Game theory and Network systems
Summer school SIDRA 2021

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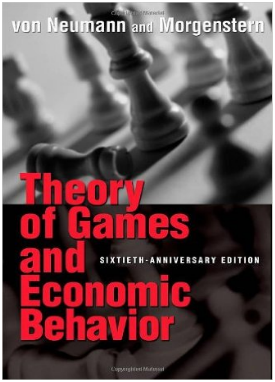
Outline of the course

1. Non-cooperative strategic games
2. Existence of Nash equilibria and potential games
3. Network games
4. Best response and noisy best response dynamics
5. Quadratic games
6. Supermodular games
7. Population games and evolutionary dynamics
8. Learning in games
9. Optimal targeting
10. Network intervention and mechanism design

Lecture I: Non-cooperative strategic games

- Historical remarks
- Fundamental examples
- Formal definitions (game, strategy, best response, Nash equilibrium)
- Discrete and continuous models
- Examples from economy
- Existence of Nash equilibria

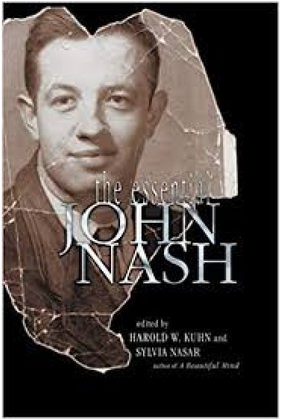
The birth of Game Theory



- 1713: Waldegrave, card games
- 1838: Cournot, theory of duopolies
- 1913: Zermelo, chess optimal strategy
- 1928: Von Neumann, dominant strategy
- 1944: Von Neumann and Mongenstern *Theory of games and economic behavior*
- game theory develops during cold war

Game Theory

John Forbes Nash (1928-2015)



- 1950: *Non-cooperative games*
  PhD Thesis (Princeton)
- 1950: *Equilibrium points in n-person games* PNAS
- 1950: *The bargaining problem*
  Econometrica
- 1951: *Non-cooperative games*
  Annals of Mathematics
- 1953: *Two-person Cooperative Games*
  Econometrica

  ...

- 1994 Nobel Price for Economics: Harsanyi, Nash, and Salten

Impact of Game Theory

in Economics:

- 1994 Nobel Price for Economics: Harsanyi, Nash, and Salten
- 2005 Nobel Price for Economics: Shelling and Aumann
- 2007 Nobel Price for Economics: Hurwicz, Maskin, and Myerson
- 2012 Nobel Price for Economics: Roth and Shapley
- 2014 Nobel Price for Economics: Tirole

and many other fields:

- Political Sciences: Downs (1957), Cuban missile crisis (1962)
- Biology: evolutionary game theory, John Maynard Smith (1970s)
- Computer Science: semantics, algorithmic mechanism design,
- Engineering: multi-agent systems, Internet, networks
- Machine Learning, Artificial Intelligence: GANs, AI planning, ...

The Prisoner Dilemma

Frank and Cora get arrested and are accused of a crime.

They get questioned separately and offered the same deal:

- if they both confess, each of them gets sentenced to 3 years
- if only one confesses (s)he gets free, the other one gets 5 years
- if neither of them confesses, they get 1 year each (a minor crime)

Formalizing the Prisoner Dilemma

Cora's choice

Frank's choice

	CONFESS	SILENT
CONFESS	-3	0
SILENT	-5	-1

	CONFESS	SILENT
CONFESS	-3	-5
SILENT	0	-1

Dominant Strategies in Prisoner Dilemma

F \ C	CONFESS	SILENT
CONFESS	-3, -3	0, -5
SILENT	-5, 0	-1, -1

- whatever Cora does, Frank is better off confessing
- Confess is a dominant strategy for Frank

Dominant Strategies in Prisoner Dilemma

F \ C	CONFESS	SILENT
CONFESS	-3, -3	0, -5
SILENT	-5, 0	-1, -1

- whatever Frank does, Cora is better off confessing
- Confess is a dominant strategy for Cora

Nash Equilibrium in Prisoner Dilemma

F \ C	CONFESS	SILENT
CONFESS	-3 (F), -3 (C)	0 (F), -5 (C)
SILENT	-5 (F), 0 (C)	-1 (F), -1 (C)

