Game theory and Network systems
Summer school SIDRA 2021

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Bertinoro, July 12-14, 2021

Lecture III: Network games

- Graphical and separable games
- Games with strategic complements and strategic substitutes
- Coordination and anticoordination games
- Best-Shot public goods game

Notation

- *Graph*: $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ where
  - $\mathcal{V}$ is the set of *nodes*
  - $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ is the set of *edges*
  - $N_i = \{j \in \mathcal{V} \,|\, (i,j) \in \mathcal{E}\}$ *(out)-neighborhood* of node i
  - $d_i = |N_i|$ *(out)-degree* of node i
- *Undirected graph*: if $(i,j) \in \mathcal{E} \Leftrightarrow (j,i) \in \mathcal{E}$
- *Weighted graph*: if $\mathcal{G} = (\mathcal{V}, \mathcal{E}, W)$, $W \in \mathbb{R}^{\mathcal{V} \times \mathcal{V}}$, $W_{ij} = 0$ if $(i,j) \notin \mathcal{E}$
- *Games in strategic form*: $(\mathcal{V}, \{\mathcal{A}_i\}, \{u_i\})$ where
  - $\mathcal{V}$ is the set of *players*,
  - $\mathcal{A}_i$ action set of player $i \in \mathcal{V}$, $\mathcal{X} = \prod_i \mathcal{A}_i$ configuration set
  - $u_i : \mathcal{X} \to \mathbb{R}$, utility of player i

Graphical games

$\mathcal{G} = (\mathcal{V}, \mathcal{E})$ graph

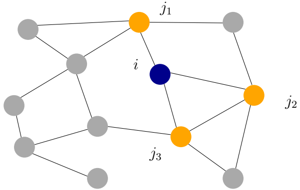
A game $(\mathcal{V}, \{\mathcal{A}_i\}, \{u_i\})$ is *graphical* with respect to $\mathcal{G}$ (a $\mathcal{G}$*-game*) if the utility of each player only depends on its own action and those of his neighbors:

$$x, y \in \prod_{i \in \mathcal{V}} \mathcal{A}_i, \; x_{|\{i\} \cup N_i} = y_{|\{i\} \cup N_i} \;\Rightarrow\; u_i(x) = u_i(y)$$

Utility can thus be expressed as $u_i(x_i, x_{-i}) = u_i(x_i, x_{|N_i})$.

Example:

$$u_i(x_i, x_{-i}) = u_i(x_i, x_{j_1}, x_{j_2}, x_{j_3})$$



Pairwise separable games

A notable example:

- An undirected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$
- An action set $\mathcal{A}_i$ for every $i \in \mathcal{V}$
- For each $i, j \in \mathcal{V}$ s.t. $(i, j), (j, i) \in \mathcal{E}$, consider a 2-players game with utility functions for i and j, respectively,

$$\lambda_{ij} : \mathcal{A}_i \times \mathcal{A}_j \to \mathbb{R}, \quad \lambda_{ji} : \mathcal{A}_j \times \mathcal{A}_i \to \mathbb{R}$$

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The game $(\mathcal{V}, \{\mathcal{A}_i\}, \{u_i\})$ where

$$u_i(x_i, x_{-i}) = \sum_{j \in N_i} \lambda_{ij}(x_i, x_j)$$

is a special instance of $\mathcal{G}$-game called *pairwise separable*

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is a special instance of $\mathcal{G}$-game called *pairwise separable*

Every player is simultaneously playing 2-player games with all its neighbors, playing the same action in all of them.

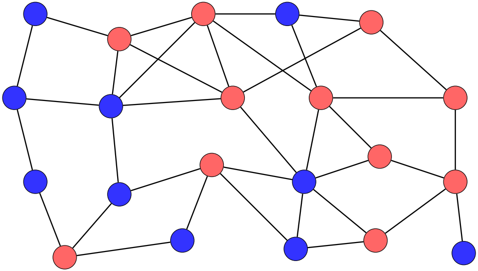
The majority game

The majority game is a pairwise graphical game with

- $\mathcal{A}_i = \{-1, +1\}$
- $\lambda_{ij}(x, y) = \begin{cases} 1 & \text{if } x = y \\ 0 & \text{if } x \neq y \end{cases}$ special coordination game

