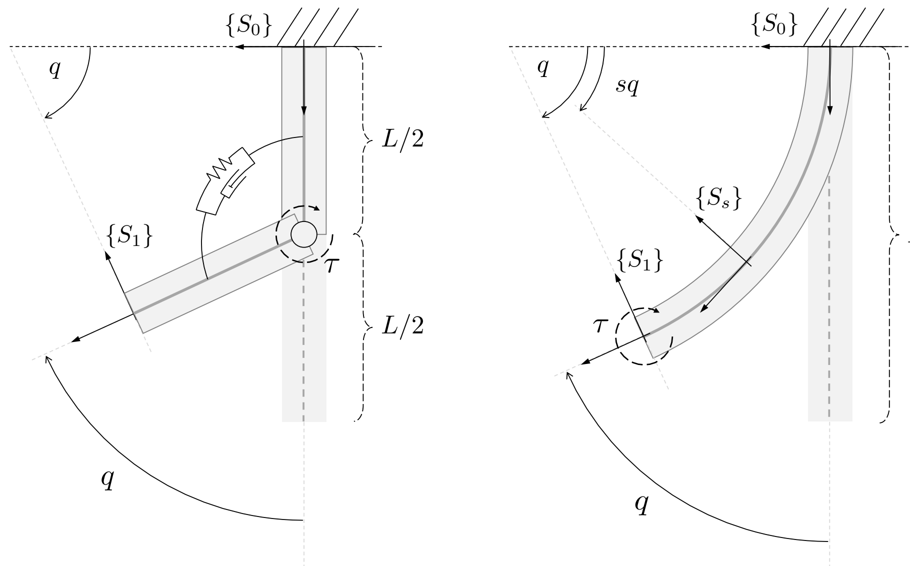


Figure 8: A constant curvature segment together with and lumped rigid-link model serving as its first order approximation. The two resulting dynamics have equivalent structural properties, but are described by substantially different dynamic equations.

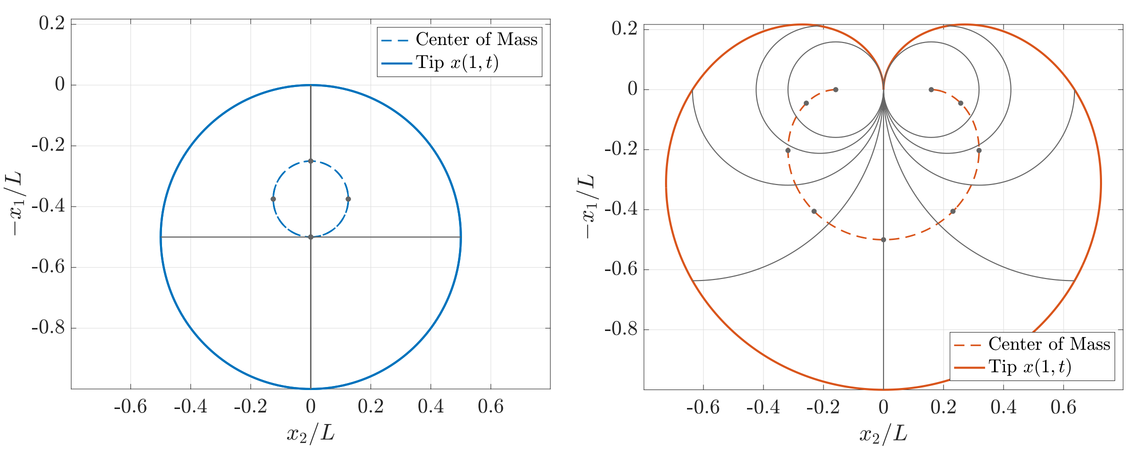


Figure 9: Geometrical characterization of a rigid robot with a single revolute joint (left) and of a constant curvature robot (right). The the behaviors are similar close to the straight configuration, but strongly depart from each other when the angle $|q|$ increases. The configurations corresponding to $q \in \{-2\pi, -3\pi/2, \ldots, 3\pi/2, 2\pi\}$ are shown with thin gray lines. The corresponding centers of mass are also reported as a gray dot. Note that this range of angles corresponds to two full rotations for the rigid links case.

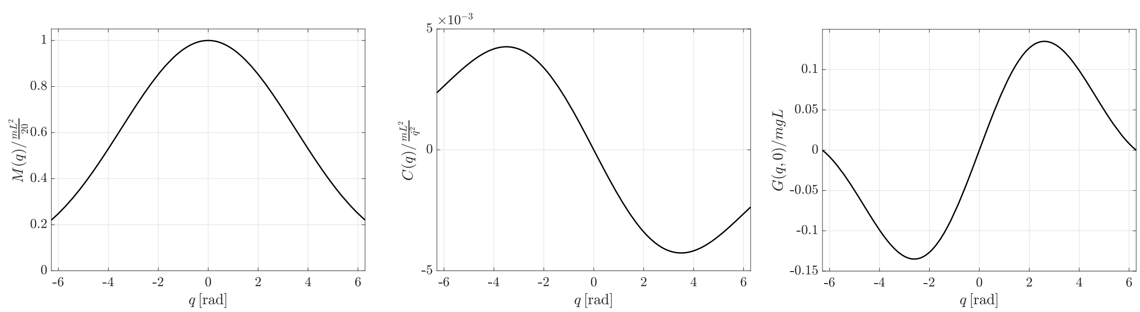


Figure 10: Evolutions of (46), (47), and (49) all normalized w.r.t. the quantities that appear linearly in their expression. A change in those quantities result in a linear scaling of the plots along the vertical axis.

