Game theory and Network systems
Summer school SIDRA 2021

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Lecture IV: Best response and noisy best response dynamics

- Recall of basic concepts from Markov chains theory
- The best response dynamics
- Asymptotic behavior for potential games
- The noisy best response dynamics
- Application to coloring and other constraint satisfaction problems

Learning dynamics in games

- Most economic theory relies on equilibrium analysis based on Nash equilibrium or its refinements.
- The traditional explanation for when and why equilibrium arises is that it results from the assumption of the rationality of the players, and that the structure of the game common shared knowledge.
- While there are situations where such elements are sufficient to determine the equilibrium (e.g. the two prisoners dilemma by the dominant strategy technique), this is not the case in many situations where there are multiple Nash equilibria (e.g. coordination games)
- In this lecture, we develop an alternative explanation why equilibrium arises as the long-run outcome of a learning process where players modify their action as time passes by.
- In contexts where the game is used as a modeling tool to solve multi-agent decision and optimization problems, such evolutionary dynamics can be interpreted as distributed algorithms.

Best Response dynamics

$(\mathcal{V}, \{\mathcal{A}_i\}, \{u_i\})$ game. $\mathcal{X} = \prod_i \mathcal{A}_i$ configuration set.

- $X(0) \in \mathcal{X}$ initial configuration
- At every discrete time instant, a player is chosen uniformly at random and becomes active;
- If player i becomes active at time t and the system is in configuration $X(t) = x \in \mathcal{X}$, he computes

$$\mathcal{B}_i(x_{-i}) = \underset{x_i \in \mathcal{A}}{\operatorname{argmax}}\, u_i(x_i, x_{-i})$$

  and then he chooses its new action as

$$\mathbb{P}(X_i(t+1) = a \,|\, X(t) = x) = \begin{cases} \frac{1}{|\mathcal{B}_i(x_{-i})|} & \text{if } \; a \in \mathcal{B}_i(x_{-i}) \\ 0 & \text{otherwise} \end{cases}$$

- $X_j(t+1) = X_j(t)$ for all $j \neq i$

$X(t)$ is a Markov chain on $\mathcal{X}$.

Markov chains

A (homogeneous) *Markov chain* (MC) on a finite set $\mathcal{X}$ is a stochastic process, namely a sequence of random variables $X(0), X(1), \dots, X(t), \dots$ taking values in $\mathcal{X}$ such that the so-called *Markov property* holds true:

$$\mathbb{P}(X(t) = x_t \mid X(1) = x_1, \dots, X(t-1) = x_{t-1})$$
$$= \mathbb{P}(X(t) = x_t \mid X(t-1) = x_{t-1}) = \mathbb{P}(X(1) = x_t \mid X(0) = x_{t-1})$$

for all possible values $x_1, \dots, x_t \in \mathcal{X}$

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A MC is completely specified by the pair (μ^0, P) where

- The *initial distribution* on $\mathcal{X}$: $\mu^0_x = \mathbb{P}(X(0) = x)$
- The *transition matrix:* $P_{xy} = \mathbb{P}(X(1) = y \mid X(0) = x)$

Markov chains

$X(t)$ MC on $\mathcal{X}$ with initial distribution and transition matrix (μ^0, P).

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All probabilities involving the MC can be computed from (μ^0, P):

- $\mathbb{P}(X(0) = x_0, X(1) = x_1, \dots, X(t) = x_t) = \mu^0_{x_0} P_{x_0 x_1} \cdots P_{x_{t-1} x_t}$
- $\mu^t_x = \mathbb{P}(X(t) = x) = ((\mu^0)' P^t)_x$
- μ^0 is called an *equilibrium distribution* if $\mu^t = \mu^0$ for every t, equivalently if $(\mu^0)' P = (\mu^0)'$.

