Modeling of Soft Robots

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Course on Modeling and Control of Soft Robots
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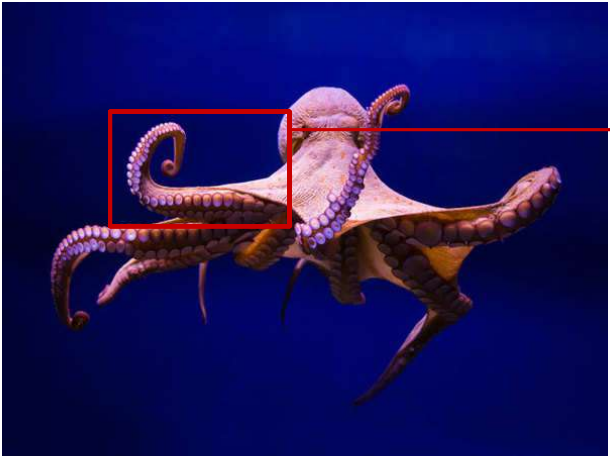
Outline

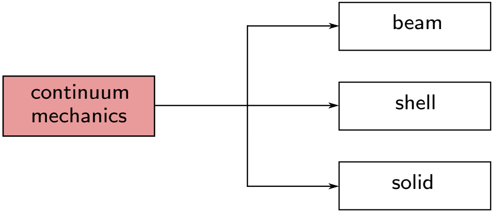
Part A:
Continuum formulation of the dynamics



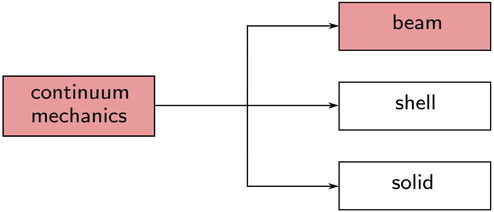
Mathematical description:

Continuum mechanics

Continuum mechanics for continuum soft robots

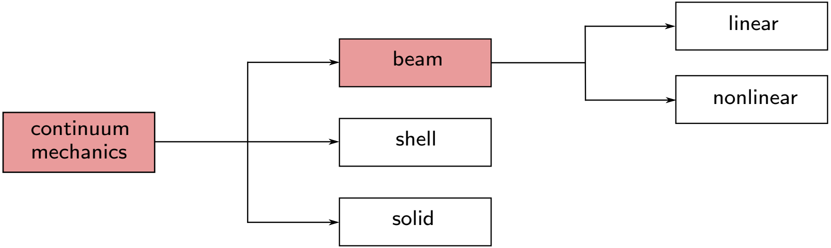


Continuum mechanics for continuum soft robots



soft robotic arm:
one dimension is dominant
over the two others

Continuum mechanics for continuum soft robots



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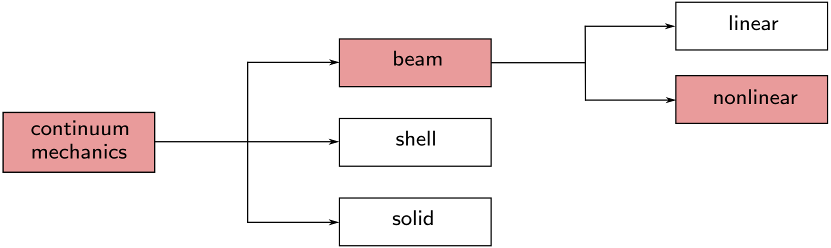
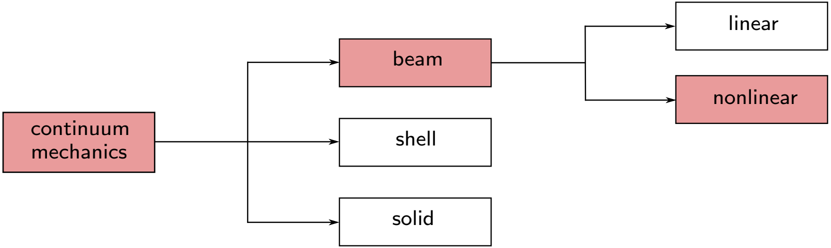


soft robotic arm:
one dimension is dominant over the two others

- large motions
- no approximation on kinematic variables

Continuum mechanics for continuum soft robots



soft robotic arm:
one dimension is dominant
over the two others

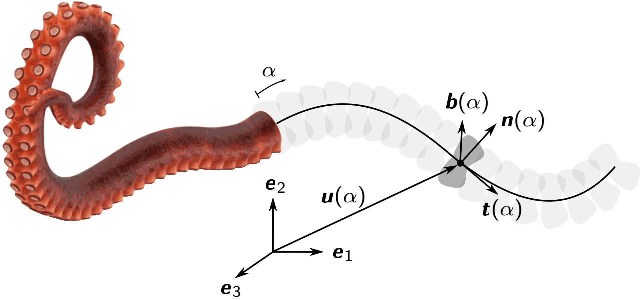
- large motions
- no approximation on kinematic variables

↓

COSSERAT THEORY

Modeling assumption

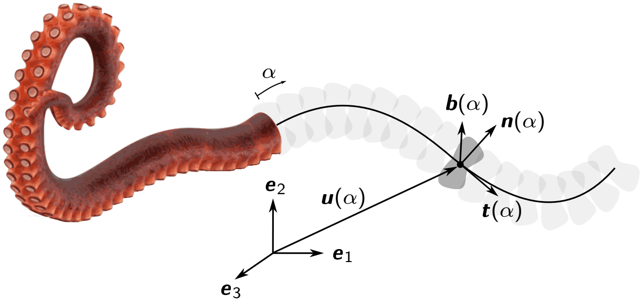
A soft robotic arm is seen as the **continuous assembly of 2D cross–sections** moving upon a **3D curve** according to **infinite rigid body transformations** which are defined by **distributed laws of internal deformations**.



Modeling strategy

We derive the **kinematics of continuum robots** from the **evolution of a 3D curve in space**

- using a **geometric local frame** approach
- with **no approximation on kinematic variables**
- including all the components of **deformation**



Position field

$$\alpha \in \mathbb{R} \mapsto \boldsymbol{H}(\alpha) = \mathcal{H}(\boldsymbol{R}(\alpha), \boldsymbol{u}(\alpha)) \in SE(3)$$

- $\alpha \in \mathbb{R}$, material abscissa along the arm
- $\boldsymbol{u}(\alpha) \in \mathbb{R}^3$, position vector of the cross-section
- $\boldsymbol{R}(\alpha) = [\boldsymbol{t}(\alpha) \ \ \boldsymbol{n}(\alpha) \ \ \boldsymbol{b}(\alpha)] \in SO(3)$, rotation matrix of the cross-section, where $\boldsymbol{t}(\alpha) =$ unit tangent vector; $\boldsymbol{n}(\alpha) =$ unit normal vector; $\boldsymbol{b}(\alpha) =$ unit binormal vector.

with

$SO(3)$ special Orthogonal group: the Lie group of the rotation matrices

$SE(3)$ special Euclidean group: the Lie group of the homogeneous matrices

Positions as Lie group elements

Change of reference frame $=$ Euclidean transformation

The space $SE(3)$ of Euclidean transformations is a **Lie Group**

$$\mathbf{H} = \mathcal{H}(\mathbf{R}, \mathbf{x}) = \begin{bmatrix} \mathbf{R} & \mathbf{x} \\ \mathbf{0}_{1x3} & 1 \end{bmatrix} \in SE(3)$$

$$SE(3) = SO(3) \times \mathbb{R}^3$$

Lie group

Definition

A <u>group</u> (G,·) is a set G of elements q together with a composition operation (·) which satisfies the four axioms of:

- *closure: the composition of two elements of the set yields an element of the set, i.e.,* $\forall q_1, q_2 \in G, q_1 \cdot q_2 = q_3 \in G$
- *associativity:* $q_1 \cdot (q_2 \cdot q_3) = (q_1 \cdot q_2) \cdot q_3$
- *neutral element: there exists an element e of the set such that* $q \cdot e = e \cdot q = q$
- *inverse element: there exists an element* q^{-1} *of the set such that* $q \cdot q^{-1} = q^{-1} \cdot q = e$

Definition

A <u>Lie group</u> is a continuous group for which the composition rule and the inverse are smooth

Lie group (cont'd)

Proposition

A matrix Lie group is a Lie group for which the composition rule is represented by the matrix product

$$\mathbf{R} \in SO(3), \textit{ the special Orthogonal group}$$
$$\mathbf{H} \in SE(3), \textit{ the special Euclidean group}$$

Deformation field

The homogeneous matrix $\boldsymbol{H}$ evolves along the material abscissa α according to the differential kinematic relationship

$$\boldsymbol{H}'(\alpha) = \boldsymbol{H}(\alpha)\widetilde{\boldsymbol{f}}(\alpha)$$

where $\widetilde{\boldsymbol{f}}(\alpha)$ is a left invariant vector field.

$$\widetilde{\boldsymbol{f}} = \begin{bmatrix} \widetilde{\boldsymbol{f}_\omega} & \boldsymbol{f}_u \\ \mathbf{0}_{1\times 3} & 1 \end{bmatrix} \in \mathfrak{se}(3)$$ deformation twist

$\boldsymbol{f}_u \in \mathbb{R}^3$ vector of linear deformations

$$\widetilde{\boldsymbol{f}_\omega} = \begin{bmatrix} 0 & -f_{\omega,3} & f_{\omega,2} \\ f_{\omega,3} & 0 & -f_{\omega,1} \\ -f_{\omega,2} & f_{\omega,1} & 0 \end{bmatrix} \in \mathfrak{so}(3)$$ skew-symmetric matrix of angular deformations

Deformations as Lie algebra elements

- $\widetilde{(\cdot)}_{SO(3)} : \mathbb{R}^3 \to \mathfrak{so}(3)$
- $\widetilde{(\cdot)}_{SE(3)} : \mathbb{R}^6 \to \mathfrak{se}(3)$

with

$\mathfrak{so}(3)$ Lie algebra associated to the Lie group SO(3)

$\mathfrak{se}(3)$ Lie algebra associated to the Lie group SE(3)

$$\Downarrow$$

$\widetilde{\mathbf{f}}(\alpha) \in \mathfrak{se}(3)$ deformation twist

$\mathbf{f}(\alpha) \in \mathbb{R}^6$ deformation vector (axial, shear, bending, torsion)

Lie derivatives

Derivative of a Lie group

The derivative of $q \in G$ with respect to $a \in \mathbb{R}$ reads

$$\begin{aligned} d_a(q) &= q\widetilde{\mathbf{a}}_L \\ &= \widetilde{\mathbf{a}}_R q \end{aligned}$$

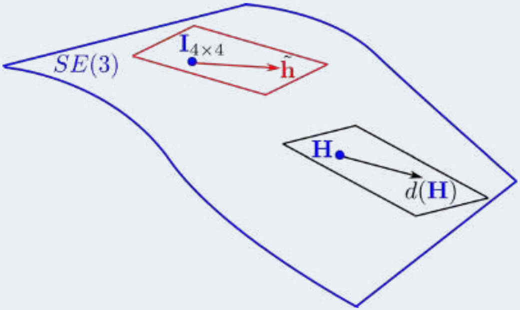
where $\widetilde{\mathbf{a}}_L \in \mathfrak{g}$ and $\widetilde{\mathbf{a}}_R \in \mathfrak{g}$ are respectively called a left and right invariant vector field. These elements represent the Lie algebra associated to the Lie group.