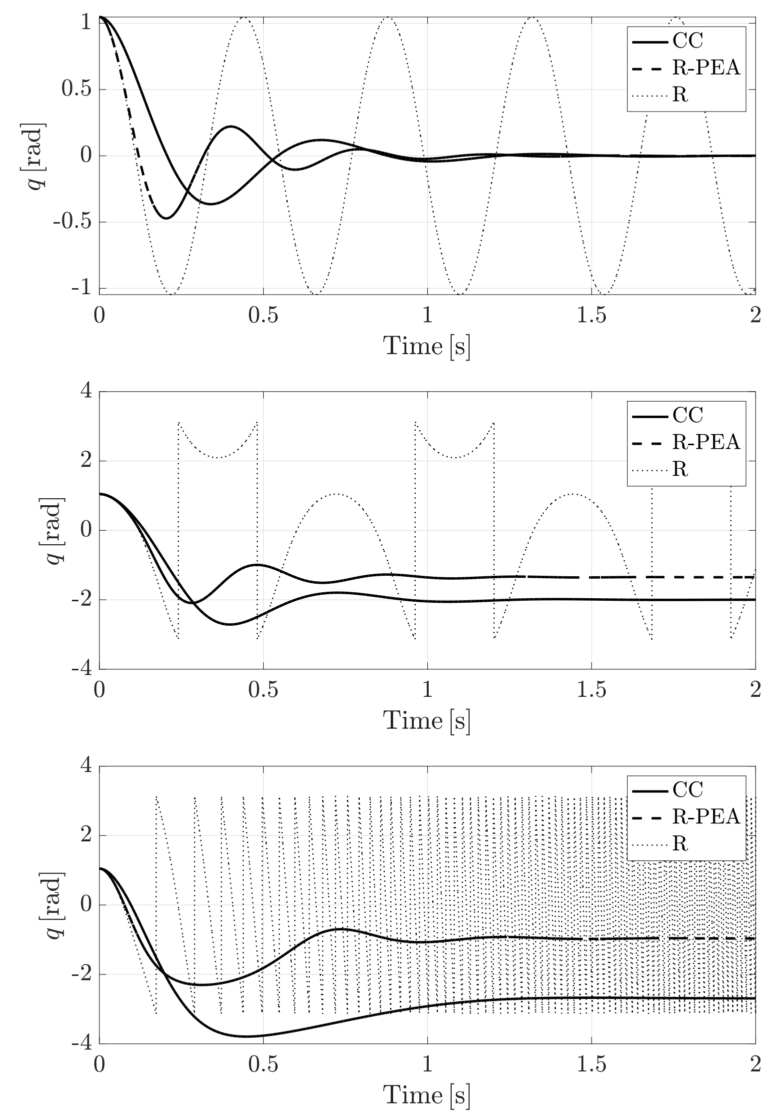
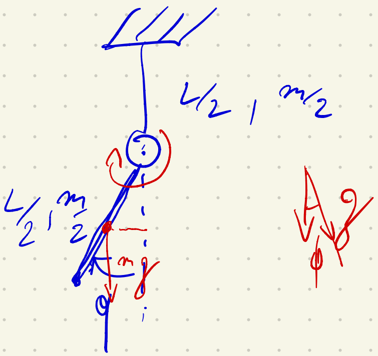


Figure 11: Examples of evolution of a constant curvature segment (CC), its lumped rigid link approximation with (R-PEA) and without (R) parallel springs. The CC dynamics is described by (46)-(52). The parameters considered here are $m = 0.5\mathrm{Kg}$, $L = 0.25\mathrm{m}$, $\int_0^1 k = 0.05\mathrm{Nm}$, $\int_0^1 sd = 0.01\mathrm{Nms}$, and $(q(0), \dot{q}(0)) = (\pi/3, 0)$. From top to bottom, the three plots show the evolutions for (τ, ϕ) equal to $(0\mathrm{Nm}, 0)$, $(-0.3\mathrm{Nm}, 0)$, and $(0\mathrm{Nm}, -pi/2)$ respectively. In all the three cases, CC and R-PEA are qualitatively similar, and both different from R.

CSR $\neq$ RR ?

SIMPLEST RIGID ROBOT THAT CAN BEND?

$\hookrightarrow$ CHANGE ITS ORIENTATION



STANDARD PENDULUM

$$\underbrace{\frac{1}{3}\,\frac{m}{2}\left(\frac{L}{2}\right)^2}_{M}\ddot{q} + 0 + \underbrace{m g \frac{L}{4} \sin(q - \phi)}_{G(q)} = \tau$$