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Transition graph associated to $X(t)$: $\Gamma = (\mathcal{X}, \mathcal{E}_P)$ $\mathcal{E}_P = \{(x, y) \,|\, P_{xy} > 0\}$

Asymptotics of Markov chains

Definition

$\mathcal{G} = (\mathcal{V}, \mathcal{E})$ directed graph. $\mathcal{W} \subseteq \mathcal{V}$

- $\mathcal{W}$ is called *trapping* if for every $(i, j) \in \mathcal{E}$ with $i \in \mathcal{W}$, it holds $j \in \mathcal{W}$. If $\mathcal{W} = \{w\}$, w is also called *absorbing*.
- $\mathcal{W}$ is called *minimal trapping* if it is trapping and no proper subset of it is trapping.

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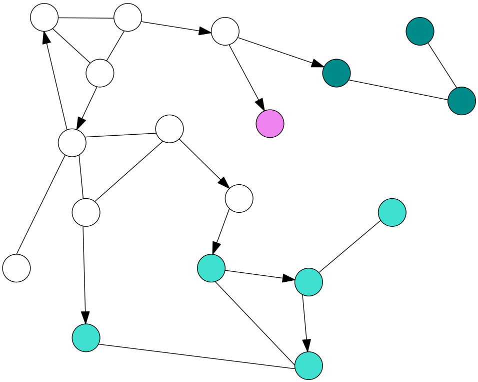
The node set can be decomposed as

$$\mathcal{V} = (\mathcal{U}_1 \cup \cdots \mathcal{U}_k) \cup \mathcal{T}$$

where $\mathcal{U}_i$ are disjoint minimal trapping sets and $\mathcal{T}$ is the remaining part called the *transient* set.

Remark: $\mathcal{W}$ minimal trapping, $\mathcal{G}_{\mathcal{W}} = (\mathcal{W}, \mathcal{E} \cap (\mathcal{W} \times \mathcal{W}))$ is strongly connected.

Asymptotics of Markov chains



Minimal trapping sets

Asymptotics of Markov chains

Theorem

If a Markov chain $X(t) \in \mathcal{X}$ has transition graph Γ, then,

- *in finite time, with probability 1, $X(t)$ will be absorbed in one of the minimal trapping sets of Γ.*
- *if $\mathcal{W} \subseteq \mathcal{X}$ is a minimal aperiodic trapping set, the Markov chain on $\mathcal{W}$ is ergodic: it possess a unique equilibrium distribution $\pi^{\mathcal{W}}$ such that*

$$X(0) \in \mathcal{W} \Rightarrow \lim_{t \to +\infty} \mathbb{P}(X(t) = i) = \pi_i^{\mathcal{W}} \, \forall i \in \mathcal{W}$$

Analysis of the BR dynamics

$(\mathcal{V}, \{\mathcal{A}_i\}, \{u_i\})$ game. $\mathcal{X} = \prod_i \mathcal{A}_i$ configuration set.

$X(t)$ BR dynamics on $\mathcal{X}$. $\Gamma_{\mathcal{B}} = (\mathcal{X}, \mathcal{E}_{\mathcal{B}})$ transition graph.

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Given $x, y \in \mathcal{X}$, $(x, y) \in \mathcal{E}_{\mathcal{B}}$ if and only if

- $\exists i \in \mathcal{V}$ such that $x_{-i} = y_{-i}$;
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$$\text{For } x \neq y \quad P_{xy} = \begin{cases} \frac{1}{n}\frac{1}{|\mathcal{B}_i(x_{-i})|} & \text{if } (x,y) \in \mathcal{E}_{\mathcal{B}}, x_{-i} = y_{-i} \\ 0 & \text{if } (x,y) \notin \mathcal{E}_{\mathcal{B}} \end{cases}$$

Asymptotics of the BR dynamics

Theorem

Consider a potential game $(\mathcal{V}, \{\mathcal{A}_i\}, \{u_i\})$*. In the transition graph* $\Gamma_{\mathcal{B}}$ *of the BR dynamics, every node of any minimal trapping component is a Nash equilibrium.*