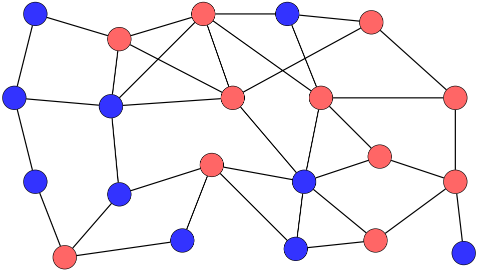
$$u_i(x_i, x_{-i}) = \sum_{j \in N_i} \lambda(x_i, x_j) = |\{j \in N_i \,|\, x_j = x_i\}| = \sum_{j \in N_i} \frac{x_i x_j + 1}{2}$$

Majority game: example



Nash equilibrium: every agent is playing an action equal to the majority of his neighbors.

Majority game: example

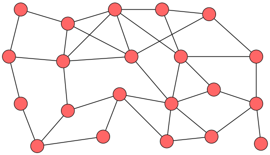


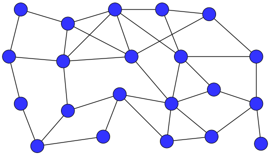
Nash equilibrium: every agent is playing an action equal to the majority of his neighbors.

The one above is not a Nash equilibrium.

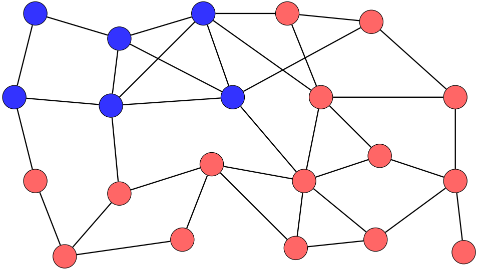
Majority game: example

The consensus configurations $+\mathbb{1}$ and $-\mathbb{1}$ are always Nash



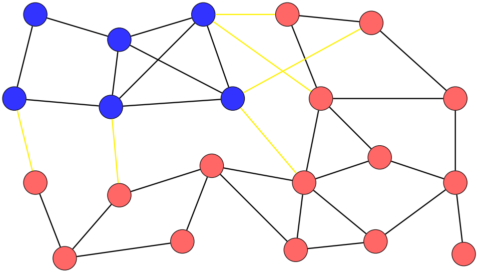


Majority game: example



There can be other Nash equilibria!

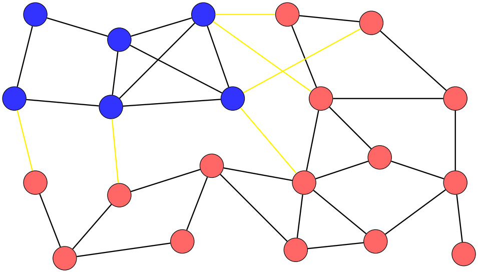
Cohesiveness and Nash



- $S \subseteq \mathcal{V}$ is *cohesive* if $|N_i \cap S| \geq |N_i \cap (\mathcal{V} \setminus S)|$ for all $i \in \mathcal{V}$.
- Fact (Morris, 2000)

  x is Nash $\Leftrightarrow$ $S = \{i : x_i = -1\}$ and S^c are cohesive.

Cohesiveness and Nash



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- However, find all Nash in a majority game is NP hard!

Network coordination games

$$u_i(x_i, x_{-i}) = \sum_{j \in N_i} x_i x_j + c_i x_i$$

Network coordination games

$$u_i(x_i, x_{-i}) = \sum_{j \in N_i} x_i x_j + c_i x_i$$

- $c_i = 0$, game *strategically equivalent* to the majority, same best response sets
- If $c_i > 0$, action $+1$ is intrinsically more convenient than action -1.
- $u_i(+1, x_{-i}) - u_i(-1, x_{-i}) = 2 \sum_{j \in N_i} x_j + 2c_i$ monotone in x_{-i} (*increasing difference property*)
- *Strategic complements effect:* the more the rest of players increase their action, the more it becomes convenient for player i to increase its own.

