Game theory and Network systems
Summer school SIDRA 2021

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Lecture V/VI: Quadratic games
Supermodular games

- Fundamental examples
- Strategic complements and substitutes
- Characterization of Nash equilibria, network centrality
- Constraint quadratic games
- Supermodular games
- Lattice structure of Nash equilibria
- Best response for supermodular games
- Comparative static

A benchmark model

$$u_i(x) = \rho_i(x_i) + \delta x_i \sum_j W_{ij} x_j$$

- $x_i \in \mathcal{A}_i \subseteq \mathbb{R}$ level of activity of player i
- $\rho_i(x_i) = a_i x_i - \frac{b_i}{2} x_i^2$ individual utility in the absence of network interactions
- W non negative interaction matrix $\leftrightarrow \mathcal{G} = (\mathcal{V}, \mathcal{E}, W)$ weighted graph

Example 1: Peer effects social models

A model to study the effect of social interactions in educational systems and in crime networks

$$u_i(x) = a_i x_i - \frac{1}{2}x_i^2 + \delta x_i \sum_j W_{ij} x_j$$

- $x_i \in \mathbb{R}^+$ level of activity of player i
- a_i marginal individual reward from level of activity x_i, $\frac{1}{2}x_i^2$ cost for providing level of activity x_i
- W_{ij} strength of the influence of j on i
- $\delta > 0$ benefit of the social interaction

Example 2: Partnership models

$$u_i(x) = -c_i(x_i) + \delta x_i \sum_j W_{ij} x_j$$

- $x_i \in [0, r_i]$ level of activity of player i
- $c_i(x_i)$ individual cost (linear, quadratic) to exert level of activity x_i. No benefit in the absence of interaction.
- W interaction matrix
- $\delta > 0$ benefit of the social interaction

Example 3: Cournot competition

n firms competing in a market, selling perfectly substitutable goods

$$u_i(x) = x_i P(\sum_j x_j) - C_i(x_i)$$

- Each firm i fixes the quantity it sells x_i
- $P(q) = (K - q)^+$ is the inverse demand function: price as function of the quantity q
- $C_i(x_i)$ cost for the firm i to produce the quantity x_i (linear or quadratic)

Example 4: Opinion formation game

$$u_i(x) = -\frac{1}{2}\left[\sum_j W_{ij}(x_i - x_j)^2 + \rho_i(x_i - c_i)^2\right]$$

- x_i opinion of agent i on a specific subject
- c_i internal belief of agent i
- ρ_i strength of the internal belief of agent i
- W social influence interaction matrix
- a model for cognitive dissonance

Quadratic games: general considerations

$$u_i(x) = \rho_i(x_i) + \delta x_i \sum_j W_{ij} x_j$$

- $\frac{\partial u_i}{\partial x_j} = \delta x_i W_{ij}$, if actions are non-negative
  - $\delta > 0$ positive externality effect
  - $\delta < 0$ negative externality effect
- $\frac{\partial^2 u_i}{\partial x_i \partial x_j} = \delta W_{ij}$
  - $\delta > 0$: strategic complements effect
  - $\delta < 0$: strategic substitutes effect
- W symmetric yields $\frac{\partial^2 u_i}{\partial x_i \partial x_j} = \frac{\partial^2 u_j}{\partial x_j \partial x_i}$. Game is potential

$$\Phi(x) = \sum_i \rho_i(x_i) + \frac{\delta}{2} \sum_{ij} W_{ij} x_i x_j$$

Quadratic games: linear best reply

$$u_i(x) = a_i x_i - \frac{1}{2} x_i^2 + \delta x_i \sum_j W_{ij} x_j$$

- $a_i > 0$ marginal utility, $b_i = 1$ normalized costs
- $x_i \in \mathbb{R}$ (unconstrained actions)
- $\mathcal{B}(x_{-i})$ consists of the scalar x_i that satisfies

$$a_i - x_i + \delta \sum_j W_{ij} x_j = 0 \Leftrightarrow x_i = a_i + \delta \sum_j W_{ij} x_j$$

