# Game Theory and Network Systems

## Summer School SIDRA 2021

## Lecture 2

## Existence of Equilibria and Potential Games

Giacomo Como and Fabio Fagnani
Department of Mathematical Sciences
Politecnico di Torino
Bertinoro, July 12-14, 2021

# Outline

- Existence results for Nash equilibria
- Mixed strategies and Nash's theorem
- Potential Games

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# Strategic Form Games

- $\mathcal{V}$ finite set of players
- $\mathcal{A}_i$ set of actions of player $i$
- $\mathcal{X} = \prod_{i \in \mathcal{V}} \mathcal{A}_i$ set of configurations
- $u_i : \mathcal{X} \to \mathbb{R}$ utility function of player $i$
- $x \in \mathcal{X}$ strategy profile or configuration
- $x_i \in \mathcal{A}_i$ action played by player $i$
- $x_{-i} \in \prod_{j \neq i} \mathcal{A}_j$ strategy profile of everyone but player $i$
- utility of player $i$ when each player $j$ plays action $x_j$:

$$u_i(x_i, x_{-i}) = u_i(x)$$

- $(\mathcal{V}, \{\mathcal{A}_i\}, \{u_i\})$ is called a (strategic form) game

# Best Response and Nash Equilibria

- rational choice for player $i$ given $x_{-i}$: best response

$$\mathcal{B}_i(x_{-i}) = \operatorname*{argmax}_{x_i \in \mathcal{A}_i} u_i(x_i, x_{-i})$$

- A pure strategy (P) Nash equilibrium (NE) is $x^* \in \mathcal{X}$ s.t.

$$x_i^* \in \mathcal{B}_i(x_{-i}^*) \,, \qquad \forall i \in \mathcal{V} \,.$$

- PNE $x^* \Leftrightarrow$ no player has strict incentive to unilaterally deviate
- PNE $x^*$ is said strict if $|\mathcal{B}_i(x_{-i}^*)| = 1$ for every player $i$
- $\mathcal{N} = \{$pure Nash equilibria$\}$
- $\mathcal{N}$ might be empty (e.g., discoordination, Rock-Scissor-Paper)
- PNE might not be unique (e.g., coordination, anti-coordination)

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When does a PNE exist? (And when is it unique?)

# Pure strategy Nash equilibria of continuous games

Continuous strategies: general results for existence/uniqueness

Theorem (Debreu, Glicksberg, Fan, '52)
Consider a game $(\mathcal{V}, \{\mathcal{A}_i\}, \{u_i\})$ such that for each $i \in \mathcal{V}$:

- $\mathcal{A}_i \subseteq \mathbb{R}^q$ is nonempty, compact, and convex;
- $u_i(x)$ is continuous in $x$ and concave in $x_i$ for all $x_{-i}$.

Then a pure strategy Nash equilibrium exists.

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Then a pure strategy Nash equilibrium exists.

- Example: Cournot oligopoly, concave non-increasing inverse demand $F(q)$ with $F(\overline{q}) = 0$, convex production costs $c_i(x_i)$

$$u_i(x) = x_i F(\sum_j x_j) - c_i(x_i)$$

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Then a pure strategy Nash equilibrium exists.

- quasi-concavity is sufficient ;
- [Rosen'65]: sufficient conditions for uniqueness of PNE: strictly diagonally concave game;
- does not apply to finite games: e.g., Matching Penny, Rock-Scissor-Paper, ...