Lie algebra

Definition

The Lie algebra $\mathfrak{g}$ $(\mathfrak{se}(3))$ *is the tangent space at the identity element of a Lie group* G $(\mathbf{H})$.

$$d_a(\mathbf{H}) = \mathbf{H}\widetilde{\mathbf{a}}$$



Lie algebra (cont'd)

Proposition

The Lie algebra $\mathfrak{g}$ *is isomorphic to* $\mathbb{R}^k$ *through the invertible linear map*

$$\widetilde{(\cdot)} : \mathbb{R}^k \to \mathfrak{g}, \qquad \mathbf{a} \in \mathbb{R}^k \mapsto \widetilde{\mathbf{a}} \in \mathfrak{g}$$

Lie algebra (cont'd)

$$SO(3) \quad \mathfrak{so}(3) \quad \mathbb{R}^3$$

$$SE(3) \quad \mathfrak{se}(3) \quad \mathbb{R}^6$$

Lie algebra (cont'd)

$$d_a(\mathbf{H}) = \mathbf{H}\widetilde{\mathbf{a}}$$

Left invariant vector field on $SE(3)$

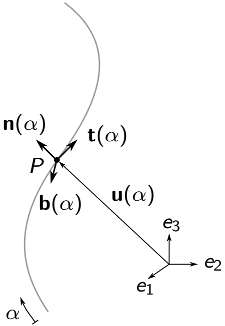
$=$

Invariant under a superimposed Euclidean transformation

$=$

Intrinsic quantity

Differential geometry of the 3D curve



Serret-Frenet Formulas

$$\mathbf{t}'(\alpha) = \kappa(\alpha)\mathbf{n}(\alpha)$$
$$\mathbf{n}'(\alpha) = -\kappa(\alpha)\mathbf{t}(\alpha) + \tau(\alpha)\mathbf{b}(\alpha)$$
$$\mathbf{b}'(\alpha) = -\tau(\alpha)\mathbf{n}(\alpha)$$

- $\kappa(\alpha)$ and $\tau(\alpha)$: curvature and torsion of the curve

$$\mathbf{R}'(\alpha) = \begin{bmatrix}\mathbf{t}'(\alpha) & \mathbf{n}'(\alpha) & \mathbf{b}'(\alpha)\end{bmatrix} \longrightarrow \mathbf{H}'(\alpha) = \begin{bmatrix}\mathbf{R}'(\alpha) & \mathbf{u}'(\alpha) \\ \mathbf{0}_{1\times 3} & 1\end{bmatrix}$$

$$\Big\downarrow \quad \textit{Invariant vector field}$$

$$\boxed{\mathbf{H}'(\alpha) = \mathbf{H}(\alpha)\widetilde{\mathbf{f}}(\alpha)}$$

- $\widetilde{\mathbf{f}}(\alpha)$: deformation twist

Compatibility equations

$$\boldsymbol{\eta}'(\alpha) - \dot{\boldsymbol{f}}(\alpha) = \widehat{\boldsymbol{\eta}}(\alpha)\boldsymbol{f}(\alpha)$$

Lie bracket

Definition

The Lie bracket operator is the bilinear operator defined as

$$[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}, \qquad \left[\widetilde{\mathbf{a}}, \widetilde{\mathbf{b}}\right] \mapsto d_b(\widetilde{\mathbf{a}}) - d_a(\widetilde{\mathbf{b}})$$

Cross derivatives

$$d_b(\widetilde{\mathbf{a}}) - d_a(\widetilde{\mathbf{b}}) = \left[\widetilde{\mathbf{a}}, \widetilde{\mathbf{b}}\right]$$

$$d_b(\mathbf{a}) - d_a(\mathbf{b}) = \widehat{\mathbf{a}}\mathbf{b} = \mathrm{ad}_{\mathbf{a}}\mathbf{b} \qquad \widehat{\mathbf{a}} = \begin{bmatrix} \widetilde{\mathbf{a}}_\omega & \widetilde{\mathbf{a}}_u \\ \mathbf{0}_{3\times 3} & \widetilde{\mathbf{a}}_\omega \end{bmatrix}$$

Definition

The linear operator $\widehat{(\cdot)}$ *is the bilinear operator defined as*

$$\widehat{(\cdot)} : \mathbb{R}^k \to \mathbb{R}^{k\times k}, \qquad \mathbf{a} \mapsto \widehat{\mathbf{a}} = \mathbf{A}$$

Strain energy, costitutive equations and stiffness matrix

Strain energy

$$\mathcal{V}_{int} = \frac{1}{2} \int_L \boldsymbol{f}^T \boldsymbol{K} \boldsymbol{f} \, \mathrm{d}\alpha$$

Costitutive equations

$$\boldsymbol{\sigma}(\alpha) = \mathbf{K}(\alpha) \, \boldsymbol{f}(\alpha)$$

Strain energy, costitutive equations and stiffness matrix (cont'd)

Stiffness matrix

$$\boldsymbol{K} = \begin{bmatrix} \boldsymbol{K}_{uu} & \boldsymbol{K}_{u\omega} \\ SYM & \boldsymbol{K}_{\omega\omega} \end{bmatrix}$$

Initially straight beam + reference curve $\equiv$ neutral axis of the beam (i.e. $\mathbf{n}^0$ and $\mathbf{b}^0$ are chosen to be the principal axes of the cross-sections) $\rightarrow$ $\mathbf{K}(\alpha_1)$ is diagonal:

$$\mathbf{K}_{uu} = \mathrm{diag}(EA, GA_2, GA_3)$$

$$\mathbf{K}_{\omega\omega} = \mathrm{diag}(GJ, EI_2, EI_3)$$

Static equilibrium equations

Principle of Virtual Work

$$\delta(\mathcal{V}_{int}) = \delta(\mathcal{V}_{ext})$$

Variations

Variations of a Lie group element

$$\delta(\mathbf{R}) = \mathbf{R}\widetilde{\delta\boldsymbol{\theta}}$$
$$\delta(\mathbf{H}) = \mathbf{H}\widetilde{\delta\mathbf{h}}$$

where $\widetilde{\delta\boldsymbol{\theta}} \in \mathfrak{so}(3)$ is an arbitrary infinitesimal rotation associated with the axial vector $\delta\boldsymbol{\theta} \in \mathbb{R}^3$ and $\delta\mathbf{h}_u = \mathbf{R}^T\delta\mathbf{u} \in \mathbb{R}^3$ is an arbitrary infinitesimal displacement

Variations of a twist element

$$\delta(\widetilde{\boldsymbol{\eta}}) - (\widetilde{\delta\mathbf{h}})^{\cdot} = \left[\widetilde{\boldsymbol{\eta}}, \widetilde{\delta\mathbf{h}}\right]$$
$$\delta(\boldsymbol{\eta}) - (\delta\mathbf{h})^{\cdot} = \widehat{\boldsymbol{\eta}}\delta\mathbf{h} = -\widehat{\delta\mathbf{h}}\boldsymbol{\eta}$$

Static equilibrium equations

Principle of Virtual Work

$$\delta(\mathcal{V}_{int}) = \delta(\mathcal{V}_{ext})$$

$$\delta(\mathbf{f}) = (\delta\mathbf{h})' + \widehat{\mathbf{f}}\delta\mathbf{h}$$

$$\begin{aligned}\delta(\mathcal{V}_{int}) &= \int_L \delta(\boldsymbol{f})^T \boldsymbol{\sigma}\,\mathrm{d}\alpha = \\ &= \left[\delta\mathbf{h}^T\boldsymbol{\sigma}\right]\Big|_0^L - \int_L \delta\mathbf{h}^T(\boldsymbol{\sigma}' - \hat{\mathbf{f}}^T\boldsymbol{\sigma}')\,\mathrm{d}\alpha\end{aligned}$$

$$\delta(\mathcal{V}_{ext}) = +\delta\mathbf{h}(0)^T\mathbf{g}_{ext}(0) - \delta\mathbf{h}(L)^T\mathbf{g}_{ext}(L) - \int_L \delta\mathbf{h}^T\mathbf{g}_{ext}(\alpha)\,\mathrm{d}\alpha$$

$$\mathbf{g}_{ext}(\alpha) = \begin{bmatrix}\mathbf{g}_{ext,u}^T & \mathbf{g}_{ext,\omega}^T\end{bmatrix}$$

Static equilibrium equations (cont'd)

Static equilibrium equations

weak form $\left[\delta \mathbf{h}^T(\boldsymbol{\sigma} - \mathbf{g}_{ext})\right]\Big|_0^L - \int_L \delta \mathbf{h}^T(\boldsymbol{\sigma}' - \hat{\mathbf{f}}^T\boldsymbol{\sigma} - \mathbf{g}_{ext})\,\mathrm{d}\alpha = 0$

strong form $\boldsymbol{\sigma}' - \hat{\mathbf{f}}^T\boldsymbol{\sigma} = \mathbf{g}_{ext}$

Velocity field

$$\dot{\boldsymbol{H}}(\alpha) = \boldsymbol{H}(\alpha)\widetilde{\boldsymbol{\eta}}(\alpha)$$

$\widetilde{\boldsymbol{\eta}} = \begin{bmatrix} \widetilde{\boldsymbol{\omega}} & \boldsymbol{v} \\ \mathbf{0}_{1\times 3} & 1 \end{bmatrix} \in \mathfrak{se}(3)$ velocity twist

$\boldsymbol{v} \in \mathbb{R}^3$ vector of linear velocities

$\widetilde{\boldsymbol{\omega}} = \begin{bmatrix} 0 & -\omega_3 & -\omega_2 \\ -\omega_3 & 0 & -\omega_1 \\ -\omega_2 & -\omega_1 & 0 \end{bmatrix} \in \mathfrak{so}(3)$ skew-symmetric matrix of angular velocities

Kinetic energy

$$\mathcal{K} = \frac{1}{2} \int_L \boldsymbol{\eta}^T \mathbf{M} \boldsymbol{\eta} \, \mathrm{d}\alpha$$

$$\mathbf{M} = \begin{bmatrix} \rho A \mathbf{I}_{3\times 3} & \mathbf{J}_I^T \\ \mathbf{J}_I & \mathbf{J}_{II} \end{bmatrix}$$

- ρ density
- A cross-section area
- $\mathbf{J}_I$ first moment of inertia of the cross section (computed in the local axes of the arm)
- $\mathbf{J}_{II}$ second moment of inertia of the cross section (computed in the local axes of the arm)

Dynamic equilibrium equations

Hamilton's principle

$$\int_{t_0}^{t_1} \left(\delta(\mathcal{K}) - \delta(\mathcal{V}_{int}) + \delta(\mathcal{V}_{ext})\right) dt = 0 \; .$$

$$\begin{aligned} \delta(\mathcal{V}_{int}) &= \int_L \delta(\boldsymbol{f})^T \boldsymbol{\sigma} \, \mathrm{d}\alpha = \\ &= \left[\delta \mathbf{h}^T \boldsymbol{\sigma}\right] |_0^L - \int_L \delta \mathbf{h}^T (\boldsymbol{\sigma}' - \hat{\mathbf{f}}^T \boldsymbol{\sigma}') \, \mathrm{d}\alpha \end{aligned}$$

$$\delta(\mathcal{V}_{ext}) = +\delta \mathbf{h}(0)^T \mathbf{g}_{ext}(0) - \delta \mathbf{h}(L)^T \mathbf{g}_{ext}(L) - \int_L \delta \mathbf{h}^T \mathbf{g}_{ext}(\alpha) \, \mathrm{d}\alpha$$

Dynamic equilibrium equations (cont'd)