- for both confessing is dominant strategy
- both have no incentive to deviate unilaterally
- if both remained silent, both would be better off

Nash Equilibrium in Prisoner Dilemma

F \ C	CONFESS	SILENT
CONFESS	F: -3, C: -3	F: 0, C: -5
SILENT	F: -5, C: 0	F: -1, C: -1

Key assumptions:

- game played only once
- Cora and Frank do not communicate
- their interest is just to minimize their time in prison

Modified Prisoner Dilemma

F \ C	CONFESS	SILENT
CONFESS	−3, −3	-2, −5
SILENT	−5, -2	−1, −1

- if Cora confesses, Frank is better off confessing
- if Cora remains silent, Frank is better off remaining silent
- no dominant strategy, best response depends on Cora's choice

Modified Prisoner Dilemma

F \ C	CONFESS	SILENT
CONFESS	-3 / -3	-5 / -2
SILENT	-2 / -5	-1 / -1

- if Frank confesses, Cora is better off confessing
- if Frank remains silent, Cora is better off remaining silent
- no dominant strategy, best response depends on Frank's choice

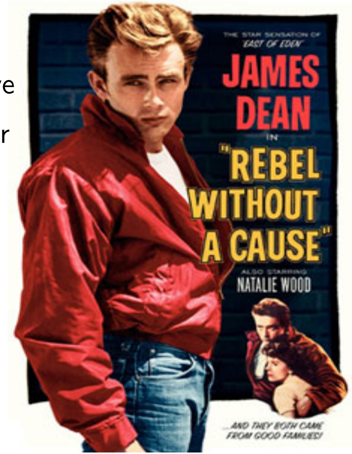
Modified Prisoner Dilemma

F \ C	CONFESS	SILENT
CONFESS	F: -3, C: -3	F: -2, C: -5
SILENT	F: -5, C: -2	F: -1, C: -1

- no dominant strategy
- best responses depend on the other player's choice
- 2 Nash equilibria: both have no incentive to deviate unilaterally

The Game of Chicken

- two cars in collision course
- each driver can go straight or swerve
- the one that swerves while the other goes straight is the “chicken”
- the one that goes straight while the other swerves is the brave
- if both go straight they both die
- if both swerve they both lose some appeal but avoid main damage



The Game of Chicken

	SWERVE	STRAIGHT
SWERVE	0, 0	-1, 1
STRAIGHT	1, -1	-100, -100

- if the other one goes straight better swerve
- if the other one swerves better go straight
- best response depends on the other player's choice
- 2 Nash equilibria: both have no incentive to deviate unilaterally

The Game of Chicken

	SWERVE	STRAIGHT
SWERVE	0, 0	-1, 1
STRAIGHT	1, -1	-100, -100

- 1959 B. Russell “Common Sense and Nuclear Warfare”
- 1973 Maynard Smith and Price, “The logic of animal conflict”
- 1964 S. Kubrik “Dr. Strangelove”

Basic definitions

- $\mathcal{V}$ set of players
- $\mathcal{A}$ set of actions
- $\mathcal{X} = \mathcal{A}^{\mathcal{V}}$ set of configurations
- $u_i : \mathcal{X} \to \mathbb{R}$ utility function
- $x \in \mathcal{X}$ action configuration
- x_i action played by player i
- x_{-i} vector of actions played by everyone but i
- utility of player i when each player j plays action x_j:

$$u_i(x_i, x_{-i}) = u_i(x)$$

$(\mathcal{V}, \mathcal{A}, \{u_i\}_{i\in\mathcal{V}})$ is called a (strategic form) game

Best response and Nash equilibrium

- each player i to be interpreted as a rational agent choosing action x_i so as to maximize her utility $u_i(x_i, x_{-i})$
- player i's utility $u_i(x_i, x_{-i})$ depends not only on her action x_i but also on the actions of the rest of the players x_{-i}
- rational choice for a player: best response

$$\mathcal{B}_i(x_{-i}) = \underset{x_i \in \mathcal{A}}{\operatorname{argmax}}\, u_i(x_i, x_{-i})$$