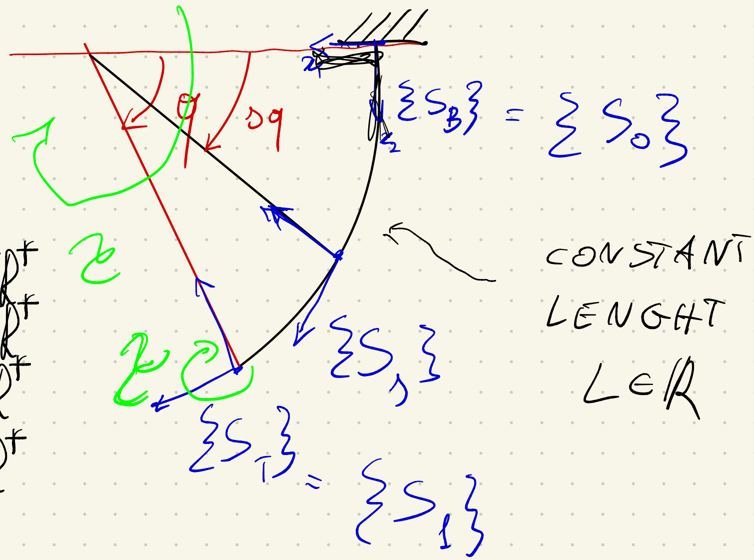
CURVATURE

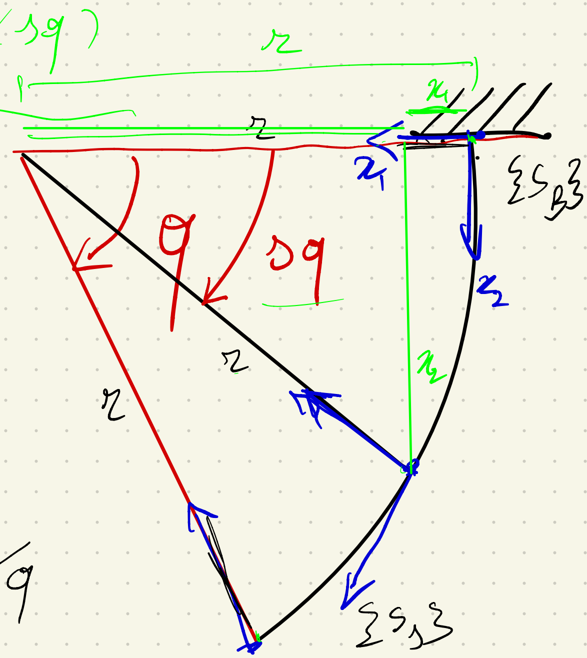
$m(s) : [0,1] \to \mathbb{R}^+$

$K(s) : [0,1] \to \mathbb{R}^+$

$\alpha(s) : [0,1] \to \mathbb{R}^+$

$j(s) : [0,1] \to \mathbb{R}^+$





$L = r q$

$\Rightarrow r = L/q$

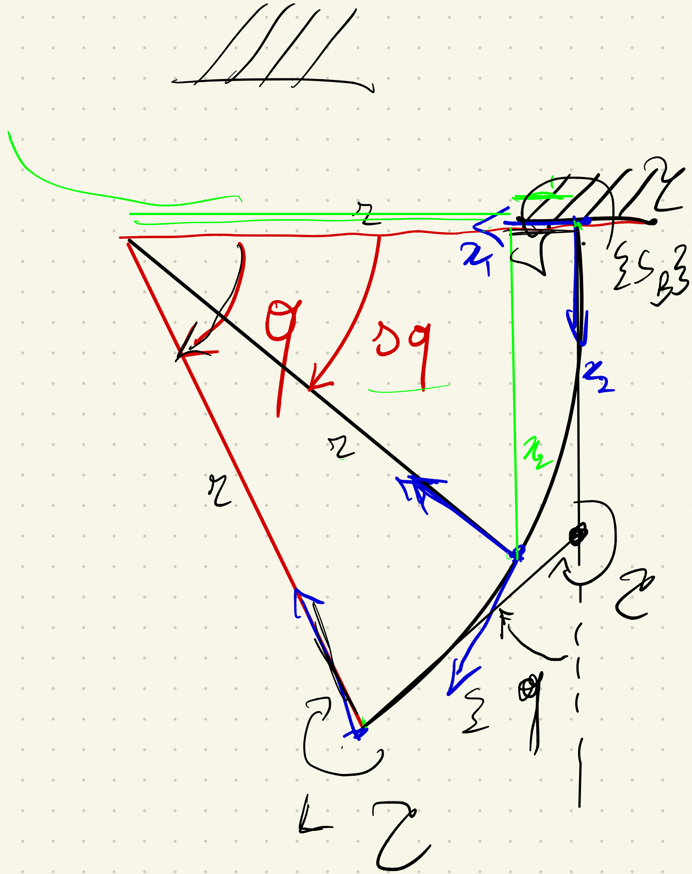
STEP 1

FORWARD KINE.

$\{S_1\} \to \{S_B\}$

$$x_2 = r \cdot \sin(\delta q) = L \frac{\sin(\delta q)}{q} \quad \left(= L\delta \ \mathrm{sinc}(\delta q)\right)$$

$$x_1 = r - r\cos(\delta q) = L \frac{1 - \cos(\delta q)}{q}$$



(1B)

$$x(s,t) = h(s, q(t))$$

$$= \begin{bmatrix} L \dfrac{1-\cos(sq)}{q} \\ L \dfrac{\sin(sq)}{q} \\ sq \end{bmatrix} \xrightarrow[q \to 0]{} \begin{bmatrix} 0 \\ sL \\ 0 \end{bmatrix}$$

STEP2 DIFFERENTIAL KINEMATICS

$$J(s,q) = \frac{\partial h}{\partial q} = \begin{bmatrix} * \\ * \\ s \end{bmatrix} \in \mathbb{R}^{3 \times 1}$$

$$\dot{q} \in \mathbb{R} \xrightarrow{J \in \mathbb{R}^{3\times 1}} \dot{x} \in \mathbb{R}^3$$