Hamilton's principle

$$\int_{t_0}^{t_1} \left(\delta(\mathcal{K}) - \delta(\mathcal{V}_{int}) + \delta(\mathcal{V}_{ext})\right) dt = 0\ .$$

$$\delta(\boldsymbol{\eta}) = (\delta\dot{\mathbf{h}}) + \widehat{\boldsymbol{\eta}}\delta\mathbf{h}$$

$$\begin{aligned}\int_{t_0}^{t_1} \delta(\mathcal{K})\,\mathrm{d}t &= \int_{t_0}^{t_1}\int_L \delta(\boldsymbol{\eta})^T \mathbf{M}\boldsymbol{\eta}\,\mathrm{d}\alpha\mathrm{d}t = \\ &= \left[\int_L \delta\mathbf{h}^T\mathbf{M}\boldsymbol{\eta}\mathrm{d}\alpha\right]_{t_0}^{t_1} - \int_{t_0}^{t_1}\int_L \delta\mathbf{h}^T(\mathbf{M}\dot{\boldsymbol{\eta}} - \hat{\boldsymbol{\eta}}^T\mathbf{M}\boldsymbol{\eta})\,\mathrm{d}\alpha\,\mathrm{d}t\end{aligned}$$

Dynamic equilibrium equations (cont'd)

Dynamic equilibrium equations

weak form $\left[\delta \mathbf{h}^T(\boldsymbol{\sigma} - \mathbf{g}_{ext})\right] \Big|_0^L - \int_L \delta \mathbf{h}^T(-\mathbf{M}\dot{\boldsymbol{\eta}} + \hat{\boldsymbol{\eta}}^T \mathbf{M}\boldsymbol{\eta} + \boldsymbol{\sigma}' - \hat{\mathbf{f}}^T \boldsymbol{\sigma} + \mathbf{g}_{ext})\, \mathrm{d}\alpha = 0$

strong form $\mathbf{M}\dot{\boldsymbol{\eta}} - \hat{\boldsymbol{\eta}}^T \mathbf{M}\boldsymbol{\eta} - \boldsymbol{\sigma}' + \mathbf{f}^T \boldsymbol{\sigma} = \mathbf{g}_{ext}$

Equations of motion

Kinematic equations	$\dot{\mathbf{H}} = \mathbf{H}\widetilde{\boldsymbol{\eta}}$
	$\mathbf{H}' = \mathbf{H}\widetilde{\boldsymbol{f}}$
Material constitutive law	$\boldsymbol{\sigma} = \mathbf{K}\boldsymbol{f}$
Compatibility equations	$\boldsymbol{\eta}' - \dot{\boldsymbol{f}} = \widehat{\boldsymbol{\eta}}\boldsymbol{f}$
Boundary conditions	$\delta\boldsymbol{h}(L)\left(\mathbf{K}(L)\boldsymbol{f}(L) - \boldsymbol{g}_{ext}(L)\right) = \delta\boldsymbol{h}(0)\left(\mathbf{K}(0)\boldsymbol{f}(0) - \boldsymbol{g}_{ext}(0)\right)$
Dynamic equilibrium equations	$\mathbf{M}\dot{\boldsymbol{\eta}} - \widehat{\boldsymbol{\eta}}^T\mathbf{M}\boldsymbol{\eta} - \boldsymbol{\sigma}' + \boldsymbol{f}^T\boldsymbol{\sigma} = \mathbf{g}_{ext}$

Part B:
Discrete formulation with piecewise constant deformation

The continuous field of deformation $\boldsymbol{f}(\alpha)$ leads to dynamics described by PDE!

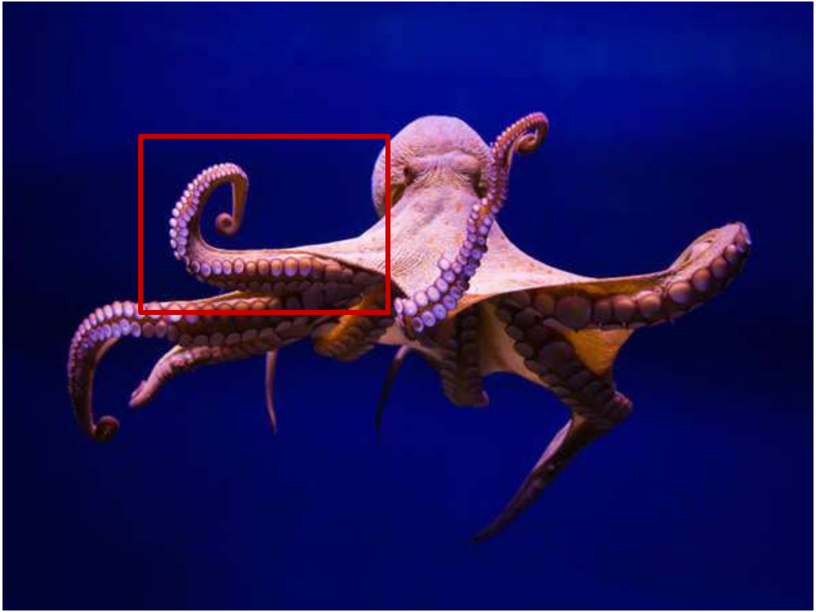
Continuum mechanics

$$\mathbf{M}\dot{\boldsymbol{\eta}} - \widehat{\boldsymbol{\eta}}^T\mathbf{M}\boldsymbol{\eta} - \boldsymbol{\sigma}' + \boldsymbol{f}^T\boldsymbol{\sigma} = \mathbf{g}_{ext}$$

Robotics

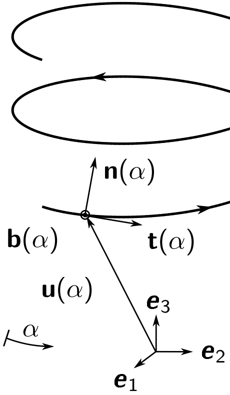
$$\mathbf{M}\ddot{\boldsymbol{q}} + \mathbf{C}\dot{\boldsymbol{q}} + \boldsymbol{g} = \boldsymbol{\tau}$$

How to close the gap?



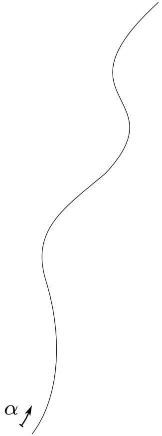



20mm

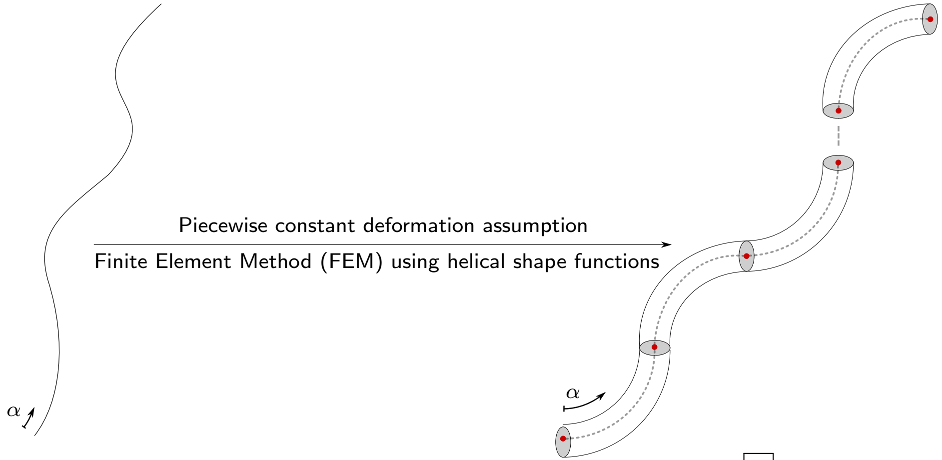
$\mathbf{n}(\alpha)$
$\mathbf{b}(\alpha)$
$\mathbf{t}(\alpha)$
$\mathbf{u}(\alpha)$
$\boldsymbol{e}_3$
α
$\boldsymbol{e}_2$
$\boldsymbol{e}_1$

From the continuum formulation ...



continuum arm: continuous field $\boldsymbol{f}(\alpha)$

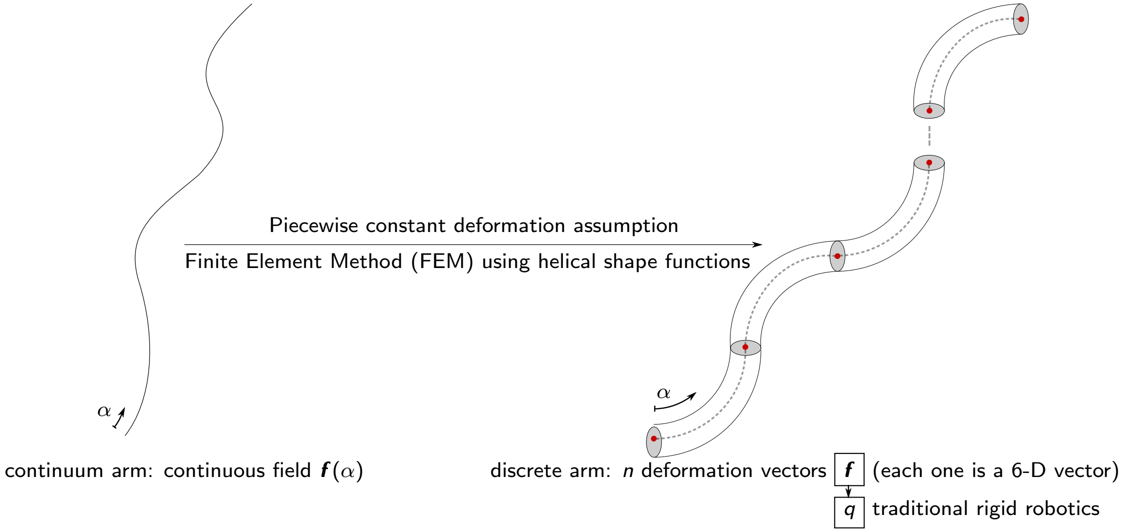
... To the discrete formulation



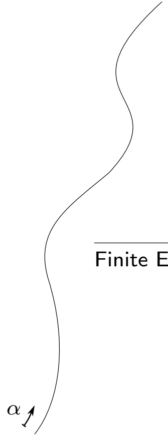
continuum arm: continuous field $\boldsymbol{f}(\alpha)$

discrete arm: n deformation vectors $\boxed{\boldsymbol{f}}$ (each one is a 6-D vector)

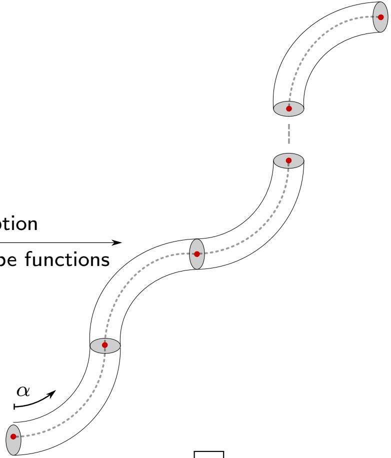
... To the discrete formulation



...To the discrete formulation



Piecewise constant deformation assumption
$\longrightarrow$
Finite Element Method (FEM) using helical shape functions



continuum arm: continuous field $\boldsymbol{f}(\alpha)$

discrete arm: n deformation vectors $\boxed{\boldsymbol{f}}$ (each one is a 6-D vector)

$\downarrow$

$\boxed{q}$ traditional rigid robotics

THE FE DEFORMATION SPACE FORMULATION

Grazioso, Di Gironimo, Siciliano "A geometrically exact model for soft continuum robots" *Soft Robotics*, 2019

Kinematics

$$\mathbf{H}'(\alpha) = \mathbf{H}(\alpha)\widetilde{\boldsymbol{f}}(\alpha)$$

- $\alpha \in [0, L]$
  L: total length of the arm
- $\boldsymbol{f}$: constant deformation vector of the discrete element of the arm

$\Downarrow$ ... since $\boldsymbol{f}$ does not depend on α

Kinematics mapping

$$\boldsymbol{H}(\alpha) = \boldsymbol{H}_0 \exp_{SE(3)}\left(\alpha\widetilde{\boldsymbol{f}}\right)$$