Linear best reply

Quadratic games: Nash equilibria

Nash equilibria are those configurations x^* such that:

$$x^* = a + \delta W x^* \Leftrightarrow (I - \delta W)x^* = a$$

- Existence and uniqueness of Nash if δ^{-1} is not an eigenvalue of W.
- If $\rho(W)$ is the spectral radius of W, $|\delta|\rho(W) < 1$ guarantees existence and uniqueness.

$$x^* = (I - \delta W)^{-1} a = \sum_k \delta^k W^k a$$

The synchronous best reply map

$$x_i(t+1) = \mathcal{B}_i(x_{-i}(t)) = a_i + \delta \sum_j W_{ij} x_j(t)$$

$$x(t+1) = a + \delta W x(t)$$

- A discrete-time linear system where a plays the role of a constant input signal
- The condition $\delta\rho(W) < 1$ guarantees that the system is asymptotically stable and that

$$\lim_{t\to+\infty} x(t) = x^* = (I - \delta W)^{-1} a$$

Convergence to the Nash equilibrium

Quadratic games with strategic complements

Assume now $\delta \geq 0$

$$x_i^* = \sum_k \delta^k \sum_j (W^k)_{ij} a_j \geq 0$$

The equilibrium can be interpreted as the *Bonacich centrality* of the weighted graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, W)$:

- a_i intrinsic centrality of node i
- the equilibrium value of a node depends on its position in the network: it aggregates the intrinsic centrality received by all the nodes through their walks to i, discounted by a factor δ^k where k is the length of the walk.

Welfare

Utility at equilibrium $x^* = a + \delta W x^*$

$$u_i(x^*) = (a_i + \delta x_i^* \sum_j W_{ij} x_j^*) x_i^* - \frac{1}{2} x_i^{*2} = b_i x_i^{*2} - \frac{1}{2} x_i^{*2} = \frac{1}{2} x_i^{*2}$$

Utility of a player proportional to the square of its Bonacich centrality: a key network externality effect!

Welfare: $U(x^*) = \frac{1}{2} ||x^*||^2 = a'(I - \delta W)'(I - \delta W) a$

x^* is not the social optimum! Exercise: compute the *PoA*.

The key player

For some applications, it is important to individuate the most critical nodes in a network. Those nodes that if removed will generate the highest level of reduction in total activity.

$$u_i(x) = x_i - \frac{1}{2}x_i^2 + \delta x_i \sum_j W_{ij} x_j$$

- x^{*-i} Nash equilibrium with player i removed,
- $y^{-i} = \sum_j x^{*-i}$ total activity in the absence of player i
- *Key player* $i^* = \operatorname{argmin} y^{-i}$

The key player

$M = (I - \delta W)^{-1}$, $b = M\mathbb{1}$ Bonacich centrality, $x^* = b$

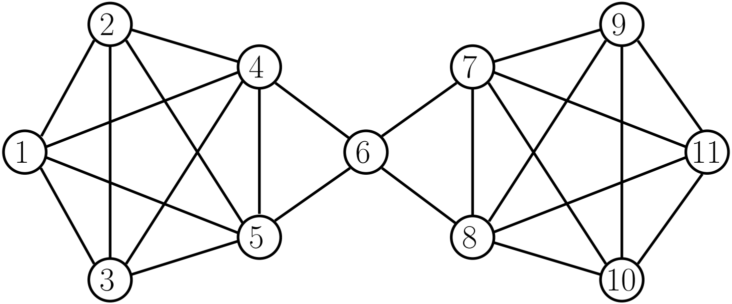
Theorem (Ballaster et al. 2004)

Assume that $\delta > 0$, W is symmetric, and $a_i = 1$. Then, the key player is

$$i^* = \operatorname*{argmax}_i \frac{b_i^2}{M_{ii}} \quad \textit{intercentrality measure}$$

$\frac{b_i^2}{M_{ii}} = b_i(1 + \frac{1}{M_{ii}} \sum_{j \neq i} M_{ij}) = b_i + \frac{b_i}{M_{ii}} \sum_{j \neq i} M_{ij}$

Who is the key player?