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# Mixed strategies

Finite game $(\mathcal{V}, \{\mathcal{A}_i\}, \{u_i\})$

- Mixed strategy for a player $i$ is a probability distribution on $\mathcal{A}_i$:

$$z_i \in \mathcal{P}(\mathcal{A}_i)$$

- $z_{ia}$ = probability of choosing action $a \in \mathcal{A}_i$
- action $a \in \mathcal{A}_i \leftrightarrow$ pure strategy $z = \delta^a$ concentrated on $a$
- mixed strategy profile space: $\mathcal{Z} = \prod_i \mathcal{P}(\mathcal{A}_i)$
- multilinear utilities: expected values assuming independent plays:

$$u_i(z) = \sum_x \left(\prod_i z_{ix_i}\right) u_i(x) \qquad \forall z \in \mathcal{Z}, \forall i \in \mathcal{V}$$

- Definition: a mixed strategy Nash equilibrium is $z^* \in \mathcal{Z}$ s.t.

$$u_i(z^*) \geq u_i(z) \qquad \forall i \in \mathcal{V},\ \forall z \in \mathcal{Z} \text{ s.t. } z_{-i} = z^*_{-i}$$

# Existence of mixed strategy Nash equilibria for finite games

Theorem (Nash, 1950)

*Every finite game admits a mixed strategy Nash equilibrium.*

Proof idea:

- for every $z \in \mathcal{Z}$ consider the best response mapping:
  $z' = f(z)$ such that $z'_i \in \underset{\zeta \in \mathcal{P}(\mathcal{A}_i)}{\operatorname{argmax}} u_i(\zeta, z_{-i}) \ \forall i \in \mathcal{V}$
- Fixed point $z^* = f(z^*)$ is mixed strategy Nash equilibrium

Problems:

- $f$ is not well defined, there can be multiple maximum points
- $f$ may not be well-behaved and lacking a fixed point

Nash's alternative solutions:

- use Kakutami's fixed point theorem for set-valued maps
- use Brouwer's fixed point theorem for continuous functions

# Proof with Brouwer's fixed point theorem (Nash's thesis)

- For given $z_{-i}$, $u_i(z_i, z_{-i}) = \sum_{a \in \mathcal{A}_i} z_{ia} u_i(\delta^a, z_{-i})$ maximized by $z_i$'s supported on $\mathcal{B}_i(z_{-i}) = \operatorname{argmax}_{a \in \mathcal{A}_i} u_i(\delta^a, z_{-i})$

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- For $\lambda > 0$, define $f^\lambda : \mathcal{Z} \to \mathcal{Z}$ by

$$f_{ib}^\lambda(z) = \frac{\rho_{ib}^\lambda(z)}{\sum_{a \in \mathcal{A}_i} \rho_{ia}^\lambda(z)}$$

$$\rho_{ib}^\lambda(z) = \left[u_i(\delta^b, z_{-i}) - \max_a u_i(\delta^a, z_{-i}) + 1/\lambda\right]_+$$

(note that $\sum_{a \in \mathcal{A}_i} \rho_{ia}^\lambda(z) \geq 1/\lambda > 0$)

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- $f^\lambda : \mathcal{Z} \to \mathcal{Z}$ continuous, $\mathcal{Z}$ nonempty, compact, and convex

$$\text{Brouwer's fixed point theorem} \implies \exists\, z^\lambda = f^\lambda(z^\lambda) \in \mathcal{Z}$$

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- $\mathcal{Z}$ compact $\implies \exists\, \lambda_n \to +\infty$ such that $z^{\lambda_n} \to z^*$
- $z^*$ is mixed strategy Nash equilibrium

# Proof with Brouwer's fixed point theorem (Nash's thesis)

Last point proven by contradiction:

- $\exists i$ s.t. $z_i^* \notin \operatorname{argmax}_{\zeta \in \mathcal{P}(\mathcal{A}_i)} u_i(\zeta, z_{-i}^*)$
- $z_{ib}^* > 0$ for some $b \notin \operatorname{argmax}_{a \in \mathcal{A}_i} u_i(\delta^a, z_{-i}^*)$
- $u_i(\delta^b, z_{-i}^\lambda) - \max_a u_i(\delta^a, z_{-i}^\lambda) + 1/\lambda < 0$ for large $\lambda$ (continuity)
- $\rho_{ib}^\lambda(z^\lambda) = 0$ for large $\lambda$
- $z_b^\lambda = f(z^\lambda)_{ib} = \dfrac{\rho_{ib}^\lambda(z^\lambda)}{\sum_{a \in \mathcal{A}} \rho_{ia}^\lambda(z^\lambda)} = 0$ which is a contradiction.