- $\boldsymbol{H}_0$ in SE(3): configuration of the arm at $\alpha = 0$

Kinematics

The exponential map on $SE(3)$

$$\exp_{SE(3)}(\cdot) : \mathbb{R}^6 \to SE(3), \qquad \boldsymbol{f} \mapsto \exp_{SE(3)}(\boldsymbol{f}) \quad (*)$$

$$\exp_{SE(3)}(\boldsymbol{f}) = \begin{bmatrix} \exp_{SO(3)}(\boldsymbol{f}_\omega) & \boldsymbol{T}^T_{SO(3)}(\boldsymbol{f}_\omega)\boldsymbol{f}_u \\ \boldsymbol{0}_{1\times 3} & 1 \end{bmatrix}$$

- $\exp_{SO(3)}(\boldsymbol{f}_\omega) = \boldsymbol{I}_{3\times 3} + \alpha(\boldsymbol{f}_\omega)\widetilde{\boldsymbol{f}}_\omega + \frac{\beta(\boldsymbol{f}_\omega)}{2}\widetilde{\boldsymbol{f}}^2_\omega$: Rodrigues' formula
- $\boldsymbol{T}_{SO(3)}(\boldsymbol{f}_\omega) = \boldsymbol{I}_{3\times 3} - \frac{\beta(\boldsymbol{f}_\omega)}{2}\widetilde{\boldsymbol{f}}_\omega + \frac{1-\alpha(\boldsymbol{f}_\omega)}{\|\boldsymbol{f}_\omega\|^2}\widetilde{\boldsymbol{f}}^2_\omega$: Tangent operator

$$\alpha(\boldsymbol{f}_\omega) = \frac{\sin(\|\boldsymbol{f}_\omega\|)}{\|\boldsymbol{f}_\omega\|} \qquad \beta(\boldsymbol{f}_\omega) = 2\frac{1-\cos(\|\boldsymbol{f}_\omega\|)}{\|\boldsymbol{f}_\omega\|^2}$$

(*) The formal definition of exponential map uses the Lie algebra $\mathfrak{se}(3)$ instead of $\mathbb{R}^6$. However, due to the isomorphism between Lie algebra $\mathfrak{se}(3)$ and $\mathbb{R}^6$, Hence, with a slight abuse of notation, we use $\mathbb{R}^6$ instead of $\mathfrak{se}(3)$

Exponential map

Definition

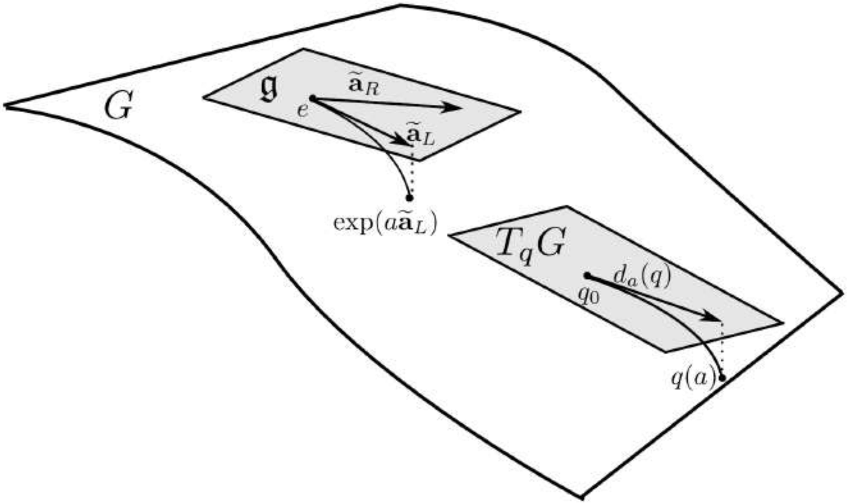
The exponential map projects an element of the Lie algebra into an element of the Lie group

$$exp : \mathfrak{g} \to G, \quad \widetilde{\mathbf{a}} \mapsto exp(\widetilde{\mathbf{a}})$$

and it is given by

$$exp(\widetilde{\mathbf{a}}) = \sum_{i=0}^{\infty} \frac{\widetilde{\mathbf{a}}^i}{i!}$$

Exponential map (cont'd)



Kinematics – multiple elements

$$\mathbf{H}'(\alpha) = \mathbf{H}(\alpha)\,\widetilde{\boldsymbol{f}}(\alpha)$$

- $\alpha \in [0, L_n] =$
  $[0, L_1), (L_1, L_2), \ldots, (L_{n-1}, L_n]$
  L_n: total length of the arm
- $\boldsymbol{f}_i$: constant deformation vector of each discrete element of the arm

$\Downarrow$ … since $\boldsymbol{f}$ does not depend on α

Product of exponentials (PoE)

$$\boldsymbol{H}(\alpha) = \boldsymbol{H}_0 \prod_{i=1}^{n} \exp_{SE(3)}\left((\min(L_i, \alpha_i) - L_{i-1})\widetilde{\boldsymbol{f}}_i\right)$$

- $\boldsymbol{H}_0$ in SE(3): configuration of the arm at $\alpha = 0$

Inverse kinematics – one element

Mapping

$$k_I(\cdot): \quad \mathbf{H}_0, \mathbf{H}_L \in SE(3) \mapsto \mathbf{f} = \log_{SE(3)}\left(\mathbf{H}_0^{-1}\mathbf{H}_L\right) \in \mathbb{R}^6$$

- $\mathbf{H}_0, \mathbf{H}_L$: the configuration of the arm at $\alpha = 0, \alpha = L$
- $\log_{SE(3)}(\cdot)$: the logarithmic map on $SE(3)$

Inverse kinematics (cont'd)

The logarithmic map on $SE(3)$

$$\log_{SE(3)}(\cdot) : SE(3) \to \mathbb{R}^6, \qquad \mathbf{H} \mapsto \log_{SE(3)}(\mathbf{H}) \quad (*)$$

$$\log_{SE(3)}(\mathbf{H}) = \begin{bmatrix} \log_{SO(3)}(\mathbf{R}) & \mathbf{T}_{SO(3)}^{-T}(\mathbf{f}_\omega)\boldsymbol{u} \\ \mathbf{0}_{1\times 3} & 1 \end{bmatrix}$$

- $\log_{SO(3)}(\mathbf{R}) = \frac{\theta}{2\sin\theta}(\mathbf{R} - \mathbf{R}^T)$, with $\theta = \mathrm{acos}\left(\frac{1}{2}(\mathrm{trace}(\mathbf{R}) - 1)\right)$, $\qquad \theta < \pi$
- $\mathbf{T}_{SO(3)}^{-1}(\mathbf{f}_\omega) = \mathbf{I}_{3\times 3} + \frac{1}{2}\widetilde{\mathbf{f}}_\omega + \frac{1-\gamma(\mathbf{f}_\omega)}{\|\mathbf{f}_\omega\|^2}\widetilde{\mathbf{f}}_\omega^2$: Inverse of the tangent operator

$\gamma(\mathbf{f}_\omega) = \frac{\|\mathbf{f}_\omega\|}{2}\cot\left(\frac{\|\mathbf{f}_\omega\|}{2}\right)$

(*) The formal definition of logarithmic map uses the Lie algebra $\mathfrak{se}(3)$ instead of $\mathbb{R}^6$.
As before, since the isomorphism $\mathfrak{se}(3) \simeq \mathbb{R}^6$ holds, with a slight abuse of notation, we use $\mathbb{R}^6$.

Logarithmic map

Definition

The logarithmic map projects an element of the Lie group into an element of the Lie algebra

$$log : G \to \mathfrak{g}, \quad q \mapsto log(q) = \widetilde{\mathbf{a}}$$

and it is given by

$$log(q) = \sum_{i=0}^{\infty} \frac{(e-q)^i}{i}$$

Tangent map

Definition

$$\mathbf{T} : \mathbb{R}^k \to \mathbb{R}^k, \quad \mathbf{u} \mapsto \mathbf{T}(\mathbf{u}) d_a(\mathbf{u}) = \mathbf{a}$$

with

$$\mathbf{T}(\mathbf{u}) = \sum_{i=0}^{\infty} (-1)^i \frac{\widehat{\mathbf{u}}^i}{(i+1)!}$$

Inverse of the tangent map

Definition

$$\mathbf{T}^{-1} : \mathbb{R}^k \to \mathbb{R}^k, \quad \mathbf{u}, \mathbf{a} \mapsto \mathbf{T}^{-1}(\mathbf{u})\mathbf{a} = d_a(\mathbf{u})$$

with

$$\mathbf{T}^{-1}(\mathbf{u}) = \sum_{i=0}^{\infty} (-1)^i B_i \frac{\widehat{\mathbf{u}}^i}{(i)!}$$

where B_i is the Bernoulli number of the first kind.

The exponential map on $SO(3)$

Exponential map	$\exp_{SO(3)}(\mathbf{h}_\omega) = \mathbf{I}_{3\times 3} + \alpha(\mathbf{h}_\omega)\widetilde{\mathbf{h}}_\omega + \frac{\beta(\mathbf{h}_\omega)}{2}\widetilde{\mathbf{h}}_\omega^2$
Logarithmic map	$\log_{SO(3)}(\mathbf{R}) = \frac{\theta}{2\sin\theta}(\mathbf{R} - \mathbf{R}^T)$
Tangent operator	$\mathbf{T}_{SO(3)}(\mathbf{h}_\omega) = \mathbf{I}_{3\times 3} - \frac{\beta(\mathbf{h}_\omega)}{2}\widetilde{\mathbf{h}}_\omega + \frac{1-\alpha(\mathbf{h}_\omega)}{\lVert\mathbf{h}_\omega\rVert^2}\widetilde{\mathbf{h}}_\omega^2$
Inverse of the tangent operator	$\mathbf{T}^{-1}_{SO(3)}(\mathbf{h}_\omega) = \mathbf{I}_{3\times 3} + \frac{1}{2}\widetilde{\mathbf{h}}_\omega + \frac{1-\gamma(\mathbf{h}_\omega)}{\lVert\mathbf{h}_\omega\rVert^2}\widetilde{\mathbf{h}}_\omega^2$

$$\alpha(\mathbf{h}_\omega) = \frac{\sin(\lVert\mathbf{h}_\omega\rVert)}{\lVert\mathbf{h}_\omega\rVert} \quad \beta(\mathbf{h}_\omega) = 2\frac{1-\cos(\lVert\mathbf{h}_\omega\rVert)}{\lVert\mathbf{h}_\omega\rVert^2} \quad \gamma(\mathbf{h}_\omega) = \frac{\lVert\mathbf{h}_\omega\rVert}{2}\cot\left(\frac{\lVert\mathbf{h}_\omega\rVert}{2}\right)$$

$$\theta = \operatorname{acos}\left(\frac{1}{2}(\operatorname{trace}(\mathbf{R}) - 1\right), \quad \theta < \pi$$

The exponential map on $SE(3)$

Exponential map	$\exp_{SE(3)}(\mathbf{h}) = \begin{bmatrix} \exp_{SO(3)}(\mathbf{h}_\omega) & \mathbf{T}^T_{SO(3)}(\mathbf{h}_\omega)\mathbf{h}_u \\ \mathbf{0}_{1\times 3} & 1 \end{bmatrix}$
Logarithmic map	$\log_{SE(3)}(\mathbf{H}) = \begin{bmatrix} \widetilde{\mathbf{h}}_\omega & \mathbf{T}^{-T}_{SO(3)}(\mathbf{h}_\omega)\mathbf{h}_u \\ \mathbf{0}_{1\times 3} & 0 \end{bmatrix}$
Tangent operator	$\mathbf{T}_{SE(3)}(\mathbf{h}) = \begin{bmatrix} \mathbf{T}_{SO(3)}(\mathbf{h}_\omega) & \mathbf{T}_{u\omega+}(\mathbf{h}_u, \mathbf{h}_\omega) \\ \mathbf{0}_{3\times 3} & \mathbf{T}_{SO(3)}(\mathbf{h}_\omega) \end{bmatrix}$
Inverse of the tangent operator	$\mathbf{T}^{-1}_{SE(3)}(\mathbf{h}) = \begin{bmatrix} \mathbf{T}^{-1}_{SO(3)}(\mathbf{h}_\omega) & \mathbf{T}_{u\omega-}(\mathbf{h}_u, \mathbf{h}_\omega) \\ \mathbf{0}_{3\times 3} & \mathbf{T}^{-1}_{SO(3)}(\mathbf{h}_\omega) \end{bmatrix}$