Network coordination games

- $u_i(+1, x_{-i}) - u_i(-1, x_{-i}) = 2 \sum_{j \in N_i} x_j + 2c_i \geq 0 \;\Leftrightarrow\; \sum_{j \in N_i} x_j \geq -c_i$
- $d_i = |N_i|, \quad d_i^+(x) = |\{j \in N_i \,|\, x_j = +1\}|$
- $\sum_{j \in N_i} x_j = 2d_i^+(x) - d_i$
- $\sum_{j \in N_i} x_j \geq -c_i \;\Leftrightarrow\; \frac{d_i^+(x)}{d_i} \geq \frac{1}{2} - \frac{c_i}{2d_i}$

Network coordination games

$$u_i(x_i, x_{-i}) = \sum_{j \in N_i} x_i x_j + c_i x_i$$

$$\mathrm{BR}(x_{-i}) = \begin{cases} \{+1\} & \frac{d_i^+(x)}{d_i} > \theta_i \\ \{-1, +1\} & \frac{d_i^+(x)}{d_i} = \theta_i \\ \{-1\} & \frac{d_i^+(x)}{d_i} < \theta_i \end{cases}$$

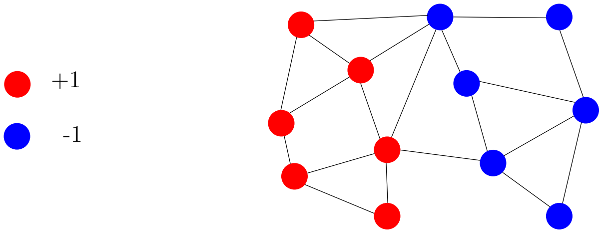
where $\theta_i = \frac{1}{2} - \frac{c_i}{2d_i}$ *threshold* of player i.

- $-d_i \le c_i \le d_i \;\Rightarrow\; \theta_i \in [0, 1]$
- In this case $x = -\mathbb{1}$ and $x = +\mathbb{1}$ are Nash equilibria
- $+\mathbb{1}$ *Pareto dominates* $-\mathbb{1}$ if $c_i > 0$
- $c_i = cd_i$ for some c yield the same threshold for all players.

Network coordination games

Example:

A coordination game where all players have threshold 3/5 exhibits this Nash equilibrium



The network anticoordination game

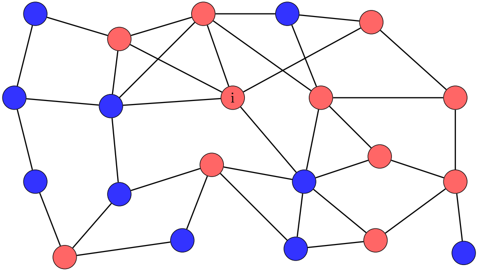
The network anticoordination game (minority game) is a pairwise graphical game with

- $\mathcal{A}_i = \{-1, +1\}$
- $\lambda_{ij}(x, y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \end{cases}$ anticoordination game

$$u_i(x_i, x_{-i}) = \sum_{j \in N_i} \lambda(x_i, x_j) = |\{j \in N_i \,|\, x_j = x_i\}| = \sum_{j \in N_i} \frac{-x_i x_j + 1}{2}$$

The network anticoordination game

-1 $+1$

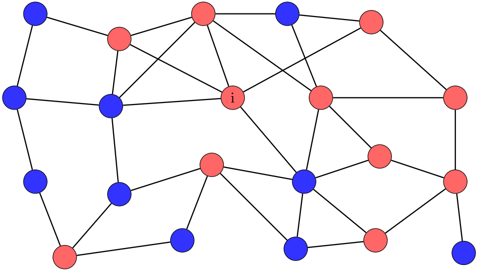


$$u_i(x_i, x_{-i}) = 2$$

Existence of Nash equilibria?

The network anticoordination game

$-1 \quad +1$



$$u_i(x_i, x_{-i}) = 2$$

Existence of Nash equilibria? Yes! Because it is a potential game!