STEP 3 INERTIA MATRIX

$$E_K = \frac{1}{2}\dot{q}^T \underline{M(q)}\,\dot{q} \quad \left(= \frac{1}{2} M(q)\,\dot{q}^2\right)$$

$$E_K = \frac{1}{2}\int_0^1 \dot{x}^T(s,t) \begin{bmatrix} m(s)\,\overset{\mathbb{R}^3}{I} & 0 \\ 0 & j(s) \end{bmatrix} \dot{x}(s,t)\,ds$$

$$= \frac{1}{2}\int_0^1 \dot{q}^T J^T(s,q) \left[\;\cdots\;\right] J(s,q)\,\dot{q}\,ds$$

$$= \frac{1}{2}\dot{q}^T \underbrace{\left[\int_0^1 J^T(s,q) \begin{bmatrix} m(s)\,I & 0 \\ 0 & j(s) \end{bmatrix} J(s,q)\,ds\right]}_{M(q)} \dot{q}$$

$$m(s) = m \quad , \; j(s) \simeq 0$$

$$M(q) = \frac{mL^2}{20}\left(\frac{20}{3}\underbrace{\frac{q^3 + 6q - 12\sin(q) + 6q\cos(q)}{q^5}}_{\xrightarrow[q \to 0]{} 1}\right)$$

STEP 4 CORIOLIS + CENTR.

$$C(q, \dot q) \underset{\underset{1\,DoF}{\uparrow}}{=} \frac{1}{2}\frac{dM}{dt}\dot q = \frac{1}{2}\frac{\partial M}{\partial q}\dot q^2 \in \mathbb{R}$$

STEP 5 POTENTIAL FORCES

$$U_G = \int_0^1 u_G(q, s)\, ds$$

$$= \int_0^1 m g \left(\underline{x(s, 0)} - x(s, q)\right)^T \underline{\begin{bmatrix} C_\phi \\ S_\phi \\ 0 \end{bmatrix}} ds$$

$$U_k = \int_0^L u_k(q,s)\,ds =$$

$$= \int_0^L \frac{1}{2} q^T K(s) q \, ds$$

$$= \frac{1}{2} q^T \left(\int_0^L K(s)\, ds \right) q$$

$$G(q) = \frac{\partial U_G}{\partial q} = -mg \int_0^L J^T(s,q) \begin{bmatrix} c\phi \\ s\phi \\ 0 \end{bmatrix} ds$$

$$K(q) = \frac{\partial U_K}{\partial q} = \left(\int_0^L K(s)\, ds \right) q$$

STEP 5

ACTUATION

$$\underbrace{J^T(q,1)}_{\begin{bmatrix}*\\ *\\ s\end{bmatrix}\Big|_{s=1}}\begin{bmatrix}0\\0\\\tau\end{bmatrix} = \tau$$

DAMPING

$$\int_0^1 J^T(s,q)\begin{bmatrix}0\\0\\d(s)\dot q\end{bmatrix} ds$$

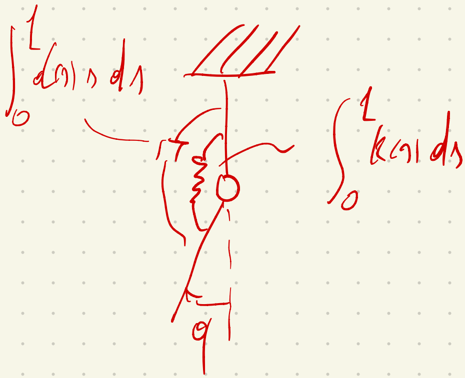
$$= \left(\int_0^1 s\, d(s)\, ds\right)\dot q$$

$$M(q)\ddot{q} + \frac{\partial M}{\partial q}\dot{q}^2 - mg\int_0^1 J^T(s,q)\begin{bmatrix} C\phi \\ S\phi \\ 0 \end{bmatrix} ds + ()q + ()\dot{q} = \tau \quad \text{CC}$$

12 12 12

$$M\ddot{q} + 0 + \frac{m}{2} g \frac{L}{4}\sin(q-\phi) + 0 + 0 = \tau \quad \text{R}$$

$$M\ddot{q} + 0 + \frac{m}{2} g \frac{L}{4}\sin(q-\phi) + ()q + ()\dot{q} = \tau \quad \text{PEA}$$



R - PEA

SHAPE REGULATION

GOAL: GIVEN A DESCRIPTION OF THE FULL SHAPE

$$\bar{q} \in \mathbb{R}^n$$

FIND A CONTROLLER $\tau(q, \dot{q}, \bar{q}, \dots)$ SUCH THAT

$$\lim_{t \to +\infty} q(t) = \bar{q} \Leftarrow AS\ \bar{q}$$

$$\frac{\partial A}{\partial q} = 0$$

SYSTEM $\quad q \in \mathbb{R}^n \qquad m \leq n$

$$\underbrace{M}_{\dot{q}^T(\dot{M} - \frac{1}{2}C)\dot{q} = 0}(q)\ddot{q} + C(q,\dot{q})\dot{q} + \underbrace{G(q)}_{\frac{\partial U_G}{\partial q}} + \underbrace{K(q)}_{\frac{\partial U_K}{\partial q}} + \underbrace{D(q)}_{D \succ 0}\dot{q} = \underbrace{A}_{\mathbb{R}^{n \times m}}\cancel{(q)}\overbrace{\tau}^{\mathbb{R}^m}$$

$$\ddot{\bar{q}} = 0, \quad \dot{\bar{q}} = 0$$

NC

$$\exists\, \bar{\tau} \in \mathbb{R}^m \ \text{s.t.} \ \underbrace{G(\bar{q}) + K(\bar{q})}_{n} = \underbrace{A}_{n \times m}\, \underbrace{\bar{\tau}}_{m}$$