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Corollary

Consider a potential game $(\mathcal{V}, \{\mathcal{A}_i\}, \{u_i\})$*. Then, the BR dynamics* $X(t)$ *is such that, for every* $X(0)$*, there exists a random time* $T > 0$ *that is finite with probability one and is such that* $X(t)$ *is Nash for all* $t \geq T$

Proof of the Theorem

- Consider the transition graph $\Gamma_{\mathcal{B}} = (\mathcal{X}, \mathcal{E}_{\mathcal{B}})$ associated to the BR dynamics. $\mathcal{N} \subseteq \mathcal{X}$ Nash equilibria.
- Notice that $(x, y) \in \mathcal{E}_{\mathcal{B}}$ implies $\Phi(y) \geq \Phi(x)$, namely the potential never decreases along the edges of the transition graph.
- By contradiction: $\mathcal{W} \subseteq \mathcal{X}$ minimal trapping and there exists $x \in \mathcal{W} \setminus \mathcal{N}$
- Then, there exists $y \in \mathcal{X}$ such that $(x, y) \in \mathcal{E}_{\mathcal{B}}$ and $\Phi(y) > \Phi(x)$.
- Since the potential never decreases, it means that there can not be a path from y back to x.
- Hence, $\mathcal{W}$ not minimal trapping! ■

More questions on the BR dynamics

- What is the behavior of the BR dynamics within the set of Nash equilibria?
- Does it converge to the maxima of the potential?
- Everything boils down to analyze the structure of the transition graph of the BR dynamics $\Gamma_{\mathcal{B}} = (\mathcal{X}, \mathcal{E}_{\mathcal{B}})$

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- What is the behavior of the BR dynamics within the set of Nash equilibria?
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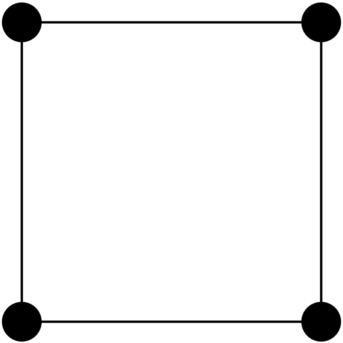
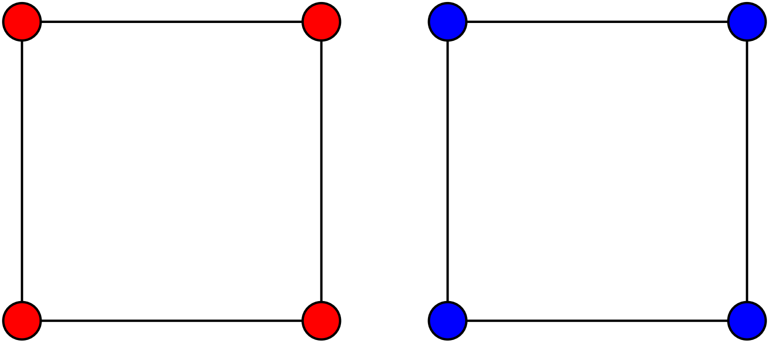
- Everything boils down to analyze the structure of the transition graph of the BR dynamics $\Gamma_{\mathcal{B}} = (\mathcal{X}, \mathcal{E}_{\mathcal{B}})$
- IMPORTANT: when we consider a graphical game $(\mathcal{V}, \{\mathcal{A}_i\}, \{u_i\})$ over a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, it is important not to confuse this graph with the transition graph $\Gamma_{\mathcal{B}}$ that is a graph on the configuration space $\mathcal{X}$!