Pairwise separable potential games

Theorem

Assume that

- $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ *undirected graph.*
- *Action sets are the same for all players:* $\mathcal{A}_i = \mathcal{A}$
- *For each edge* $(i, j) \in \mathcal{E}$*, there exists a symmetric potential 2-player game having utilities* $\lambda_{ij}(a, b) = \lambda_{ji}(a, b)$ *(*$a, b \in \mathcal{A}$*) and potential* $\phi_{\{i,j\}}$*.*

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Then, the graphical game $(\mathcal{V}, \{\mathcal{A}_i\}, \{u_i\})$ *with utilities* $u_i(x) := \sum_{j \in N_i} \lambda_{ij}(x_i, x_j)$ *is potential with potential function:*

$$\Phi(x) = \frac{1}{2} \sum_{(i,j) \in \mathcal{E}} \phi_{\{i,j\}}(x_i, x_j)$$

Pairwise separable potential games

- Majority game $\lambda_{ij}(x,y) = \begin{cases} 1 & \text{if } x = y \\ 0 & \text{if } x \neq y \end{cases}$ $\phi_{\{ij\}} = \lambda_{\{ij\}}$

  $$\Phi(x) = \frac{1}{2} \sum_{(i,j) \in \mathcal{E}} \frac{x_i x_j + 1}{2}$$

  number of edges whose nodes are in agreement

- Anticoordination game $\lambda_{ij}(x,y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \end{cases}$ $\phi_{\{ij\}} = \lambda_{ij}$

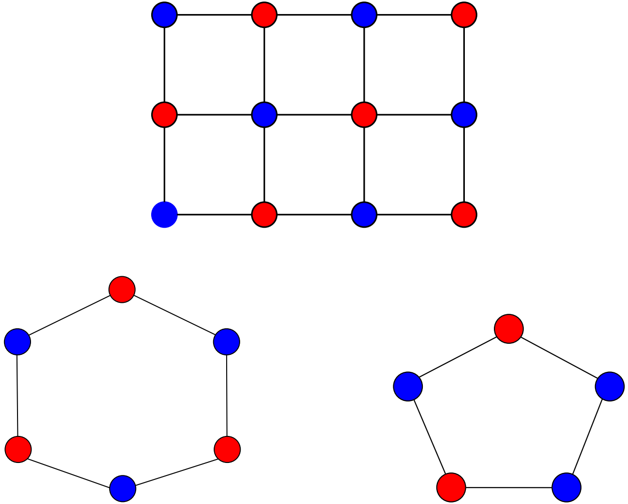
  $$\Phi(x) = \frac{1}{2} \sum_{(i,j) \in \mathcal{E}} \frac{-x_i x_j + 1}{2}$$

  number of edges whose nodes are in disagreement

Back to the anticoordination game

- We can use the FIP to practically find Nash equilibria in the anticoordination game
- $\max \Phi = \frac{|\mathcal{E}|}{2}$ if and only if the graph is bipartite. In this case there are Nash equilibria where all edges connect nodes in disagreement.

Nash equilibria in the anticoordination game



Coloring graphs

$\mathcal{G} = (\mathcal{V}, \mathcal{E})$ simple undirected graph

- A *coloring* of $\mathcal{G}$ with k colors is a configuration $x \in \mathcal{X} = \mathcal{A}^{\mathcal{V}}$ where $\mathcal{A} = \{1, 2, \dots, k\}$ such that

$$(i, j) \in \mathcal{E} \;\Rightarrow\; x_i \neq x_j$$

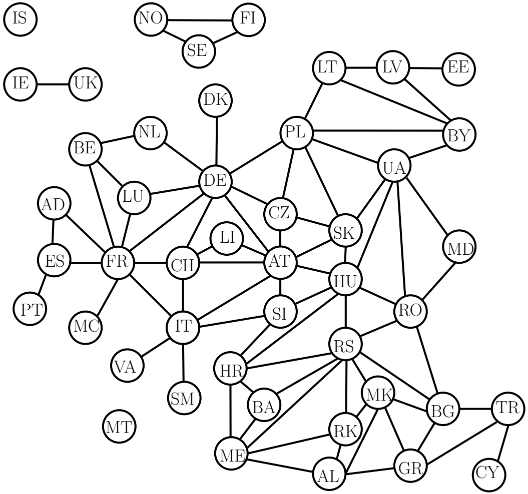
  the requirement is that neighbor nodes must have different colors.
- $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is said to be *k-colorable* if there exists a coloring with k colors.
- The minimum number of colors k for which $\mathcal{G}$ is k-colorable, is called the *chromatic number* of $\mathcal{G}$.