# Outline

- Existence results for Nash equilibria
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# Potential games

- Definition: A game $(\mathcal{V}, \{\mathcal{A}_i\}, \{u_i\})$ is an (exact) potential game if there exists $\Phi : \mathcal{A}^{\mathcal{V}} \to \mathbb{R}$ (called potential function) such that

$$u_i(y_i, x_{-i}) - u_i(x_i, x_{-i}) = \Phi(y_i, x_{-i}) - \Phi(x_i, x_{-i}),$$

for every $x \in \mathcal{X}$, $i \in \mathcal{V}$, and $y_i \in \mathcal{A}$.

- In an exact potential game, for any configuration $x$, the utility variation incurred by player $i$ when changing action unilaterally is the same as the corresponding variation in the potential function

# Ordinal potential games

- Definition: A game $(\mathcal{V}, \{\mathcal{A}_i\}, \{u_i\})$ is an ordinal potential game if there exists $\Phi : \mathcal{A}^{\mathcal{V}} \to \mathbb{R}$ (called ordinal potential function) s.t.

$$u_i(y_i, x_{-i}) - u_i(x_i, x_{-i}) > 0 \iff \Phi(y_i, x_{-i}) - \Phi(x_i, x_{-i}) > 0$$

for every $x \in \mathcal{X}$, $i \in \mathcal{V}$, and $y_i \in \mathcal{A}$.

- In an ordinal potential game, the sign of the utility variation incurred by player $i$ when changing action unilaterally is the same as the sign of corresponding variation in the potential function:

$$\operatorname{sgn}\left(u_i(y_i, x_{-i}) - u_i(x_i, x_{-i})\right) = \operatorname{sgn}\left(\Phi(y_i, x_{-i}) - \Phi(x_i, x_{-i})\right)$$

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$$\text{exact potential} \;\; \underset{\not\Longleftarrow}{\Longrightarrow} \;\; \text{ordinal potential}$$

# Example: Symmetric Cournot Oligopoly

- arbitrary inverse demand (price) function $F(q)$
- identical linear cost functions $c_i(x_i) = cx_i$ for firms $i = 1, \ldots, n$
- profit (utility) of firm $i$ producing $x_i > 0$ is

$$u_i(x) = x_i F\left(\sum_j x_j\right) - cx_i$$

- this is ordinal potential game with potential function

$$\Phi(x) = \left(\prod_i x_i\right)\left(F\left(\sum_i x_i\right) - c\right)$$

# Example: Cournot Oligopoly with Affine Inverse Demand

- inverse demand (price) function $F(q) = \alpha - \beta q$
- arbitrary differentiable cost functions $c_i(x_i)$ for firms $i = 1, \ldots, n$
- profit (utility) of firm $i$ producing $x_i > 0$ is

$$u_i(x) = \alpha x_i - \beta x_i \sum_j x_j - c_i(x_i)$$

- this is exact potential game with potential function

$$\Phi(x) = \alpha \sum_i x_i - \beta \sum_i x_i^2 - \frac{\beta}{2} \sum_i \sum_{j \neq i} x_i x_j - \sum_i c_i(x_i)$$

# Potential games have Pure Strategy Nash Equilibria

Proposition: For an ordinal potential game, every global max point of the ord. potential function $\Phi(x)$ is a pure Nash equilibrium, i.e.,

$$\mathcal{N} \supseteq \mathcal{N}_{max} := \operatorname*{argmax}_{x \in \mathcal{X}} \Phi(x)$$