Majority game on the 4-cycle



Majority game on the 4-cycle: Nash



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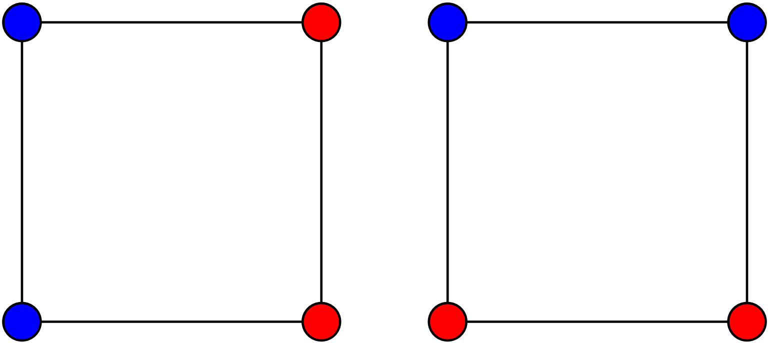

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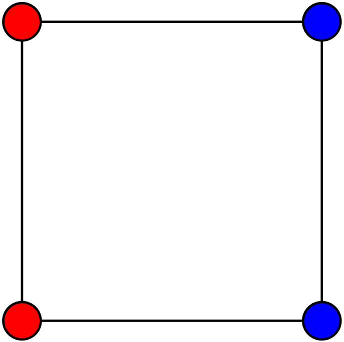
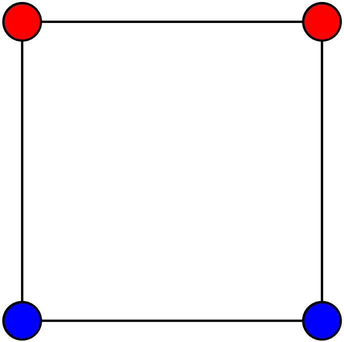
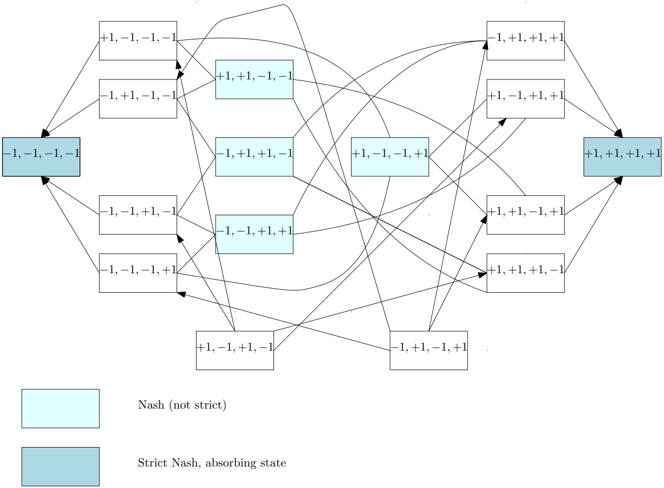




Transition graph of the Majority dynamics over the 4-cycle



Behavior of the BR dynamics

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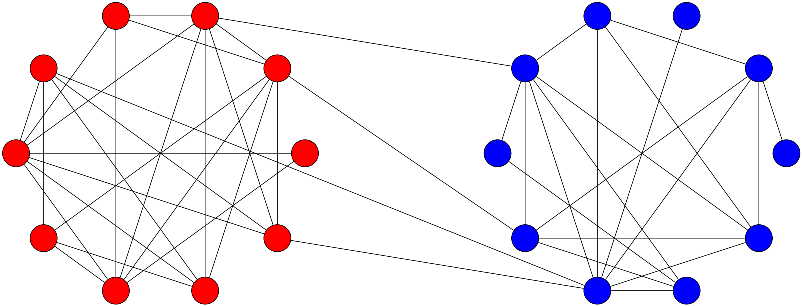
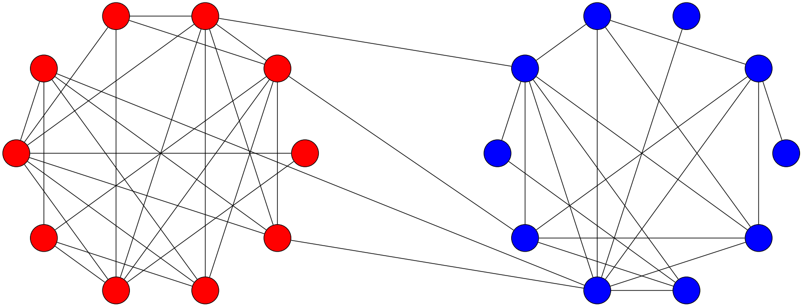
- There may be Nash equilibria that are not in minimal trapping sets.
- The BR dynamics may enter and leave the set of Nash equilibria.
- There is a subset of the Nash equilibria that is trapping but in general does not coincide with the maxima of the potential.