The exponential map on $SE(3)$ (cont'd)

$$\mathbf{T}_{u\omega+}(\mathbf{h}_\omega,\mathbf{h}_u) = \frac{-\beta}{2}\widetilde{\mathbf{h}}_\omega + \frac{1-\alpha}{\|\mathbf{h}_\omega\|^2}[\mathbf{h}_\omega,\mathbf{h}_u] + \frac{\mathbf{h}_u^T\mathbf{h}_\omega}{\|\mathbf{h}_u\|^2}\left((\beta-\alpha)\widetilde{\mathbf{h}}_u + (\frac{\beta}{2} - \frac{3(1-\alpha}{\|\mathbf{h}_u\|^2})\widetilde{\mathbf{h}}_u^2\right)$$

$$\mathbf{T}_{u\omega-}(\mathbf{h}_\omega,\mathbf{h}_u) = \frac{1}{2}\widetilde{\mathbf{h}}_\omega + \frac{1-\gamma}{\|\mathbf{h}_\omega\|^2}[\mathbf{h}_\omega,\mathbf{h}_u] + \frac{\mathbf{h}_u^T\mathbf{h}_\omega}{\|\mathbf{h}_u\|^4}\left((\frac{1}{\beta}+\gamma-2)\widetilde{\mathbf{h}}_u^2\right)$$

Geometric interpretation of the interpolated reference curve

$$\mathbf{H}(\alpha) = \mathbf{H}_A \mathbf{H}_{A0} \exp_{SE(3)} \left(\frac{\alpha}{L} \widetilde{\mathbf{d}} \right)$$

$$\Downarrow$$

$$\mathbf{u}'(\alpha) = \mathbf{R}(\alpha) \frac{\widetilde{\mathbf{d}}_u}{L} \quad \text{and} \quad \mathbf{R}'(\alpha) = \mathbf{R}(\alpha) \frac{\widetilde{\mathbf{d}}_\omega}{L}$$

Local triad

$$\mathbf{t}(\alpha) = \frac{\mathbf{u}'(\alpha)}{\|\mathbf{u}(\alpha)\|} = \mathbf{R}(\alpha) \frac{\mathbf{d}_u}{L}$$

$$\mathbf{n}(\alpha) = \frac{1}{\kappa} \mathbf{t}'(\alpha) = \mathbf{R}(\alpha) \frac{\widetilde{\mathbf{d}}_\omega \mathbf{d}_u}{\|\widetilde{\mathbf{d}}_\omega \mathbf{d}_u\|}$$

$$\mathbf{b}(\alpha) = \frac{1}{\|\mathbf{d}_u\| \|\widetilde{\mathbf{d}}_\omega \mathbf{d}_u\|} \widetilde{\mathbf{R}(\alpha) \mathbf{d}_u} \mathbf{R}(\alpha) \mathbf{d}_\omega \mathbf{d}_u = \mathbf{R}(\alpha) \frac{\widetilde{\mathbf{d}}_u \widetilde{\mathbf{d}}_\omega \mathbf{d}_u}{\|\mathbf{d}_u\| \|\widetilde{\mathbf{d}}_\omega \mathbf{d}_u\|}$$

Geometric interpretation of the interpolated reference curve (cont'd)

$$\mathbf{H}(\alpha) = \mathbf{H}_A \mathbf{H}_{A0} \exp_{SE(3)}\left(\frac{\alpha}{L}\widetilde{\mathbf{d}}\right)$$

$$\Downarrow$$

$$\mathbf{u}'(\alpha) = \mathbf{R}(\alpha)\frac{\widetilde{\mathbf{d}}_u}{L} \quad \text{and} \quad \mathbf{R}'(\alpha) = \mathbf{R}(\alpha)\frac{\widetilde{\mathbf{d}}_\omega}{L}$$

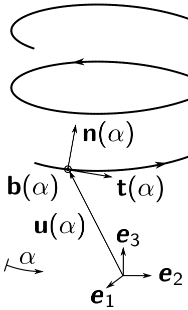
Curvature and torsion of the curve

$$\kappa = \frac{\|\widetilde{\mathbf{d}}_\omega \mathbf{d}_u\|}{L^2} = \frac{\|\mathbf{d}_\omega\| \sin(\mathbf{d}_\omega, \mathbf{d}_u)}{L} = \text{cost}$$

$$\tau = \frac{(\mathbf{d}_\omega^T \mathbf{d}_u)}{L^2} = \frac{\|\mathbf{d}_\omega\| \cos(\mathbf{d}_\omega, \mathbf{d}_u)}{L} = \text{cost}$$

$$\kappa_g = \sqrt{\kappa^2 + \tau^2} = \frac{\|\mathbf{d}_\omega\|}{L} = \frac{\|\mathbf{d}_\omega\|}{\|\mathbf{d}_u\|} = \text{cost}$$

Geometric interpretation of the interpolated reference curve (cont'd)

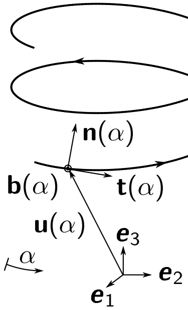


Curvature and torsion of the curve

$$\kappa = \frac{\|\widetilde{\mathbf{d}}_\omega \mathbf{d}_u\|}{L^2} = \frac{\|\mathbf{d}_\omega\| \sin(\mathbf{d}_\omega, \mathbf{d}_u)}{L} = \text{cost}$$

$$\tau = \frac{(\mathbf{d}_\omega^T \mathbf{d}_u)}{L^2} = \frac{\|\mathbf{d}_\omega\| \cos(\mathbf{d}_\omega, \mathbf{d}_u)}{L} = \text{cost}$$

Geometric interpretation of the interpolated reference curve (cont'd)



Physical meaning: **Helix**!

Example

- Computing the shape of a soft arm with one element ($n = 1$) with internal deformations $\boldsymbol{f}_u = [1\,0\,0]^T$; $\boldsymbol{f}_\omega = [\tau\,0\,\kappa]^T$ (no axial and shear deformations; bending about z; torsion about x)

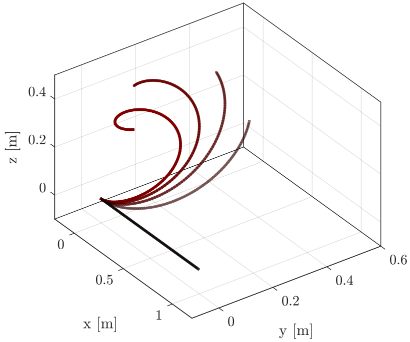
$$\Downarrow$$

$$\boldsymbol{H}(\alpha) = \begin{bmatrix} 1-(1-\cos(\alpha\kappa_g))\frac{\kappa^2}{\kappa_g^2} & -\sin(\alpha\kappa_g)\frac{\kappa}{\kappa_g} & (1-\cos(\alpha\kappa_g))\frac{\kappa\tau}{\kappa_g^2} & \alpha+(\sin(\alpha\kappa_g)-\alpha\kappa_g)\frac{\kappa^2}{\kappa_g^3} \\ \sin(\alpha\kappa_g)\frac{\kappa}{\kappa_g} & \cos(\alpha\kappa_g) & -\sin(\alpha\kappa_g)\frac{\tau}{\kappa_g} & (1-\cos(\alpha\kappa_g))\frac{\kappa}{\kappa_g^2} \\ (1-\cos(\alpha\kappa_g))\frac{\kappa\tau}{\kappa_g^2} & \sin(\alpha\kappa_g)\frac{\tau}{\kappa_g} & 1-(1-\cos(\alpha\kappa_g))\frac{\tau^2}{\kappa_g^2} & (\alpha\kappa_g-\sin(\alpha\kappa_g))\frac{\kappa\tau}{\kappa_g^3} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

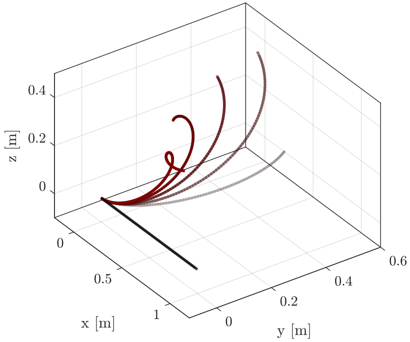
- $\kappa_g = \sqrt{\kappa^2+\tau^2}$: Gaussian curvature of the arm

Grazioso, Di Gironimo, Siciliano "From differential geometry of curves to helical kinematics of continuum robots using exponential mapping" *ARK*, 2018

Example (cont'd)



(a) $\tau = 3\,\mathrm{m}^{-1}$; $\kappa = 0 : \pi/2 : 2\pi\ \mathrm{m}^{-1}$.



(b) $\kappa = 3\,\mathrm{m}^{-1}$; $\tau = 0 : \pi/2 : 2\pi\ \mathrm{m}^{-1}$

Figure: Whole arm screw motion of a manipulator with constant curvature and torsion. $L = 1\,\mathrm{m}$.

Grazioso, Di Gironimo, Siciliano "From differential geometry of curves to helical kinematics of continuum robots using exponential mapping" *ARK*, 2018

Differential kinematics

- Mapping between the velocities along the arm and the time derivatives of the states of the manipulator, i.e. the internal deformations

Differential kinematics

$$\boldsymbol{\eta}(\alpha) = \boldsymbol{J}(\alpha, \boldsymbol{d})\dot{\boldsymbol{f}}$$

Soft Geometric Jacobian

$$\begin{aligned}\boldsymbol{J}(\alpha, \boldsymbol{d}) = {} & \mathrm{Ad}_{\exp_{SE(3)}(-\frac{\alpha}{L}\mathbf{d})\mathbf{H}_{A0}^{-1}} \mathbf{T}_{SE(3)}^{-1}(-\boldsymbol{d})\mathrm{Ad}_{\mathbf{H}_{A0}^{-1}} \dots \\ & \dots \left(\mathbf{T}_{SE(3)}^{-1}(-\boldsymbol{d})\mathrm{Ad}_{\mathbf{H}_{A0}^{-1}})^2 + (\mathbf{T}_{SE(3)}^{-1}(\boldsymbol{d})\mathrm{Ad}_{\mathbf{H}_{B0}^{-1}})^2)^{-1} \right) \dots \\ & \dots + \frac{\alpha}{L}\mathbf{T}_{SE(3)}\left(\frac{\alpha}{L}\boldsymbol{d}\right)\end{aligned}$$

Adjoint representation

Definition

The adjoint representation of a Lie algebra element is defined as

$$Ad_q : \mathfrak{g} \to \mathfrak{g}, \qquad \widetilde{\mathbf{a}} \mapsto q\widetilde{\mathbf{a}}q^{-1}$$

Adjoint representation of a $\mathfrak{se}(3)$ element

$$\mathrm{Ad}_{\mathbf{H}}(\widetilde{\mathbf{a}}) = \mathbf{H}\widetilde{\mathbf{a}}\mathbf{H}^{-1}$$

$$\mathrm{Ad}_{\mathbf{H}}(\mathbf{a}) = \begin{bmatrix} \mathbf{R} & \widetilde{\mathbf{u}}\mathbf{R} \\ \mathbf{0}_{3\times3} & \mathbf{R} \end{bmatrix} \mathbf{a}$$