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + \overbrace{G(q) + K(q)}^{(G+K)(q)} + D(q)\dot{q} = A \bar{\tau}$$

$$= \underbrace{G(\bar{q}) + K(\bar{q})}$$

$$\Longleftrightarrow$$

$$\underbrace{M(q)\ddot{q} + C(q,\dot{q})\dot{q}}_{\text{RIGID ROBOT}} = \underbrace{(G+K)(\bar{q}) - (G+K)(q)}_{\text{NONLINEAR P}} + \underbrace{D(q)(-\dot{q})}_{\text{NONLINEAR D}}$$

$\mathbb{R}$

RR W.O. GRAVITY CONTROLLED WITH A PD

LYAPUNOV ANALYSIS IN A NUTSHELL

$$\dot{x} = f(x) \text{ s.t. } \underline{f(\bar{x}) = 0}, \; x, \bar{x} \in \mathbb{R}^n$$

YOU WANT TO PROVE THAT $\bar{x}$ AS

1. INTRODUCE A LYAPUNOV CANDIDATE

$$V: \mathbb{R}^n \longrightarrow \mathbb{R}$$

POSITE DEFINITE FUNCTION
$$\begin{cases} V(\bar{x}) = 0 \\ V(x) > 0, \; \forall x \in B(\bar{x}) \setminus \{\bar{x}\} \end{cases}$$

2. LOOK AT THE TIME DERIVATIVE

$$\dot{V} = \left\langle \frac{\partial V}{\partial x}, f(x) \right\rangle$$

3A. IF $\dot{V}$ N.D. $(-\dot{V}$ IS P.D.$)$

THEN $\bar{x}$ A.S. WITH RAS $B(\bar{x})$

LASALLE PRINCIPLES

3B. IF $\dot{V}$ S.N.D.

THEN $x \longrightarrow \{x : \dot{V}(x) = 0\} = I$

AND $\bar{x}$ AS IT IS THE ONLY $x \in I$

s.t. $\underline{x(t + \delta t)} \in I$

Lecture 2

$$V(q,\dot q) = \underbrace{\frac{1}{2}\,\dot q^T M(q)\,\dot q}_{} \quad \text{K.E.}$$

$$+ \left[(U_K+U_G)(q) - (U_K+U_G)(\bar q)\right] \quad \text{P.E.} \quad \text{P.E.}$$

$$- \left((K+G)(\bar q)\right)^T (q-\bar q) \qquad > 0$$

STEP 1

$$V(\bar q, 0) = \cancel{\frac{1}{2}\, 0^T M\, 0} + \left[\cancel{(U_K+U_G)(\bar q) - (U_K+U_G)(\bar q)}\right] - (\,)^T \cancel{(\bar q - \bar q)}$$

$$= 0 \;\checkmark$$

$$V(q,\dot q) > 0 \quad \forall (q,\dot q) \in B(\bar q, 0) \setminus \{(\bar q, 0)\}$$

$$M(q) > 0 \;\; \forall q \implies \dot q^T M \dot q > 0 \;\; \forall \dot q \neq 0$$

H2

$$\underline{(U_K+U_G)(q) - (U_K+U_G)(\bar q)} > \underline{(q-\bar q)^T\left[(K+G)(\bar q)\right]}$$

$$\forall q \in B(\bar q, 0) \setminus \{\bar q, 0\}$$

$$(K+G)(q) = P q \longrightarrow P \geq 0$$

$$F = (U_K + U_G)(q) - (U_K + U_G)(\bar{q}) - \left[(K+G)(\bar{q})\right]^T (q - \bar{q}) > 0$$

$$\left.\frac{\partial F}{\partial q}\right|_{q=\bar{q}} = 0 \quad , \quad \left.\frac{\partial^2 F}{\partial q^2}\right|_{q=\bar{q}} \succ 0$$

$$\left.\frac{\partial F}{\partial q}\right|_{q=\bar{q}} = \cancel{K}(q) + G(\cancel{q}) - \cancel{K}(\bar{q}) - G(\cancel{\bar{q}}) = 0$$

$$\left.\frac{\partial^2 F}{\partial q^2}\right|_{q=\bar{q}} = \left(\frac{\partial K}{\partial q} + \frac{\partial G}{\partial q}\right)_{q=\bar{q}} \succ 0$$