Behavior of the BR dynamics



This is a Nash configuration for the majority game that is trapping but is not a maximum of the potential.

Behavior of the BR dynamics



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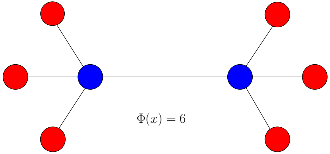
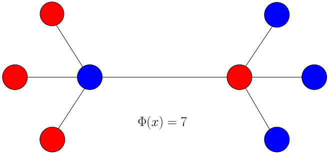
BR dynamics can get stacked in 'local maxima' and not converge to the global maxima of the potential!

Behavior of the BR dynamics

Two strict Nash equilibria for the network anticoordination game





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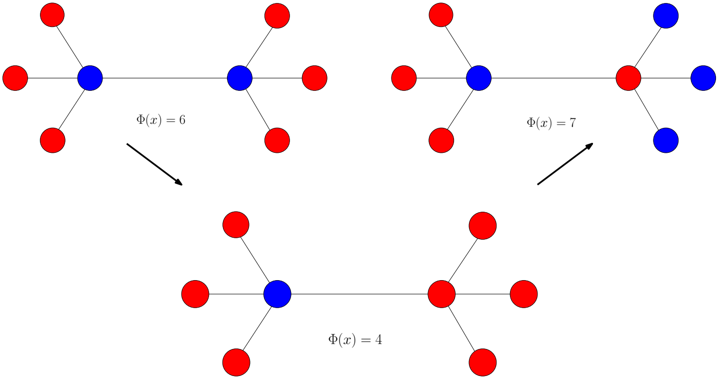

The system will need to go through lower potential configurations in order to finally reach a maximum.

Noisy Best Response dynamics (log-linear learning)

$(\mathcal{V}, \{\mathcal{A}_i\}, \{u_i\})$ game. $\mathcal{X} = \prod_i \mathcal{A}_i$ configuration set.

- $X(0) \in \mathcal{X}$ initial configuration
- At every discrete time instant, a player is chosen uniformly at random and becomes active;
- If node i becomes active at time t and the system is in configuration $X(t) = x \in \mathcal{X}$, he chooses its new action with probability

$$\mathbb{P}(X_i(t+1) = a \,|\, X(t) = x) = \frac{1}{Z_i(x)} e^{\beta u_i(a; x_{-i})}$$

  where

$$Z_i(x) = \sum_{a' \in \mathcal{A}_i} e^{\beta u_i(a'; x_{-i})}$$

- $X_j(t+1) = X_j(t)$ for all $j \neq i$

Noisy Best Response dynamics (log-linear learning)

The NBR $X(t)$ is a Markov chain on $\mathcal{X}$.