Opinion formation game

$$u_i(x) = -\frac{1}{2}\left[\sum_j W_{ij}(x_i - x_j)^2 + \rho_i(x_i - c_i)^2\right]$$

$w_i = \sum_j W_{ij}$ out-degree of node i, $P_{ij} = \frac{W_{ij}}{w_i}$, $\quad D = \mathrm{diag}\left(\frac{\rho_i}{w_i+\rho_i}\right)$

Nash equilibrium $x^* = (I - D)Px^* + Dc$

Opinion formation game

$$x^* = (I - D)Px^* + Dc$$

- If $\rho_i = 0$ for every i, $D = 0$, $x^* = Px^*$: infinite solutions forming a subspace whose dimensionality is given by the number of connected components in the graph described by W. If graph is connected Nash equilibria are the consensus configurations.
- If $\rho_i > 0$ for at least one i, then $(I - D)P$ is sub-stochastic. If the set $\mathcal{W} = \{i \in \mathcal{V} \,|\, \rho_i > 0\}$ is globally reachable in the graph, then $(I - D)P$ is asymptotically stable: the Nash equilibrium is unique and
$$x^* = (I - (I - D)P)^{-1}Dc$$

Opinion dynamics

The best reply linear system corresponds, in the two cases, to two celebrated opinion dynamics models

1. The French-De Groot model: $x(t+1) = Px(t)$
2. The Friedkin-Johnsen model $x(t+1) = (I - D)Px(t) + Dc$

Constrained quadratic games

$$u_i(x) = a_i x_i - \frac{1}{2}x_i^2 + \delta x_i \sum_j W_{ij} x_j$$

- $x_i \in [0, w_i]$ (constrained actions)
- Form of the best response set

$$\mathcal{B}_i(x_{-i}) = \min\left\{\max\left\{\delta \sum_j W_{ij} x_j + a_i, 0\right\}, w_i\right\}$$

Saturation terms → nonlinear...

Constrained quadratic games

Equation for Nash equilibria

$$x_i^* = \min\left\{\max\left\{\delta\sum_j W_{ij}x_j^* + a_i, 0\right\}, w_i\right\}$$

- Conditions for the existence and uniqueness of Nash equilibria

Constrained quadratic games

Equation for Nash equilibria

$$x_i^* = \min\left\{ \max\left\{ \delta \sum_j W_{ij} x_j^* + a_i, 0 \right\}, w_i \right\}$$

- Conditions for the existence and uniqueness of Nash equilibria
- W symmetric $\Rightarrow$ potential $\Rightarrow$ existence of Nash equilibria
- The strategic complements case $\delta > 0$ $\longleftarrow$ Next lecture!

Supermodular games

Supermodular games

Definition

A game $(\mathcal{V}, \{\mathcal{A}_i\}, \{u_i\})$ is called *super-modular* if

- $\mathcal{A}_i \subseteq \mathbb{R}$ compact subsets
- $u_i(x_i, x_{-i})$ upper semicontinuous in (x_i, x_{-i})
- $u_i(x_i, x_{-i})$ satisfy the increasing difference property:

$$u_i(b_i, x_{-i}) - u_i(a_i, x_{-i}) \leq u_i(b_i, y_{-i}) - u_i(a_i, y_{-i})$$

if $a_i \leq b_i$ and, componentwise, $x_{-i} \leq y_{-i}$

Supermodular games

Supermodular games model the so called *strategic complements effect*: the increase of one player's action makes more profitable for the others also to increase theirs.

Examples:

- Network coordination games
- Quadratic games with positive interaction term δ

Best response in supermodular games

Most of the properties of supermodular games stem from the following key fact:

Theorem

For a supermodular game $(\mathcal{V}, \{\mathcal{A}_i\}, \{u_i\})$, *the following facts hold:*

- *For every* $i \in \mathcal{V}$ *and* x_{-i}, *the best response set* $\mathcal{B}_i(x_{-i})$ *has a maximum and a minimum element denoted, respectively,* $\mathcal{B}_i^+(x_{-i})$ *and* $\mathcal{B}_i^-(x_{-i})$
- $\mathcal{B}_i^+(x_{-i})$ *and* $\mathcal{B}_i^-(x_{-i})$ *are monotone non-decreasing in* x_{-i}

Ideas of the proof.....