STIFFNESS

$\succ 0$

$\not\succ 0$

$\nprec 0$

STEP 2   EVALUATE $\dot{V}$

$$
\begin{aligned}
V(q,\dot q) = \; & \tfrac{1}{2}\,\dot q^T M(q)\,\dot q \quad \leftarrow \text{K.E.} \\
& + \big[(U_K + U_G)(q) - (U_K + U_G)(\bar q)\big] \quad \text{P.E.} \quad \text{P.E.} \\
& - \big((K+G)(\bar q)\big)^T (q - \bar q)
\end{aligned}
$$

$$\frac{d}{dt}() = \dot q^T \frac{\partial}{\partial q}()$$

$$
\begin{aligned}
\dot V = \; & \dot q^T M(q)\,\ddot q + \tfrac{1}{2}\dot q^T \dot M(q)\,\dot q \\
& + \dot q^T \big((G+K)(q) - (G+K)(\bar q) \big)
\end{aligned}
$$

$$
\begin{aligned}
= \; & \dot q^T \big(\cancel{-C(q,\dot q)\,\dot q} + \cancel{(G+K)(\bar q)} - \cancel{(G+K)(q)} - D(q)\,\dot q \big) \\
& + \cancel{\tfrac{1}{2}\dot q^T \dot M(q)\,\dot q} + \dot q^T \big(\cancel{(G+K)(q)} - \cancel{(G+K)(\bar q)} \big)
\end{aligned}
$$

$$= \dot q^T \left(\tfrac{1}{2}\dot M - C \right) \dot q = 0$$

$$= -\,\dot q^T \underbrace{D(q)}_{>0}\,\dot q \le 0$$

$$I = \{ (q,\dot q) \mid \dot q = 0 \}$$

$$(\bar{\bar q}, \underline{0}) \xrightarrow{S_t} (q^{*}, \underline{\dot q^{*}}) \qquad \text{IF } \bar{\bar q} \neq \bar q$$

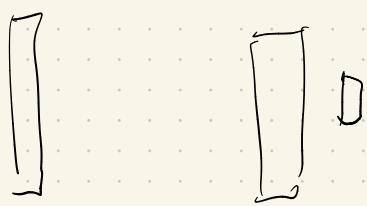
$$\ddot{\bar{\bar q}} = \underbrace{M^{-1}(q)}_{\det \neq 0} \Big(-\cancel{C(\bar{\bar q}, 0)\, 0} + (G+K)(\bar{\bar q}) - (G+K)(\bar q) + \cancel{D(q)\, 0} \Big) \underset{?}{\neq} 0$$

H3

$$\underline{(G+K)(\bar{\bar q}) \neq (G+K)(\bar q)} \qquad \forall\, \bar{\bar q} \in \underline{B(\bar q)} \setminus \{\bar q\}$$

$(G+K)(\bar q) = A\bar z$ <— EQ. CONDITION

$(G+K)(\bar{\bar q}) \neq A\bar z$ $\square$



SELF - STABILIZATION OF CSR

$\bar{q} \in \mathbb{R}^n$ IS AN A.S.E. OF THE CSR UNDER THE CONSTANT ACTION $\bar{\tau}$ IF

- $\underline{A\bar{\tau} = G(\bar{q}) + K(\bar{q})}$ $\longleftarrow$
- $\exists\, B(\bar{q})$ s.t.

$$G(\bar{q}) + K(\bar{q}) \neq G(\bar{\bar{q}}) + K(\bar{\bar{q}}), \ \forall \bar{\bar{q}} \in B() \setminus \{\}$$

- $(U_K + U_G)(q) - (U_K + U_G)(\bar{q})$

$$> [(K+G)(\bar{q})]^T (q - \bar{q}) \quad \forall q \in B() \setminus \{\}$$

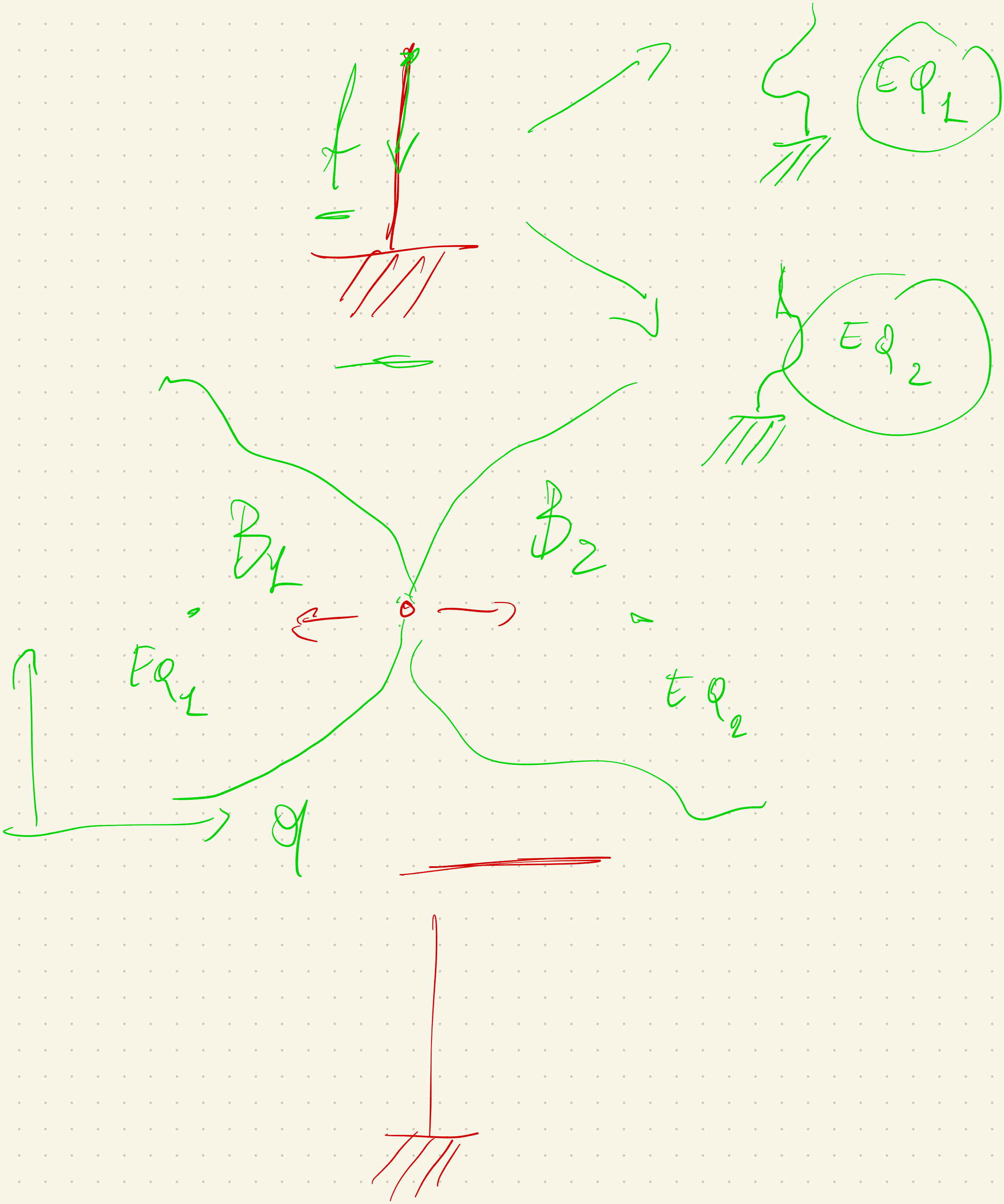
AND $B(\bar{q})$ IS A LOWER APPROXIMATION OF THE R.A.S