The transition graph $\Gamma_{\mathrm{NBR}} = (\mathcal{X}, \mathcal{E}_{\mathrm{NBR}})$ is the *hypercube graph* (with self-loops):

$$x \neq y \in \mathcal{X},\ (x, y) \in \mathcal{E}_{\mathrm{NBR}} \ \Leftrightarrow \exists i \in \mathcal{V}\, \mathrm{s.t.}\ x_{-i} = y_{-i},\ x_i \neq y_i$$

The transition matrix is

$$\mathrm{For}\, x \neq y \quad P_{xy} = \begin{cases} \frac{1}{n} \frac{1}{Z_i(x)} e^{\beta u_i(y_i; x_{-i})} & \mathrm{if}\, (x, y) \in \mathcal{E}_{\mathcal{B}}, x_{-i} = y_{-i} \\ 0 & \mathrm{if}\, (x, y) \notin \mathcal{E}_{\mathcal{B}} \end{cases}$$

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- $\beta = 0$: NBR is a pure random walk on the hypercube graph of $\mathcal{X}$ (in a transition only one component changes);
- $\beta \to +\infty$: NBR $\to$ BR.

Behavior of the NBR dynamics

Theorem

Consider a potential game $(\mathcal{V}, \{\mathcal{A}_i\}, \{u_i\})$ *with a potential* $\Phi : \mathcal{X} \to \mathbb{R}$*. Then, the NBR dynamics* $X(t)$ *is a reversible ergodic Markov chain whose unique equilibrium distribution is given by the Gibbs measure:*

$$\pi_x = \frac{e^{\beta\Phi(x)}}{\sum_{y \in \mathcal{X}} e^{\beta\Phi(y)}}$$

Moreover, considering that π *depends on* β*:*

$$\lim_{\beta \to +\infty} \pi = \textit{ uniform distribution over } \operatorname{argmax} \Phi$$

The proof

- The graph $\mathcal{G}^{\mathrm{NBR}}$ is strongly connected: there is no transient part and $\mathcal{V}$ itself is a minimal trapping set
- From the general theory, we know that the NBR $X(t)$ possesses just one equilibrium distribution π
- To find π one needs to solve the eigenvalue relation $\pi_x = \sum_y \pi_y P_{yx}$
- There are special MC (called reversible) for which there exists a positive vector ρ such that the balancing equation is satisfied:

$$\rho_x P_{xy} = \rho_y P_{yx} \quad \forall x, y$$

  Note that

$$\sum_y \rho_y P_{yx} = \sum_y \rho_x P_{xy} = \rho_x$$

  Thus, $\pi_x = \rho_x / (\sum_y \rho_y)$ is the wanted equilibrium distribution.
- Balancing equation trivially true if $x = y$ or if $P_{xy} = P_{yx} = 0$.

The proof

- In our case, the only case we need to check is for $x, y \in \mathcal{X}$ such that $x_{-i} = y_{-i}$ and $x_i \neq y_i$:

$$\frac{P_{xy}}{P_{yx}} = \frac{\frac{1}{n}\frac{1}{Z_i(x)}e^{\beta u_i(y_i;x_{-i})}}{\frac{1}{n}\frac{1}{Z_i(y)}e^{\beta u_i(x_i;y_{-i})}} = e^{\beta[u_i(y_i;x_{-i})-u_i(y_i;y_{-i})]} = e^{\beta[\Phi(x)-\Phi(y)]} = \frac{e^{\beta\Phi(x)}}{e^{\beta\Phi(y)}}$$

- This says that the NBR is reversible and that $\pi_x = \frac{e^{\beta\Phi(x)}}{\sum_{y\in\mathcal{X}} e^{\beta\Phi(y)}}$ is the equilibrium distribution

- $$\lim_{\beta\to+\infty} \frac{e^{\beta\Phi(x)}}{\sum_{y\in\mathcal{X}} e^{\beta\Phi(y)}} = \begin{cases} \frac{1}{|\operatorname{argmax}\Phi|} & \text{if } x \in \operatorname{argmax}\Phi \\ 0 & \text{otherwise} \end{cases}$$

Behavior of the NBR dynamics

The NBR dynamics $X(t)$ can be effectively used to find the maxima of the potential:

$$\lim_{\beta\to+\infty}\lim_{t\to+\infty}\mathbb{P}(X(t)=x) = \lim_{\beta\to+\infty}\pi_x = \begin{cases}\dfrac{1}{|\operatorname{argmax}\Phi|} & \text{if } x\in\operatorname{argmax}\Phi \\ 0 & \text{otherwise}\end{cases}$$

- If β is large, $X(t)$ will be, after a transient, with high probability on a maximum of the potential.
- As time passes by, $X(t)$ approximately samples uniformly on $\operatorname{argmax}\Phi$.