Definition: A (pure strategy) Nash equilibrium (NE) for the game $(\mathcal{V}, \mathcal{A}, \{u_i\}_{i\in\mathcal{V}})$ is a configuration $x^* \in \mathcal{A}^{\mathcal{V}}$ such that

$$x_i^* \in \mathcal{B}_i(x_{-i}^*)\,, \qquad \forall i \in \mathcal{V}\,.$$

- NE x^* is a configuration from which no player has strict incentive to unilaterally change her action
- $\mathcal{N}$ the set of NE of a game
- NE x^* is said strict if $|\mathcal{B}_i(x_{-i}^*)| = 1$ for every player i

Price of Anarchy (PoA)

- The welfare of a game $(\mathcal{V}, \mathcal{A}, \{u_i\}_{i\in\mathcal{V}})$ in a configuration x is the sum of utilities for that configuration:

$$U(x) = \sum_{i\in\mathcal{V}} u_i(x)$$

- We can consider the optimization problem

$$\max_{x\in\mathcal{X}} U(x)$$

A maximum x of U is called a social optimum

- Nash equilibria may not be social optima and viceversa.

Definition: The price of Anarchy is defined as

$$PoA = \frac{\max_{x\in\mathcal{X}} U(x)}{\min_{x\in\mathcal{N}} U(x)} \geq 1$$

It measures how bad is the worse Nash equilibrium with respect to a social optimum

Two-player games

- when $\mathcal{V} = \{1, 2\}$: two utility functions $u_i(r, s)$, for $i = 1, 2$ (r = action played by i and s = action played by opponent of i)

- Two-player symmetric game:

$$u_1(r, s) = u_2(r, s) = \phi(r, s)$$

(the role of the two players is exchangeable).

- Table representation: rows $\leftrightarrow$ actions of player 1, columns $\leftrightarrow$ action of player 2, (r, s)-th entry displays the pair $u_1(r, s), u_2(s, r)$

- 2×2 symmetric game

	-1	$+1$
-1	a,a	d,c
$+1$	c,d	b,b

Example 1: Coordination game

$$a > c\,, \qquad b > d$$

	-1	+1
-1	a,a	d,c
+1	c,d	b,b

- Best response for both: copy the other player

$$\mathcal{B}_i(-1) = -1\,, \qquad \mathcal{B}_i(+1) = +1\,, \qquad i = 1, 2$$

- Two Nash equilibria: $\mathcal{N} = \{(-1,-1), (+1,+1)\}$
- Two NE not equally good: $a > b \Rightarrow (-1,-1)$ payoff dominant
- The social optimum is $(-1,-1)$, $PoA = \frac{a}{b}$

Example 2: Prisoner's dilemma

$$a > c, \qquad b < d$$

	-1	+1
-1	a,a	d,c
+1	c,d	b,b

- Interpretation: Action $-1 \leftrightarrow$ Betraying, Action $+1 \leftrightarrow$ Silent
- Not a coordination game!
- Best response for both:

$$\mathcal{B}_i(+1) = \mathcal{B}_i(-1) = -1 \qquad i = 1, 2$$

- Betraying $= -1$ is a dominant action for both
- One Nash equilibrium: $x^* = (-1, -1)$
- In the example: the social optimum is $(1, 1)$ and $PoA = \frac{b}{a}$

Example 3: Anti-coordination game

$$a < c\,, \qquad b < d$$

	-1	+1
-1	a,a	d,c
+1	c,d	b,b

- Best response for both: do the opposite of the other player

$$\mathcal{B}_i(-1) = +1\,, \qquad \mathcal{B}_i(+1) = -1\,, \qquad i = 1, 2$$

- Two Nash equilibria: $\mathcal{N} = \{(-1, +1), (+1, -1)\}$
- Notable example: Game of Chicken (Hawk-Dove game) where $a > d$. ($-1 =$ Swerve, $+1=$ Straight)

Example 4: Discoordination game

$$a > c\,, \qquad d > b$$

	-1	+1
-1	a,b	c,d
+1	c,d	a,b

- Discoordination game not symmetric
- Best response

$$\mathcal{B}_1(-1) = \mathcal{B}_2(1) = -1\,, \qquad \mathcal{B}_1(+1) = \mathcal{B}_2(-1) = +1\,.$$

- No (pure strategy) Nash equilibrium!
- Notable example: the matching penny game where $a = d = 1$ and $b = c = -1$. Instance of a 0 sum game.