SC

$$\underline{\left(\frac{\partial K}{\partial q} + \frac{\partial G}{\partial q}\right)}_{q=\bar{q}} > 0$$

NC

$$\underline{\left(\ \right)_{q=\bar{q}} \geq 0}$$



EQ1
EQ2
B1
B2
EQ1
EQ2

IS THERE ANY ROOM FOR FEEDBACK CONTROL?

- STABILIZE U.E.
- ENLARGE B

$$\tau = \bar{\tau} + \underbrace{\alpha A^T (\overbrace{\bar{q} - q}^{\in \mathbb{R}^n}) + \beta A^T (-\dot{q})}_{\sim PD} \quad \in \mathbb{R}^m$$

$$\alpha, \beta \in \mathbb{R}^{m \times m}$$

$$m\left[\ \ \right] = m\overbrace{\left[\quad A^T \quad\right]}^{n} \left[\ \ \right] \qquad n\begin{pmatrix} \alpha \succ 0 \\ \beta \succeq 0 \end{pmatrix}$$

CLOSED LOOP DYNAMICS

$$M\ddot{q} + C\dot{q} = \overbrace{(G+K)(\bar{q}) - (G+K)(q) + A\alpha A^T(\bar{q} - q)}^{P} + \overbrace{D(q)(-\dot{q}) + A\beta A^T(-\dot{q})}^{D}$$

$$V(q,\dot{q}) = V_{OLD}(q,\dot{q}) + \frac{1}{2}(q-\bar{q})^T A \alpha A^T (q - \bar{q})$$

• V D.P.

$$(U_K + U_G)(q) - (U_K + U_G)(\bar{q}) + \frac{1}{2}(q-\bar{q})^T A\alpha A^T (q-\bar{q})$$

$$> [(K+G)(\bar{q})]^T (q-\bar{q})$$

$$\Uparrow$$

COLLOCATION

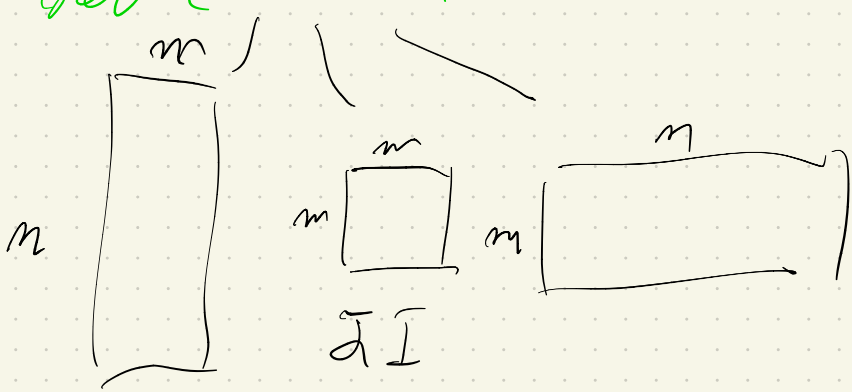
$$\left(\frac{\partial G}{\partial q} + \frac{\partial K}{\partial q}\right)_{q=\bar{q}} + A\alpha A^T > 0$$

• $$\dot{V} = -\dot{q}^T \underbrace{\left(D(q) + A\beta A^T\right)}_{>0} \dot{q} \leq 0$$