Nash equilibria in supermodular games

Definition

A *lattice* is a partially ordered set A for which every pairs $a, b \in \mathcal{A}$ possess a supremum (least upper bound) $a \vee b$ and a infimum (greatest lower bound) $a \wedge b$. A lattice A is *complete* if every non empty subset $B \subseteq A$ admits supremum and infimum.

Examples of lattices

- totally ordered sets $a \vee b = \max\{a, b\}$, $a \wedge b = \min\{a, b\}$
- $\mathcal{X} = \prod_i \mathcal{A}_i$ where $\mathcal{A}_i \subseteq \mathbb{R}$ compacts, with the order $x \leq y$ iff $x_i \leq y_i$ for all i. $(a \vee b)_i = \max\{a_i, b_i\}$, $(a \wedge b)_i = \min\{a_i, b_i\}$. This is also complete

Nash equilibria in supermodular games

Theorem

For a supermodular game $(\mathcal{V}, \{\mathcal{A}_i\}, \{u_i\})$, the following facts hold:

- *The set of pure Nash equilibria $\mathcal{N}$ is always non empty*
- *$\mathcal{N} \subseteq \mathcal{X}$ is a complete (sub)lattice*
- *In particular, there exists a minimum $\underline{x}^*$ and a maximum $\bar{x}^*$ pure Nash equilibria.*

Ideas in the proof

- Define $f^+ : \mathcal{X} \to \mathcal{X}$ by

$$f^+(x)_i = \mathcal{B}_i^+(x_{-i})$$

  largest best response map
- $x \leq y \;\Rightarrow\; f^+(x) \leq f^+(y)$
- Consider $x(t+1) = f^+(x(t))$
- If $x(0) = \sup \mathcal{X}$, then $x(t+1) \leq x(t)$ for all t
- $x(t) \downarrow x^* \in \mathcal{X}$ *iterated dominance technique* $\to$ algorithm
- x^* is a Nash
- x^* is the largest Nash
- All points can be proven through the map f^+ and the analogous f^-

Applications

$$u_i(x) = \rho_i(x_i) + \delta x_i \sum_j W_{ij} x_j$$

- $x_i \in [0, w_i]$
- $\delta > 0$: strategic complements
- game is supermodular
- Nash equilibria always exist
- $x_i^* = \min\left\{\max\left\{\delta \sum_j W_{ij} x_j^* + a_i, 0\right\}, w_i\right\}$
- Non uniqueness in general!
- A fundamental study for the case W row or column stochastic by [Como et al. 2021]

Partnership model with linear cost

$\mathcal{G} = (\mathcal{V}, \mathcal{E})$, W adjacency matrix of $\mathcal{G}$

$$u_i(x) = x_i(-c_i + \sum_j W_{ij}x_j), \quad x_i \in \mathcal{A}_i = [0, r_i]$$

- $\mathcal{B}_i(x_{-i}) = \begin{cases} 0 & \text{if } \sum_j W_{ij}x_j < c_i \\ \mathcal{A}_i & \text{if } \sum_j W_{ij}x_j = c_i \\ r_i & \text{if } \sum_j W_{ij}x_j > c_i \end{cases}$
- Without loss of generality and up to normalization, we assume that $\mathcal{A}_i = \{0, 1\}$ for all i

Partnership model with linear cost

$$u_i(x_i, x_{-i}) = x_i(-c_i + \sum_j W_{ij} x_j), \qquad x_i \in \mathcal{A}_i = \{0, 1\}$$

- $\mathcal{B}_i(x_{-i}) = \begin{cases} 0 & \text{if } \sum_j W_{ij} x_j < c_i \\ \mathcal{A}_i & \text{if } \sum_j W_{ij} x_j = c_i \\ 1 & \text{if } \sum_j W_{ij} x_j > c_i \end{cases}$

Partnership model with linear cost

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- a threshold model with similarities with the majority game

Partnership model with linear cost

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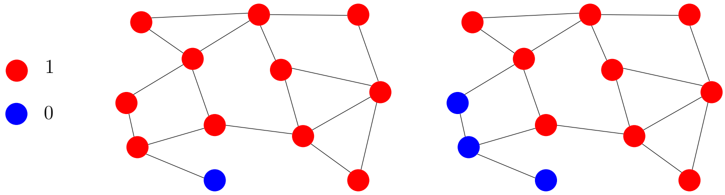
- a threshold model with similarities with the majority game
- $0\mathbb{1}$ is always a Nash equilibrium. $\mathbb{1}$ is a Nash equilibrium if $w_i \geq c_i$ for all $i \in \mathcal{V}$.