The coloring problem: examples

- Bipartite graphs (trees, grids) have chromatic number 2.
- Planar graphs have chromatic number at most 4 (the famous 4-colors problem).

The coloring problem: examples

The coloring problem: examples



The coloring game

$\mathcal{G} = (\mathcal{V}, \mathcal{E})$ undirected graph, $\mathcal{A} = \{1, \ldots, k\}$ colors, $\mathcal{X} = \mathcal{A}^{\mathcal{V}}$.

We consider a game where the players are the nodes of the graph, the actions are the colors, and the costs are the number of local violations:

$$x \in \mathcal{X}, \quad u_i(x_i; x_{-i}) = -\sum_{j \in N_i} \mathbb{1}_{x_i = x_j}$$

This is a pairwise $\mathcal{G}$-game and is potential with potential function

$$\Phi(x) = -\frac{1}{2} \sum_{(i,j) \in \mathcal{E}} \mathbb{1}_{x_i = x_j}$$

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$$\Phi(x) = -\frac{1}{2} \sum_{(i,j) \in \mathcal{E}} \mathbb{1}_{x_i = x_j}$$

- $\Phi(x)$ counts the number of 'coloring violations' in x
- $\max \Phi(x) = 0$ if and only if $\mathcal{G}$ is k-colorable
- Elements in $\operatorname{argmax} \Phi$ are exactly the coloring with k colors
- When $k = 2$, it is essentially the anticoordination game

Best-Shot public goods game

$\mathcal{G} = (\mathcal{V}, \mathcal{E})$ undirected graph

$(\mathcal{V}, \{\mathcal{A}_i\}, \{u_i\})$ $\mathcal{G}$-game with $\mathcal{A}_i = \{0, 1\}$ for all i

- Action 1 might be learning how to do something, where that information is easily communicated, or buying a book or other product that is easily lent from one player to another
- If a player or some in its neighborhood play action 1, then he gets a reward 1. Who plays action 1 pays a cost $0 < c < 1$.

$$u_i(x_i, x_{-i}) = \begin{cases} 1 - c & \text{if } x_i = 1 \\ 1 & \text{if } x_i = 0,\ \exists j \in N_i \,:\, x_j = 1 \\ 0 & \text{if } x_i = 0,\ \forall j \in N_i \,:\, x_j = 0 \end{cases}$$

Best-Shot public goods games

Utilities $u_i(x_i, x_{-i})$ satisfy the *decreasing difference property*:

$$u_i(b_i, x_{-i}) - u_i(a_i, x_{-i}) \geq u_i(b_i, y_{-i}) - u_i(a_i, y_{-i})$$

if $x_{-i} \leq y_{-i}$ and $a_i \leq b_i$.

In economy, such games model the so called *strategic substitutes effect*: the increase of a player's action, makes less profitable for the others to increase theirs.

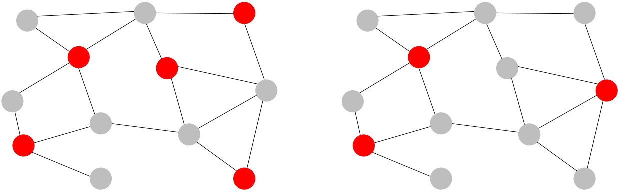
Best-Shot public goods games

Theorem

For the Best-Shot public goods game, Nash equilibria always exist:

$x \in \{0,1\}^n$ *is a Nash equilibrium if and only if* $\{i \in \mathcal{A} : x_i = 1\}$ *forms a maximal independent set of the graph* $\mathcal{G}$

Example:



General considerations on graphical games

Is graphicality an important concept?

- All games are graphical (with respect to the complete graph)
- However, for every game, there exists a minimal graph $\mathcal{G}$ for which that game is a $\mathcal{G}$-game
- Graphical games with respect to sparse graphs are describable in terms of 'few' parameters. This makes inferential and learning problems more tractable.
- Pairwise separability can be generalized to hypergraph separability

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