Statics

Variation of internal energy (continuum)

$$\delta(\mathcal{V}_{int}) = \int_L \delta(\boldsymbol{f}(\alpha))^T \mathbf{K}(\alpha)\boldsymbol{f}(\alpha)\,\mathrm{d}\alpha$$

$\Downarrow$

- $\boldsymbol{f}$ = cost for the finite element

Variation of internal energy (discrete)

$$\delta(\mathcal{V}_{int}) = \delta(\boldsymbol{f})^T \mathbf{K}_L \boldsymbol{f}, \qquad \mathbf{K}_L = \int_L \mathbf{K}(\alpha)\,\mathrm{d}\alpha; \quad \mathbf{K}_L \boldsymbol{f} = \mathbf{T}$$

Statics

Variation of external energy (continuum)

$$\delta(\mathcal{V}_{ext}) = +\delta \boldsymbol{h}(0)^T \mathbf{g}_{ext}(0) - \delta \boldsymbol{h}(L)^T \mathbf{g}_{ext}(L) - \int_L \delta \boldsymbol{h}^T \mathbf{g}_{ext} \, \mathrm{d}\alpha$$

$$\Downarrow$$

- $\boldsymbol{f}$ = cost for the finite element
- $\delta \boldsymbol{h}(\alpha) = \boldsymbol{J}(\alpha, \boldsymbol{d})\delta(\boldsymbol{f})$

Variation of external energy (discrete)

$$\delta(\mathcal{V}_{ext}) = -\delta(\boldsymbol{f})^T \boldsymbol{J}_L^T(\mathbf{d})\mathbf{F}, \qquad \mathbf{F} = \int_L \mathbf{g}_{ext} \, \mathrm{d}\alpha; \quad \boldsymbol{J}_L(\boldsymbol{d}) = \int_L \boldsymbol{J}(\alpha, \boldsymbol{d}) \, \mathrm{d}\alpha$$

Statics

PVV

$$\delta(\mathcal{V}_{int}) = \delta(\mathcal{V}_{ext}) \qquad \forall\delta(\boldsymbol{f})$$

Static model

$$\mathbf{T} = \boldsymbol{J}_L^T(\boldsymbol{d})\mathbf{F}$$

$\rightarrow$ Kineto-statics duality!

Dynamics

Weak form of dynamic equilibrium equations (continuum)

$$\left[\delta\mathbf{h}^T(\boldsymbol{\sigma} - \mathbf{g}_{ext})\right]\Big|_0^L - \int_L \delta\mathbf{h}^T(-\mathbf{M}\dot{\boldsymbol{\eta}} + \hat{\boldsymbol{\eta}}^T\mathbf{M}\boldsymbol{\eta} + \boldsymbol{\sigma}' - \hat{\mathbf{f}}^T\boldsymbol{\sigma} + \mathbf{g}_{ext})\,\mathrm{d}\alpha = 0$$

- $\boldsymbol{\eta}(\alpha) = \boldsymbol{J}(\alpha, \boldsymbol{d})\dot{\boldsymbol{f}}$
- $\dot{\boldsymbol{\eta}}(\alpha) = \boldsymbol{J}(\alpha, \mathbf{d})\ddot{\boldsymbol{f}} + \dot{\boldsymbol{J}}(\alpha, \mathbf{d})\dot{\boldsymbol{f}}$
- $\delta\boldsymbol{h}(\alpha) = \boldsymbol{J}(\alpha, \boldsymbol{d})\delta(\boldsymbol{f})$

Weak form of dynamic equilibrium equations (discrete)

$$\delta(\boldsymbol{f})^T \int_L \boldsymbol{J}^T(\mathbf{M}(\boldsymbol{J}\ddot{\boldsymbol{f}} + \dot{\boldsymbol{J}}\dot{\boldsymbol{f}}) - \widehat{\boldsymbol{J}\dot{\boldsymbol{f}}}^T\mathbf{M}\boldsymbol{J}\dot{\boldsymbol{f}} + \widehat{\boldsymbol{\epsilon}}^T\mathbf{K}\boldsymbol{f} - \mathbf{g}_{ext})\,\mathrm{d}\alpha = 0$$

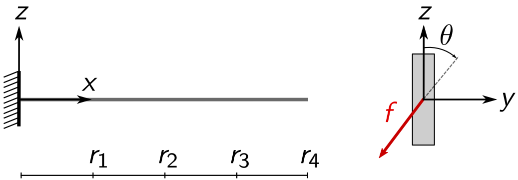
Dynamics

- $\int_L \boldsymbol{J}^T \mathbf{M} \boldsymbol{J} \, \mathrm{d}\alpha = \mathbb{M}$, the 6×6 discretized mass matrix.
- $\int_L \boldsymbol{J}^T \mathbf{M} \dot{\boldsymbol{J}} \, \mathrm{d}\alpha = \mathbb{C}_1$, the 6×6 velocity matrix which contributes only if $\dot{\epsilon}$ does not vanish, i.e., only when the deformation of the arm changes in time.
- $\int_L \boldsymbol{J}^T \widehat{\boldsymbol{J}\dot{\boldsymbol{f}}}^T \mathbf{M} \boldsymbol{J} \, \mathrm{d}\alpha = \mathbb{C}_2$, the 6×6 velocity matrix related to gyroscopic effects, contributes also in the case of a rigid body motion of the soft arm.
- $\int_L \boldsymbol{J}^T \widehat{\boldsymbol{f}}^T \mathbf{K} \, \mathrm{d}\alpha = \mathbb{K}$, the 6×6 discretized stiffness matrix.
- $\int_L \boldsymbol{J}^T \mathbf{g}_{ext} \, \mathrm{d}\alpha = \mathbb{F}$, the 6×1 vector of generalized applied forces. It also includes actuation loads and gravity field.

Dynamic model

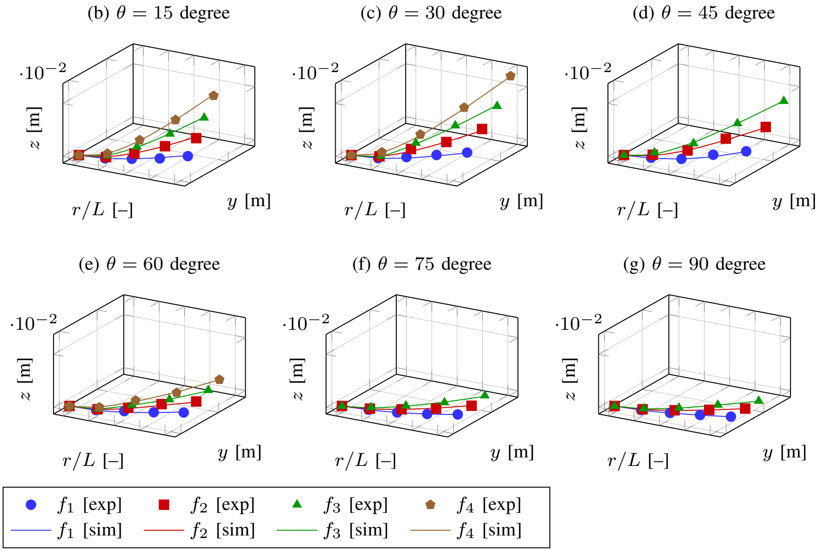
$$\mathbb{M}(\alpha, \boldsymbol{f})\ddot{\boldsymbol{f}} + \left(\mathbb{C}_1\left(\alpha, \boldsymbol{f}, \dot{\boldsymbol{f}}\right) - \mathbb{C}_2(\alpha, \boldsymbol{f}, \dot{\boldsymbol{f}})\right)\dot{\boldsymbol{f}} - \mathbb{K}\boldsymbol{f} = \mathbb{F}$$

The Princeton Experiment

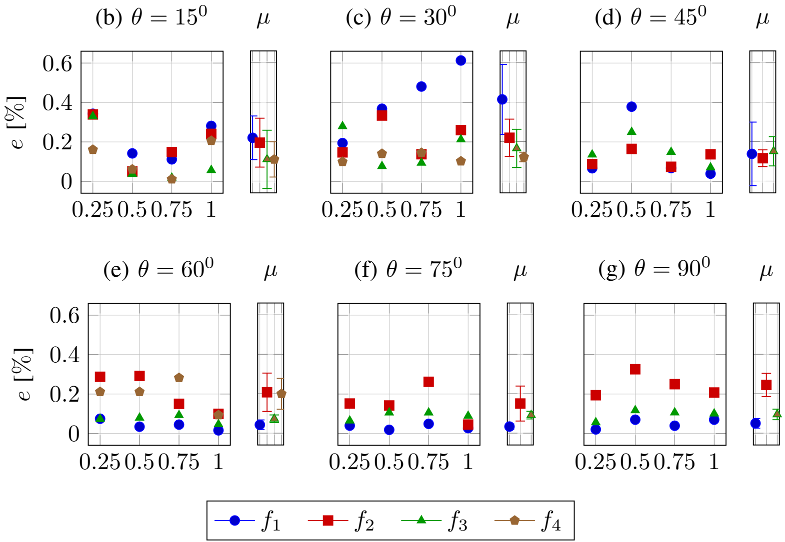


- θ: 15, 30, 45, 60, 75, 90 degree
- $\boldsymbol{f}$: 4.448, 8.896, 13,344, 17.792 N

The Princeton Experiment



The Princeton Experiment



Part C:
Piecewise constant curvature as special case of the deformation space formulation

Looking for simplified models!

- Under reference external forces
- Using a particular actuation source (tendons / pneumatic actuation)
- We look for experimental prototypes and we see that they exhibit some relevant shape configuration

Constant moment applied at the tip of the robotic arm

Figure: Cantilever soft arm subject to a torque $\boldsymbol{\tau}_y$ at its free end

Grazioso, Di Gironimo, Siciliano "Analytic solutions for the static equilibrium configurations of externally loaded cantilever soft robotic arms" *ROBOSOFT*, 2018

Constant moment applied at the tip of the robotic arm (cont'd)

Equations

Kinematic equations	$\mathbf{H}'(\alpha) = \mathbf{H}(\alpha)(\boldsymbol{f})^{\sim}$
Material constitutive law	$\boldsymbol{\sigma}(\alpha) = \mathbf{K}(\alpha)\boldsymbol{f}(\alpha)$
Boundary conditions	$\delta\boldsymbol{h}(L)\left(\mathbf{K}(L)\boldsymbol{f}(L) - \boldsymbol{g}_{ext}(L)\right) - \delta\boldsymbol{h}(0)\left(\mathbf{K}(0)\boldsymbol{f}(0) - \boldsymbol{g}_{ext}(0)\right) = 0$
Statics	$\boldsymbol{\sigma}'(\alpha) - \widehat{\boldsymbol{f}}^{T}(\alpha)\boldsymbol{\sigma}(\alpha) = \boldsymbol{g}_{ext}(\alpha)$

Constant moment applied at the tip of the robotic arm (cont'd)

Boundary conditions

$$\boldsymbol{\sigma}(L) = \boldsymbol{K}(L)\boldsymbol{f}(L) = \boldsymbol{g}_{ext}(L)$$

Constant moment applied at the tip of the robotic arm (cont'd)

Deformation field

Since the stiffness matrix is constant over the arm, the equilibrium equations in the static configuration become

$$\boldsymbol{K}\boldsymbol{f}' - \widehat{\boldsymbol{f}^0}^T \boldsymbol{K}\boldsymbol{f} = \boldsymbol{0}_{6\times 1}$$

In this case, the solution for the deformation field can be expressed in closed form and it is given by

$$\boldsymbol{f}(\alpha) = \boldsymbol{K}^{-1}\boldsymbol{F}(\alpha)\boldsymbol{K}\boldsymbol{f}_0$$

where $\boldsymbol{f}_0$, the deformation at $\alpha = 0$, is a constant of integration and

$$\boldsymbol{F}(\alpha) = \begin{bmatrix} \boldsymbol{L}^T(\alpha) & \boldsymbol{0}_{3\times 3} \\ \left(\boldsymbol{T}_{SO(3)}(\alpha\boldsymbol{f}^0_\omega)\,\alpha\boldsymbol{f}^0_u\right)^{\sim}\boldsymbol{L}^T(\alpha) & \boldsymbol{L}^T(\alpha) \end{bmatrix}$$

with $\boldsymbol{L}(\alpha) = \exp_{SO(3)}\left(\alpha\boldsymbol{f}^0_\omega\right)$.