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Proof: Since

$$\operatorname{sgn}(u_i(y_i, x_{-i}) - u_i(x_i, x_{-i})) = \operatorname{sgn}(\Phi(y_i, x_{-i}) - \Phi(x_i, x_{-i}))$$

we have that $x^* \in \mathcal{X}$ is a pure strategy Nash equilibrium iff

$$\Phi(y_i, x^*_{-i}) \leq \Phi(x^*_i, x^*_{-i}) \qquad \forall i \in \mathcal{V},\ \forall y_i \in \mathcal{A}_i$$

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- Note: There might be pure Nash equilibria outside $\operatorname*{argmax}_{x \in \mathcal{X}} \Phi(x)$

$\mathcal{N} =$ "local maximum points"

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- Corollary 1: Every finite ordinal potential game admits (at least) a pure Nash equilibrium

- Corollary 2: Every continuous ordinal potential game with compact strategy space admits (at least) a pure Nash equilibrium

# Finite 2-player potential games

Every symmetric $2 \times 2$ game is an exact potential game

| | - | $+$ |
|---|---|---|
| - | a,a | d,c |
| $+$ | c,d | b,b |

| $\Phi$ | - | $+$ |
|---|---|---|
| - | a-c | 0 |
| $+$ | 0 | b-d |

- Coordination and anti-coordination games are potential games
- Prisoner's dilemma is a potential game
- Discoordination games and Rock-Scissor-Paper are NOT ordinal potential games, as they do not admit Nash equilibria.

# Finite Improvement Property

- length-$l$ path: sequence of strategy profiles $(x^{(0)}, x^{(1)}, \ldots, x^{(l)})$ such that there exist deviating players $i_1, i_2, \ldots, i_l$ with

$$x_{i_k}^{(k-1)} \neq x_{i_k}^{(k)} \qquad x_{-i_k}^{(k-1)} = x_{-i_k}^{(k)} \qquad \forall k = 1, \ldots, l$$

- improvement path if deviating players have positive utility gain

$$u_{i_k}(x_{i_k}^{(k)}, x_{-i_k}^{(k)}) > u_{i_k}(x_{i_k}^{(k-1)}, x_{-i_k}^{(k)}) \qquad \forall k = 1, \ldots, l$$

- useful to model myopic behavior of the players
- Finite Improvement Property (FIP): every improv. path is finite

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- useful to model myopic behavior of the players
- Finite Improvement Property (FIP): every improv. path is finite
- Lemma: FIP $\implies \exists$ pure Nash equilibrium

  Proof: maximal path terminates in pure Nash equilibrium

# Finite Improvement Property

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- Converse is NOT true: e.g., the following 2×2 game has the FIP

| | - | $+$ |
|---|---|---|
| - | 1,0 | 2,0 |
| $+$ | 2,0 | 0,1 |

but if it existed an every ordinal potential function $\Phi$ should satisfy

$$\Phi(-,-) < \Phi(+,-) < \Phi(+,+) < \Phi(-,+) = \Phi(-,-)$$

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- Definition: Game is generalized ordinal potential if $\exists \Phi : \mathcal{X} \to \mathbb{R}$

$$u_i(y_i, x_{-i}) - u_i(x_i, x_{-i}) > 0 \implies \Phi(y_i, x_{-i}) - \Phi(x_i, x_{-i}) > 0$$

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- Proposition: For finite games

Generalized Ordinal Potential $\iff$ Finite Improvement Property

# Properties of Exact Potential Games

- Lemma: Game is exact potential $\Leftrightarrow$ strategically equivalent to game with identical utilities:

$$u_i(x) = \Phi(x) + \psi_i(x_{-i})$$

- Lemma: If $\Phi(x)$ and $\overline{\Phi}(x)$ exact potential functions for same game, then $\exists$ constant $C$

$$\Phi(x) = \overline{\Phi}(x) + C \qquad \forall x \in \mathcal{X}$$

- Theorem: Game is exact potential if and only if

$$\sum_{i=1}^{4} u_{i_k}(x^{(k)}) - u_{i_k}(x^{(k-1)}) = 0$$

for every length-4 closed path $(x^{(0)}, x^{(1)}, x^{2)}, x^{(3)}, x^{(4)} = x^{(0)})$