Example 5: Rock-Scissor-Paper

- Action space $\mathcal{A} = \{R, S, P\}$

	R	S	P
R	0,0	1,-1	-1,1
S	-1,1	0,0	1,-1
P	1,-1	-1,1	0,0

- Zero-sum game
- No (pure strategy) Nash equilibrium!

Example 6: Economic models

A firm producing and selling a certain good

- Market consumers determine an elastic demand: $q(p)$ is the quantity sold at price p
- Constant marginal cost of production c
- Utility $u(p) = pq(p) - cq(p) = (p - c)q(p)$
- Assume $q(p) = \max(0, K - p)$ with $K > c$
- $u(p)$ assumes its maximum value at $p^{mon} = \frac{K+c}{2}$ monopoly price

How can we model competition among firms?

Example 6a: Cournot competition

A. A. Cournot (1801-1877) philosopher, mathematician

n firms competing in a market, producing and selling the same good. They fix the quantities.

- $x_i \in A = [0, +\infty)$ amount of good produced by firm i
- $p(\sum_i x_i)$ price of good *(inverse demand function)*, c marginal cost (the same for both firms)
- $u_i(x_i, x_{-i}) = x_i p(\sum_i x_i) - cx_i$

Example 6a: Cournot competition (two firms)

$u_i(x_1, x_2) = x_i p(x_1 + x_2) - cx_i$

Assume $p(x_1 + x_2) = \max(0, K - x_1 - x_2)$.

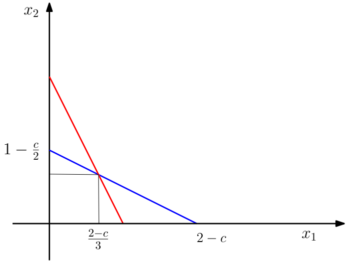
Best response:

$$\mathcal{B}_i(x_{-i}) = \begin{cases} \frac{1}{2}(K - c - x_{-i}) & \text{if } x_{-i} \leq K - c \\ 0 & \text{if } x_{-i} > K - c. \end{cases}$$

Nash equilibria: $x_1^* = x_2^* = \frac{K-c}{3}$,

Price $c < p^{*cou} = \frac{K+2c}{3} < p^{mon}$

Equilibrium profit: $u_i(x_i, x_{-i}) > 0$



Example 6b: Bertrand competition

J. Bertrand (1822-1900) mathematician (number theory, probability, mechanics...)

Two firms competing in a market, producing and selling the same good. They fix the prices.

- $x_i \in A = [0, +\infty)$ price of good fixed by firm i
- Consumers by at the smallest price

$$u_i(x_i, x_{-i}) = \begin{cases} 0 & \text{if } x_i > x_{-i} \\ (x_i - c)\frac{q(x_i)}{2} & \text{if } x_i = x_{-i} \\ (x_i - c)q(x_i) & \text{if } x_i < x_{-i} \end{cases}$$

- $q(x) = \max(0, K - x)$ quantity of good sold at price x, c marginal cost (the same for both firms)

Example 6b: Bertrand competition

$$u_i(x_i, x_{-i}) = \begin{cases} 0 & \text{if } x_i > x_{-i} \\ (x_i - c)\dfrac{\max(0, K - x_i)}{2} & \text{if } x_i = x_{-i} \\ (x_i - c)\max(0, K - x_i) & \text{if } x_i < x_{-i} \end{cases}$$

- There is only one Nash equilibrium $x_1^* = x_2^* = c$
- Equilibrium price is $p^{*ber} = c$
- Firms make zero utility! (Bertrand paradox)

Example 6: Economic models

- $c = p^{*ber} < p^{*cou} < p^{mon} \Rightarrow$ Competition lowers prices
- Bertrand model lower prices at their minimum value.

Many possible generalizations:

- Firms have different costs
- Goods produced are not completely substitutable
- Many firms and possibly many markets

Where are we?

The concept of pure Nash equilibrium poses a number of problems:

- Which conditions guarantee the existence of Nash equilibria?
- When Nash equilibria do not exist, is there an equilibrium concept that can suitably replace them?
- When Nash equilibria are not unique, can we say which of them the players are more likely to reach?

Next lecture

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2. Existence of Nash equilibria and potential games
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A bibliography

- J. Nash, *Non-cooperative games*, PhD Thesis, 1950
- M. J. Osborne, *An introduction to game theory*, 2009 (partial draft available online)