NBR for coloring graphs

$\mathcal{G} = (\mathcal{V}, \mathcal{E})$ undirected graph. Is $\mathcal{G}$ k-colorable?

- k-coloring game: $\mathcal{A} = \{1, \ldots, k\}$ colors, $\mathcal{X} = \mathcal{A}^{\mathcal{V}}$
- $u_i(x_i; x_{-i}) = -\sum_{j \in N_i} \mathbb{1}_{x_i = x_j}$
- Potential: $\Phi(x) = -\frac{1}{2} \sum_{(i,j) \in \mathcal{E}} \mathbb{1}_{x_i = x_j}$
- $\max \Phi(x) = 0$ if and only if $\mathcal{G}$ is k-colorable
- In this case $\operatorname{argmax} \Phi$ are exactly the coloring with k colors
- NBR can be used as an algorithm

Graphical games and optimization: a general viewpoint

Many combinatorial problems in AI lead to optimization problems for functionals $\Phi : \prod_{i \in \mathcal{V}} \mathcal{A}_i \to \mathbb{R}$ of type

$$\Phi(x) = \sum_{F \in \mathcal{F}} \Phi^F(x_{|F})$$

where $\mathcal{F} \subseteq 2^{\mathcal{V}}$ is a family of subsets of $\mathcal{V}$.

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The game theoretic approach:

- Construct a game $(\mathcal{V}, \{\mathcal{A}_i\}, \{u_i\})$ that has potential Φ;
- Construct a dynamics where players *learn* a Nash equilibrium that is a maximum for the potential.

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- Construct a dynamics where players *learn* a Nash equilibrium that is a maximum for the potential.

In the coloring problem $\mathcal{F}$ is the set of edges of the graph.

Graphical games and optimization: a general viewpoint

$$\Phi(x) = \sum_{F \in \mathcal{F}} \Phi^F(x_{|F})$$

There is a standard way to construct a game having Φ as potential: given a player i we consider all the contributions in the decomposition of Φ relative to subsets F containing i.

$$u_i(x) = \sum_{F \ni i} \Phi^F(x_{|F})$$

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$x, y \in \mathcal{X}$, $x_{-i} = y_{-i}$.

$$\begin{aligned}\Phi(x) - \Phi(y) = \sum_{F \in \mathcal{F}} [\Phi^F(x_{|F}) - \Phi^F(y_{|F})] &= \sum_{F \ni i} [\Phi^F(x_{|F}) - \Phi^F(y_{|F})] \\ &= u_i(x) - u_i(y)\end{aligned}$$

Graphical games and optimization: a general viewpoint

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There is a standard way to construct a game having Φ as potential: given a player i we consider all the contributions in the decomposition of Φ relative to subsets F containing i.

$$u_i(x) = \sum_{F \ni i} \Phi^F(x_{|F})$$

It is a graphical game with respect to the *proximity* graph determined by $\mathcal{F}$: two players i, j are connected by a link if there exists $F \in \mathcal{F}$ such that $i, j \in F$.

The sets in $\mathcal{F}$ are cliques in this proximity graph.

An interesting converse

The result above admits an interesting converse recently proven by Babichenko-Tamuz (2016):

Theorem

Let $\mathcal{G}$ be an undirected graph and let $(\mathcal{V}, \{\mathcal{A}_i\}, \{u_i\})$ be a game that is graphical with respect to $\mathcal{G}$ and that is potential. Then the potential Φ admits a decomposition

$$\Phi(x) = \sum_{F \in \mathcal{F}} \Phi^F(x_{|F})$$

where $\mathcal{F}$ is the set of cliques in the graph $\mathcal{G}$.