null "cirquitation" on 4-cycles

# Finite 2-player potential games

- Proposition: Symmetric two player game with utilities

$$u_1(a,b) = U_{ab} \qquad u_2(a,b) = U_{ba}$$

is exact potential if and only if

$$U = S + C$$

where $S$ symmetric and $C$ constant on columns

# Smooth potential games

- Proposition: Game $(\mathcal{V}, \{\mathcal{A}_i\}, \{u_i\})$, $\mathcal{A}_i \subseteq \mathbb{R}$ interval, $u_i \in \mathcal{C}^2$.

Then, the game is an exact potential if and only if

$$\frac{\partial^2}{\partial x_i x_j} u_i(x) = \frac{\partial^2}{\partial x_j x_i} u_j(x)$$

for every $i, j \in \mathcal{V}$ and $x \in \mathcal{X}$. Moreover, in this case a potential function is

$$\Phi(x) = \sum_i \int_{\Gamma_{\overline{x} \to x}} \frac{\partial u_i}{\partial x_i}(s) \cdot \mathrm{d}s$$

# Congestion games

For player set $\mathcal{V}$, action set $\mathcal{A}$ and $c_a : \mathbb{Z}_+ \to \mathbb{R}$ for $a \in \mathcal{A}$.

$$x \in \mathcal{X},\ a \in \mathcal{A} \quad n_a^x = |\{i \in \mathcal{V} \,|\, x_i = a\}|$$

Utility of unit $i$: $u_i(x) = -c_{x_i}(n_{x_i}^x)$.

The game $(\mathcal{V}, \mathcal{A}, \{u_i\})$ is called a congestion game.

- utility of a player only depends on total number of players playing the same action.

- Actions $\leftrightarrow$ shared resources. If $c_a$'s are non-decreasing, the more units use the same resource, the worse the performance.

# Congestion games (cont'd)

An important extension:

- set of resources $\mathcal{E}$ (e.g., links in a transportation network) and, for $e \in \mathcal{E}$, congestion costs $c_e : \mathbb{Z}_+ \to \mathbb{R}_+$,
- action set $\mathcal{A} \subseteq 2^{\mathcal{E}}$ consists of a family of subsets of $\mathcal{E}$.

$$x \in \mathcal{A}^{\mathcal{V}},\ e \in \mathcal{E} \quad n_e^x = |\{i \in \mathcal{V} \,|\, e \in x_i\}|$$

- $x \in \mathcal{A}^{\mathcal{V}}$, $u_i(x) = -\sum_{e \in x_i} c_e(n_e^x)$.

# Congestion games are exact potential games

Theorem

$\Phi(x) = -\sum_{e\in\mathcal{E}} \sum_{h=1}^{n_e^x} c_e(h)$ *is a potential*

Proof: Let $y, x \in \mathcal{A}^{\mathcal{V}}$ be such that $y_j = x_j$ for every $j \neq i$.

$$
\begin{aligned}
\Phi(y) - \Phi(x) &= -\sum_{e\in\mathcal{E}} \left[\sum_{h=1}^{n_e^y} c_e(h) - \sum_{h=1}^{n_e^x} c_e(h)\right] \\
&= -\sum_{e\in y_i\setminus x_i} \left[\sum_{h=1}^{n_e^y} c_e(h) - \sum_{h=1}^{n_e^y-1} c_e(h)\right] \\
&\quad -\sum_{e\in x_i\setminus y_i} \left[\sum_{h=1}^{n_e^x-1} c_e(h) - \sum_{h=1}^{n_e^x} c_e(h)\right] \\
&= -\sum_{e\in y_i\setminus x_i} c_e(n_e^y) + \sum_{e\in x_i\setminus y_i} c_e(n_e^x) \\
&= -\sum_{e\in y_i} c_e(n_e^y) + \sum_{e\in x_i} c_e(n_e^x) \\
&= u_i(y) - u_i(x)
\end{aligned}
$$