Constant moment applied at the tip of the robotic arm (cont'd)

Deformation field

The boundary conditions become

$$\boldsymbol{\sigma}(L) = \boldsymbol{K}\boldsymbol{K}^{-1}\boldsymbol{F}(L)\boldsymbol{K}\boldsymbol{f}_0 = \boldsymbol{g}_{ext}(L)$$

such that the constant of integration $\boldsymbol{f}_0$ is given by

$$\boldsymbol{f}_0 = \boldsymbol{K}^{-1}(\boldsymbol{F}(L))^{-1}\boldsymbol{g}_{ext}(L)$$

Therefore, the solution for the deformation field reads

$$\boldsymbol{f}(\alpha) = \boldsymbol{K}^{-1}\boldsymbol{F}(\alpha)(\boldsymbol{F}(L))^{-1}\boldsymbol{g}_{ext}(L)$$

Constant moment applied at the tip of the robotic arm (cont'd)

Deformation field

In the special cases of pure bending/torsion solicitations, the external forces are given by

$$\boldsymbol{g}_{ext,u}(L) = \mathbf{0}_{3\times 1}$$
$$\boldsymbol{g}_{ext,\omega}(L) = \tau \boldsymbol{a}$$

where $\tau \in \mathbb{R}$ and $\boldsymbol{a} \in \mathbb{R}^3$ is an arbitrary vector. For an initially straight arm, we have $\boldsymbol{F}(\alpha)(\boldsymbol{F}(L))^{-1} = \boldsymbol{I}_{6\times 6}$. Hence, the deformation field becomes

$$\boldsymbol{f} = \boldsymbol{K}^{-1} \begin{bmatrix} \mathbf{0}_{3\times 1} \\ \tau \boldsymbol{a} \end{bmatrix}$$

Constant moment applied at the tip of the robotic arm (cont'd)

Deformation field

Thus, it results that $\boldsymbol{f}$ is constant along the continuum arm. The solution reads

$$\begin{bmatrix} \gamma \\ \boldsymbol{\kappa} \end{bmatrix} = \begin{bmatrix} \mathbf{0}_{3\times 1} \\ \boldsymbol{K}_{\omega\omega}^{-1}(\tau \boldsymbol{a}) \end{bmatrix}$$

where $\boldsymbol{K}_{\omega\omega} = \mathrm{diag}(GJ, EI_y, EI_z)$ contains the torsional and bending stiffnesses of the cross section.

Constant moment applied at the tip of the robotic arm (cont'd)

Position field

The position and orientation fields are obtained by solving the kinematic equations. Since the deformation field obtained above involves constant deformations, the kinematic equations can be integrated analitically and the solution for the $SE(3)$ field is given by

$$\boldsymbol{H}(\alpha) = \boldsymbol{H}_0 \exp_{SE(3)}(\alpha \boldsymbol{f})$$

where $\boldsymbol{H}_0 = \mathcal{H}(\boldsymbol{R}_0, \boldsymbol{u}_0)$ is a constant of integration and $\exp_{SE(3)}(\cdot)$ is the exponential map on $SE(3)$. Explicitly, it reads

$$\begin{aligned}
\boldsymbol{u}(\alpha) &= \boldsymbol{u}_0 + \boldsymbol{R}_0 \boldsymbol{T}^T_{SO(3)}(\alpha(\boldsymbol{f}^0_\omega + \boldsymbol{\kappa}))\alpha(\boldsymbol{f}^0_u + \gamma) \\
\boldsymbol{R}(\alpha) &= \boldsymbol{R}_0 \exp_{SO(3)}(\alpha(\boldsymbol{f}^0_\omega + \boldsymbol{\kappa}))
\end{aligned}$$

Since we are considering initially straight arm, we have that $\boldsymbol{u}_0 = \mathbf{0}_{3\times1}$, $\boldsymbol{R}_0 = \mathbf{I}_{3\times3}$ and $\boldsymbol{f}^0_u = [1 \;\; 0 \;\; 0]^T$, $\boldsymbol{f}^0_\omega = \mathbf{0}_{3\times1}$.

Constant moment applied at the tip of the robotic arm (cont'd)

Position field

In the case of the figure above, $\boldsymbol{g}_{ext,\omega}(L) = \tau[0 \;\; 1 \;\; 0]^T$. According to the deformation field solution, the deformations are given by

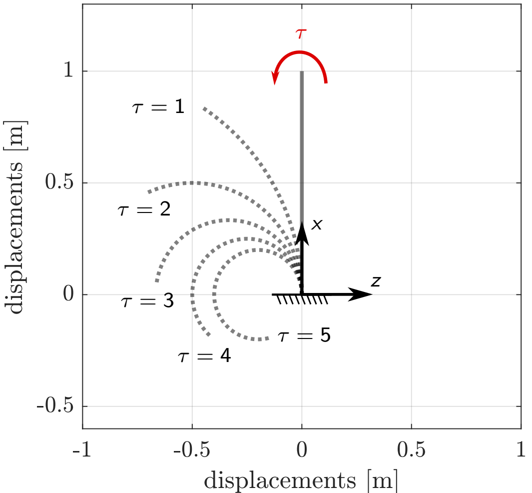
$$\boldsymbol{\gamma} = [0 \;\; 0 \;\; 0]^T$$
$$\boldsymbol{\kappa} = [0 \;\; \kappa_y \;\; 0]^T$$

where $\kappa_y = \tau_y/(EI_y)$. Indeed, according to the $SE(3)$ field solution, the position and rotation fields are given by

$$\boldsymbol{u}(\alpha) = \begin{bmatrix} \frac{1}{\kappa_y}\sin(\alpha\kappa_y) \\ 0 \\ -\frac{1}{\kappa_y}(1-\cos(\alpha\kappa_y)) \end{bmatrix}$$
$$\boldsymbol{R}(\alpha) = \begin{bmatrix} \cos(\alpha\kappa_y) & 0 & \sin(\alpha\kappa_y) \\ 0 & 1 & 0 \\ -\sin(\alpha\kappa_y) & 0 & \cos(\alpha\kappa_y) \end{bmatrix}$$

Constant moment applied at the tip of the robotic arm (cont'd)

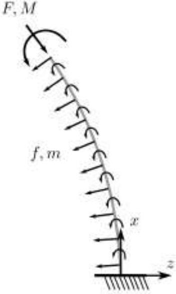
By considering $EI_y = 1\,\text{Nm}^2$ and $L = 1\,\text{m}$, the soft arm's steady-state shape, for bending tip loads $\tau = 1, 2, 3, 4, 5$ N m, is:



Constant moment applied at the tip of the robotic arm (cont'd)

If a constant moment is applied at the tip of a robotic arm $\rightarrow$ the mechanics predict a constant curvature result !

Tendon–driven planar soft continuum arm



(a) Full coupled model

(b) Point moment model

Assumptions:

- Tendon modeled as an extensible string which can not support internal moments or shear forces, but only tension T
- Frictionless interaction between the tendon and the channel $\rightarrow$ T is constant
- Location of the tendons is not varying under the applied loads
- Constant cross-section properties (mass and stiffness properties)

  $\rightarrow$ **Equivalence between full coupled model (a) and point moment model (b).**

Soft continuum arm with constant curvature

- $\mathbf{f}_u = [1\;0\;0]^T$
- $\mathbf{f}_\omega = [0\;0\;\kappa]^T$

$$\xrightarrow{FKINE} \qquad \xleftarrow{IKINE}$$

$$\mathbf{H}(\alpha) = \begin{bmatrix} \cos(\alpha\kappa) & -\sin(\alpha\kappa) & 0 & \frac{1}{\kappa}\sin(\alpha\kappa) \\ \sin(\alpha\kappa) & \cos(\alpha\kappa) & 0 & \frac{1}{\kappa}(1-\cos(\alpha\kappa)) \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Soft continuum arm with constant curvature

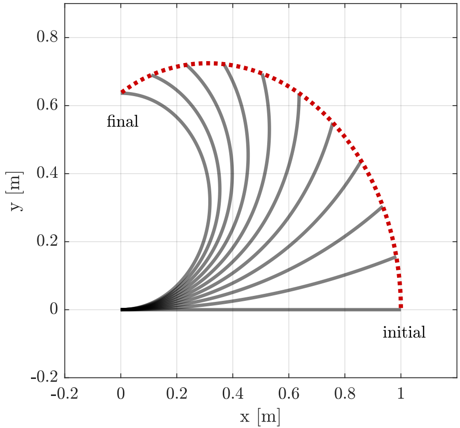


Figure: Whole arm shape configuration under a desired circular end point trajectory. $L = 1\,\mathrm{m}$; $\kappa = \pi/10 : \pi/10 : \pi$ m^{-1}.

An alternative approach for kinematic shape recostruction

Kinematics using the exponential map

$$\mathbf{f} \in \mathbb{R}^6 \mapsto \mathbf{H}(\alpha) = \exp_{SE(3)}(\alpha \mathbf{f}) \in SE(3)$$

- $\exp_{SE(3)}(\mathbf{f}) = \begin{bmatrix} \exp_{SO(3)}(\mathbf{f}_\omega) & \mathbf{T}^T_{SO(3)}(\mathbf{f}_\omega)\mathbf{f}_u \\ \mathbf{0}_{1\times 3} & 1 \end{bmatrix}$
- $\exp_{SO(3)}(\mathbf{f}_\omega) = \mathbf{I}_{3\times 3} + \alpha(\mathbf{f}_\omega)\widetilde{\mathbf{f}}_\omega + \frac{\beta(\mathbf{f}_\omega)}{2}\widetilde{\mathbf{f}}^2_\omega$
- $\mathbf{T}_{SO(3)}(\mathbf{f}_\omega) = \mathbf{I}_{3\times 3} - \frac{\beta(\mathbf{f}_\omega)}{2}\widetilde{\mathbf{f}}_\omega + \frac{1-\alpha(\mathbf{f}_\omega)}{\|\mathbf{f}_\omega\|^2}\widetilde{\mathbf{f}}^2_\omega$

$$\alpha(\mathbf{f}_\omega) = \frac{\sin(\|\mathbf{f}_\omega\|)}{\|\mathbf{f}_\omega\|} \qquad \beta(\mathbf{f}_\omega) = 2\frac{1-\cos(\|\mathbf{f}_\omega\|)}{\|\mathbf{f}_\omega\|^2}$$

Kinematics using the exponential map (cont'd)

$$\mathbf{f} \in \mathbb{R}^6 \mapsto \mathbf{H}(\alpha) = \exp_{SE(3)}(\alpha \mathbf{f}) \in SE(3)$$

- $\exp_{SE(3)}(\mathbf{f}) = \begin{bmatrix} \exp_{SO(3)}(\mathbf{f}_\omega) & \mathbf{T}^T_{SO(3)}(\mathbf{f}_\omega)\mathbf{f}_u \\ \mathbf{0}_{1\times 3} & 1 \end{bmatrix}$
- $\exp_{SO(3)}(\mathbf{f}_\omega) = \mathbf{I}_{3\times 3} + \alpha(\mathbf{f}_\omega)\widetilde{\mathbf{f}}_\omega + \frac{\beta(\mathbf{f}_\omega)}{2}\widetilde{\mathbf{f}}^2_\omega$
- $\mathbf{T}_{SO(3)}(\mathbf{f}_\omega) = \mathbf{I}_{3\times 3} - \frac{\beta(\mathbf{f}_\omega)}{2}\widetilde{\mathbf{f}}_\omega + \frac{1-\alpha(\mathbf{f}_\omega)}{\|\mathbf{f}_\omega\|^2}\widetilde{\mathbf{f}}^2_\omega$

$$\alpha(\mathbf{f}_\omega) = \frac{\sin(\|\mathbf{f}_\omega\|)}{\|\mathbf{f}_\omega\|} \qquad \beta(\mathbf{f}_\omega) = 2\frac{1-\cos(\|\mathbf{f}_\omega\|)}{\|\mathbf{f}_\omega\|^2}$$

Problems

- Exponential mapping involves **trascendental functions** at each time step
- There exists conditions in which soft continuum robots does not undergone large deformations, but **low deformations**, even if yet finite.