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The result above admits an interesting converse recently proven by Babichenko-Tamuz (2016):

Theorem

Let $\mathcal{G}$ be an undirected graph and let $(\mathcal{V}, \{\mathcal{A}_i\}, \{u_i\})$ be a game that is graphical with respect to $\mathcal{G}$ and that is potential. Then the potential Φ admits a decomposition

$$\Phi(x) = \sum_{F \in \mathcal{F}} \Phi^F(x_{|F})$$

where $\mathcal{F}$ is the set of cliques in the graph $\mathcal{G}$.

- This results has been refined in [Arditti et al. 2021] for games that admit a separability structure of the utility functions.
- Connections with Markov random fields and Hammersley-Clifford th.

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Problems:

- Determine the maximum number of jobs k that can be executed by a single machine.
- Find all subsets $\mathcal{W} \subseteq \mathcal{V}$ consisting of k jobs that can be executed by a single machine.

Another optimization problem

- Consider the undirected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ where $\mathcal{E}$ consists of the forbidden pairs
- Notice that $\mathcal{W} \subseteq \mathcal{V}$ consists of jobs that can be executed by a single machine if and only if, given any $i, j \in \mathcal{W}$ it holds that $(i, j) \notin \mathcal{E}$, namely $\mathcal{W}$ is an independent set.
- We look for independent sets of $\mathcal{V}$ of *maximum* cardinality.
- We already met a game whose Nash equilibria are the *maximal* independent sets (and thus including those of maximum cardinality):

  $\mathcal{A} = \{0, 1\}$, $x \in \mathcal{X} \leftrightarrow \{i \,|\, x_i = 1\}$ = set of jobs selected.

$$u_i(x_i, x_{-i}) = \begin{cases} 1 - c & \text{if } x_i = 1 \\ 1 & \text{if } x_i = 0,\ \exists j \in N_i \,:\, x_j = 1 \\ 0 & \text{if } x_i = 0,\ \forall j \in N_i \,:\, x_j = 0 \end{cases}$$

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However, it can be shown that this game is not potential.

A game theoretic approach to the independent set problem

We follow our general approach:

$\mathcal{G} = (\mathcal{V}, \mathcal{E})$ undirected graph, $\mathcal{A} = \{0, 1\}$, $\mathcal{X} = \mathcal{A}^{\mathcal{V}}$,

For every $i \in \mathcal{V}$, we consider $\Phi^{(i)} : \mathcal{A}^{N_i \cup \{i\}} \to \mathbb{R}$,

$$\Phi^{(i)}(y) = \begin{cases} 0 & \text{if } y_i = 0 \\ 1 & \text{if } y_i = 1,\ y_{-i} = 0 \\ -1 & \text{if } y_i = 1,\ y_{-i} \neq 0 \end{cases}$$

$$\Phi(x) = \sum_{i \in \mathcal{V}} \Phi^{(i)}(x_{|N_i \cup \{i\}})$$

Fact: $\operatorname{argmax} \Phi =$ independent sets of maximum cardinality.

A game theoretic approach to the independent set problem

The functional Φ has the form

$$\Phi(x) = \sum_{F \in \mathcal{F}} \Phi^F(x_{|F})$$

where $\mathcal{F} \subseteq 2^{\mathcal{V}}$ is the family of augmented neighborhood in the graph: $F = N_i \cup \{i\}$.

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Therefore, we can design a network game having Φ as potential, by considering:

$$u_i(x) = \sum_{F \ni x} \Phi^F(x_{|F}) = \Phi^{(i)}(x_{|N_i \cup \{i\}}) + \sum_{j \in N_i} \Phi^{(j)}(x_{|N_j \cup \{j\}})$$