• $$G(\bar{q}) + K(\bar{q}) \neq G(\bar{\bar{q}}) + K(\bar{\bar{q}}) + \alpha A^T(\bar{q} - \bar{\bar{q}})$$

$$\mathbb{R}^{n\times m} \quad \mathbb{R}^{m\times m} \quad \mathbb{R}^{m\times n} \qquad n \geq m$$

$$\det(A\alpha A^T) = 0$$



$$\underset{\in \mathbb{R}^\sigma}{\underline{x}} = h(\underset{\in \mathbb{R}^n}{\underline{q}})$$

$\sigma \leq n$

Lecture 3

$$\lim_{t \to \infty} \left\| h(q(t)) - \bar{x}(t) \right\| = 0$$

$$x = h(q) \implies \underline{\dot{x} = J(q)\dot{q}}$$

CONTROL INPUT

$$\underline{\dot{q}} = \underline{J^{+}(q)\left(\dot{\bar{x}} + K_x(\bar{x} - h(q)\right)})$$

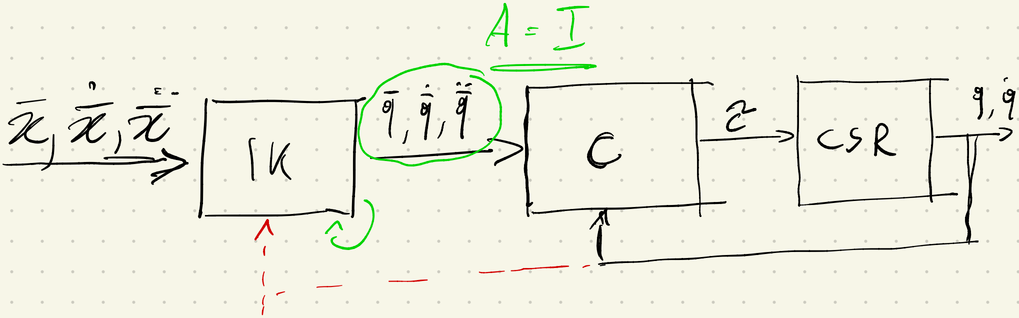
$\mathbb{K}$

$$\dot{x} = \overbrace{J J^{+}}^{I} \left(\dot{\bar{x}} + K_x(\bar{x} - \underbrace{h(q)}_{x})\right)$$

$\iff$

$$(\dot{x} - \dot{\bar{x}}) = -K_x(x - \bar{x}) \implies x - \bar{x} \to 0$$

$$\bar{q} = \int_0^t \dot{q}$$



SOFT ROBOT DYNAMICS IS NEGLIGIBLE W.R.T. ACTUATORS

$$\underline{M(q)\ddot{q} + C(q,\dot{q})\dot{q} + K(q) + G(q) + D(q)\dot{q}} + \underline{\frac{\partial U_c}{\partial q}(q,\eta)} = \underline{0}$$

$$\underline{\eta} \in \mathbb{R}^m \qquad B\underline{\ddot{\eta}} + \underline{\frac{\partial U_c}{\partial \eta}(q,\eta)} = \underline{\tau}$$

$$U_c = \frac{1}{2}(h_c(q) - \eta)^T K_c (h_c(q) - \eta)$$

A1 $\longrightarrow$ A $\tau(\cdots)$ EXISTS SUCH THAT

$$\lim_{t\to\infty} \eta = \bar{\eta}$$ "IN A SHORT TIME"

$$\tau(\bar{\eta}, \eta, \hat{\eta})$$

A2 $\longrightarrow$ IF $\eta \simeq \bar{\eta}$ THEN $q \longrightarrow \bar{q}$!

A3 $\longrightarrow$ $\exists\, q(\eta)$ s.t. $\underline{\bar{q} = q(\bar{\eta})}$

$\eta \in \mathbb{R}^m$ $\quad q \in \mathbb{R}^n$ $\quad n > m$

$$\dot{q} = J_\eta(\eta)\,\hat{\eta} \quad \text{WITH} \quad J_\eta = \frac{\partial q(\eta)}{\partial \eta}$$

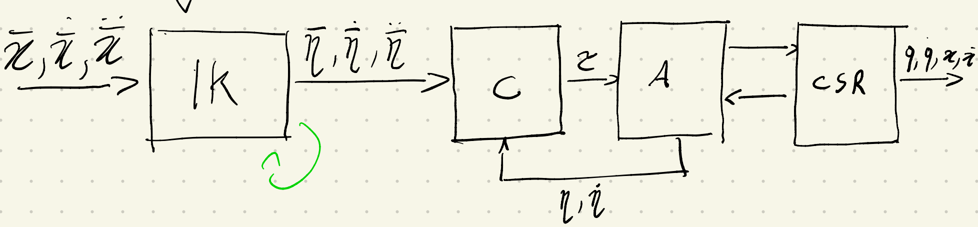
$$x = h(q(\eta))$$

$$\Rightarrow \boxed{\dot{x}} = J(q(\eta))\,\dot{q}$$

$$\boxed{= J(q(\eta))\,J_\eta(\eta)\,\dot{\eta}}$$

$$\dot{\eta} = \left(J(q(\eta))\,J_\eta(\eta)\right)^{+}\left(\dot{\bar{x}} + K_x\left(\bar{x} - \underbrace{h(q(\eta))}_{x}\right)\right)$$

$$\bar{\eta} = \int_0^t \dot{\eta}$$



$J \qquad J_\eta \in \mathbb{R}^{\sigma \times m} \qquad m \geq \sigma$

$\in \mathbb{R}^{n \times m}$

$\in \mathbb{R}^{\sigma \times n}$