Kinematics using the Cayley map

Cayley map: definition

$$\begin{aligned}\mathrm{cay}(\widetilde{\mathbf{a}}) = (\mathbf{I} + \widetilde{\mathbf{a}})(\mathbf{I} - \widetilde{\mathbf{a}})^{-1} = \\ = (\mathbf{I} - \widetilde{\mathbf{a}})^{-1}(\mathbf{I} + \widetilde{\mathbf{a}}) = \mathbf{A}\end{aligned}$$

- $\boldsymbol{A} \in G$, Lie group
- $\widetilde{\mathbf{a}} \in \mathfrak{g}$, its corresponding Lie algebra
- $\mathbf{I}$, identity matrix

Geometric interpretation

$$\exp(\widetilde{\mathbf{a}}) \approx \mathrm{Pade}(\widetilde{\mathbf{a}}) = (\mathbf{I} - \frac{1}{2}\widetilde{\mathbf{a}})^{-1}(\mathbf{I} + \frac{1}{2}\widetilde{\mathbf{a}})$$

- **Cay map: Pade diagonal approximation of the exp map**

Comparisons

... using the Exponential map

$$\mathbf{f} \in \mathbb{R}^6 \mapsto \mathbf{H}(\alpha) = \exp_{SE(3)}(\alpha\mathbf{f}) \in SE(3)$$

- $\exp_{SE(3)}(\mathbf{f}) = \begin{bmatrix} \exp_{SO(3)}(\mathbf{f}_\omega) & \mathbf{T}^T_{SO(3)}(\mathbf{f}_\omega)\mathbf{f}_u \\ \mathbf{0}_{1\times 3} & 1 \end{bmatrix}$
- $\exp_{SO(3)}(\mathbf{f}_\omega) = \mathbf{I}_{3\times 3} + \alpha(\mathbf{f}_\omega)\widetilde{\mathbf{f}}_\omega + \frac{\beta(\mathbf{f}_\omega)}{2}\widetilde{\mathbf{f}}^2_\omega$
- $\mathbf{T}_{SO(3)}(\mathbf{f}_\omega) = \mathbf{I}_{3\times 3} - \frac{\beta(\mathbf{f}_\omega)}{2}\widetilde{\mathbf{f}}_\omega + \frac{1-\alpha(\mathbf{f}_\omega)}{\|\mathbf{f}_\omega\|^2}\widetilde{\mathbf{f}}^2_\omega$

$$\alpha(\mathbf{f}_\omega) = \frac{\sin(\|\mathbf{f}_\omega\|)}{\|\mathbf{f}_\omega\|} \qquad \beta(\mathbf{f}_\omega) = 2\frac{1-\cos(\|\mathbf{f}_\omega\|)}{\|\mathbf{f}_\omega\|^2}$$

... using the Cayley map

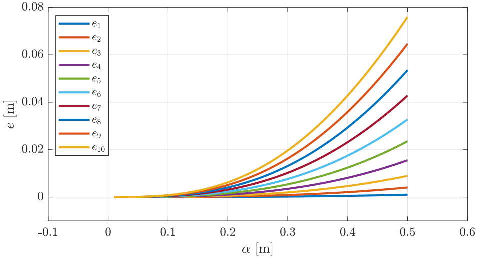
$$\mathbf{f} \in \mathbb{R}^6 \mapsto \mathbf{M}(\alpha) = \mathrm{cay}_{SE(3)}(\alpha\mathbf{f}) \in SE(3)$$

- $\mathrm{cay}_{SE(3)}(\mathbf{f}) = \begin{bmatrix} \mathrm{cay}_{SO(3)}(\mathbf{f}_\omega) & \mathrm{dcay}_{SO(3)}(\mathbf{f}_\omega)\mathbf{f}_u \\ \mathbf{0}_{1\times 3} & 1 \end{bmatrix}$
- $\mathrm{cay}_{SO(3)}(\alpha\mathbf{f}_\omega) = \boldsymbol{I}_{3\times 3} + \frac{4\kappa}{4\kappa^2+\|\boldsymbol{f}_\omega\|^2}(\alpha\widetilde{\mathbf{f}}_\omega + \alpha\widetilde{\mathbf{f}}^2_\omega)$; k=1,1/2
- $\mathrm{dcay}_{SO(3)}(\alpha\mathbf{f}_\omega) = \mathrm{cay}_{SO(3)}(\alpha\mathbf{f}_\omega) + \boldsymbol{I}_{3\times 3}$

Difference between $\mathbf{H}(\alpha)$ and $\mathbf{M}(\alpha)$?

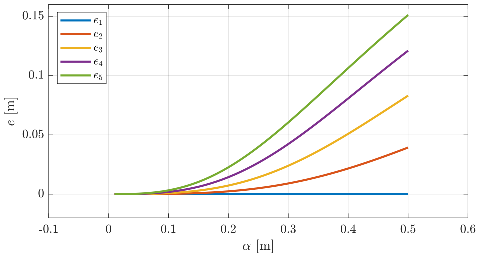
Comparisons (cont'd)

- **Planar bending**: $\mathbf{f} = [1 \ \ 0 \ \ 0 \ \ 0 \ \ 0 \ \ K]^T$



Ten cases with increasing curvature. $1 : K = \pi/10\,\mathrm{m}^{-1}; \ldots; 10\!: \ K = \pi\,\mathrm{m}^{-1}$

- **Twist motion**: $\mathbf{f} = [1 \ \ 0 \ \ 0 \ \ T \ \ 0 \ \ K]^T$



Five cases with increasing curvature and fixed torsion. $1 : T = 3\,\mathrm{m}^{-1}$ and $K = 0; \ldots; 5\!:$ $T = 3\,\mathrm{m}^{-1}$ and $K = 2\pi\,\mathrm{m}^{-1}$

Comparisons (cont'd)

- Geometric mechanics of soft robots
- Cayley transform for parametrization of soft robot kinematics, in case of low, yet finite deformations $\rightarrow$ advantages also for compound robots; finite element models using Cayley map as shape function
- Cayley map for recovering the geometrically exact motion, i.e. $\exp_{SE(3)}(\widetilde{\boldsymbol{f}}) = \mathrm{cay}_{SE(3)}(\widetilde{\zeta})$ if we consider the change of deformation coordinates

$$\begin{bmatrix} \zeta_u \\ \zeta_\omega \end{bmatrix} = \begin{bmatrix} \lambda \boldsymbol{I}_{3\times 3} & \tau\lambda' \boldsymbol{I}_{3\times 3} \\ \boldsymbol{0}_{3\times 3} & \lambda \boldsymbol{I}_{3\times 3} \end{bmatrix} \begin{bmatrix} \boldsymbol{f}_u \\ \boldsymbol{f}_\omega \end{bmatrix}$$

with

$$\lambda = 2\kappa \frac{\tan(||\boldsymbol{f}_\omega||/2)}{||\boldsymbol{f}_\omega||}$$

$$\tau = \boldsymbol{u}\boldsymbol{f}_\omega = \boldsymbol{T}^T_{SO(3)}(\boldsymbol{f}_\omega)\boldsymbol{f}_u\boldsymbol{f}_\omega$$

SimSOFT: a finite element solver for soft robots dynamics

SimSOFT

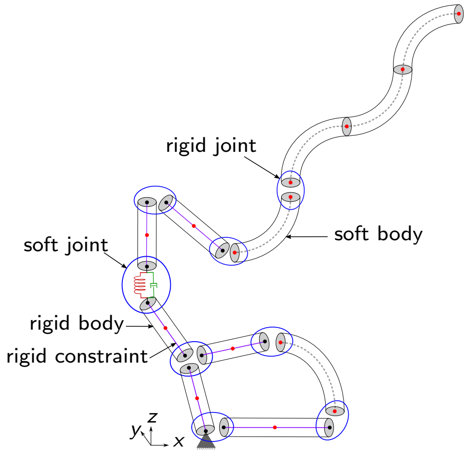
Input

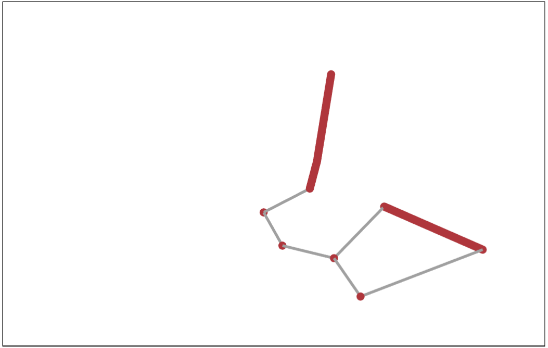
- Geometry description and behavior (rigid / soft)
- Joints
- Actuation sources
- Planner
- Boundary conditions
- Solving parameters

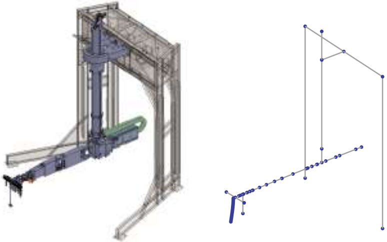
Output

- Positions, velocities and accelerations of nodes and joints
- Forces and torques at boundaries and joint locations
- Stresses and strains of the elements
- Animations

Soft articulated robot composed by soft continuum elements





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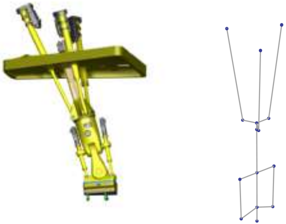

Conclusions

- Geometric approach for modeling soft robots
- Soft continuum robots modeled as Cosserat rod elements
- The deformation space formulation for soft robots dynamics
- The special case of constant curvature
- An alternative approach based on Cayley map
- A finite element solver for soft robots dynamics and relative applications

References

[1] S. Grazioso, G. Di Gironimo, B. Siciliano "A geometrically exact model for soft continuum robots: the finite element deformation space formulation", *Soft Robotics*, 2019.

[2] S. Grazioso, G. Di Gironimo, B. Siciliano "From differential geometry of curves to helical kinematics of continuum robots using exponential mapping", *International Symposium on Advances in Robot Kinematics*, 2018.

[3] S. Grazioso, G. Di Gironimo, B. Siciliano "Analytic solutions for the static equilibrium configurations of externally loaded cantilever soft robotic arms", *2018 IEEE International Conference on Soft Robotics*, 2018.

[4] S. Grazioso, G. Di Gironimo, B. Siciliano "On the use of Cayley transform for kinematic shape reconstruction of soft continuum robots", *2019 International Symposium on Robotics Research*, 2019.

Modeling of Soft Robots

Course on Modeling and Control of Soft Robots
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