Partnership model: examples

$$u_i(x_i, x_{-i}) = x_i(-c_i + \sum_j W_{ij} x_j), \qquad x_i \in \mathcal{A}_i = \{0, 1\}$$

Two Nash equilibria for $c_i = 1.5$ for all i



Partnership model: examples

$$u_i(x_i, x_{-i}) = x_i(-c_i + \sum_j W_{ij} x_j), \qquad x_i \in \mathcal{A} = \{0, 1\}$$

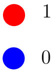
- The minimal Nash equilibrium is always $\underline{x}^* = 0$
- The maximal Nash equilibrium can be found with the iterated dominance technique $f^+(x)_i = \max \mathcal{B}_i(x_{-i})$

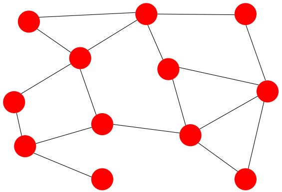
Example: $c_i = 1.5$ for all i

$x(t+1) = f^+(x(t))$

$x(0) = \mathbb{1}$

$t = 0$





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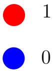
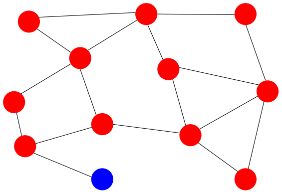
Example: $c_i = 1.5$ for all i

$x(t+1) = f^+(x(t))$
$x(0) = \mathbb{1}$

$t = 1$





Partnership model: examples

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- The minimal Nash equilibrium is always $\underline{x}^* = 0$
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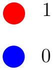
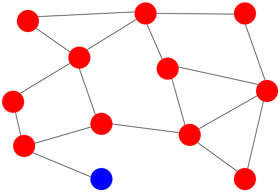
Example: $c_i = 1.5$ for all i

$x(t+1) = f^+(x(t))$ $\qquad t = 1$

$x(0) = \mathbb{1}$

$\bar{x}^* = x(1)$





Partnership model: examples

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- The minimal Nash equilibrium is always $\underline{x}^* = 0$
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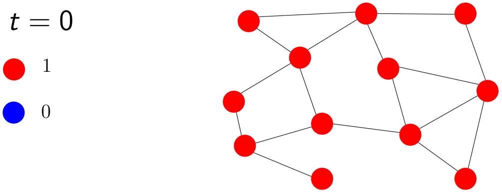
Example: $c_i = 2.5$ for all i

$x(t+1) = f^+(x(t))$
$x(0) = \mathbb{1}$

$t = 0$



Partnership model: examples

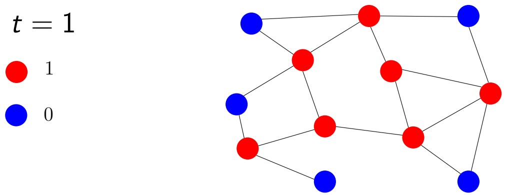
$$u_i(x_i, x_{-i}) = x_i(-c_i + \sum_j W_{ij}x_j), \qquad x_i \in \mathcal{A} = \{0, 1\}$$

- The minimal Nash equilibrium is always $\underline{x} = 0$
- The maximal Nash equilibrium can be found with the iterated dominance technique.

Example: $c = 2.5$

$x(t+1) = f^+(x(t))$ $\qquad t = 1$

$x(0) = \mathbb{1}$



Partnership model: examples

$$u_i(x_i, x_{-i}) = x_i(-c_i + \sum_j W_{ij} x_j), \qquad x_i \in \mathcal{A} = \{0, 1\}$$

- The minimal Nash equilibrium is always $\underline{x} = 0$
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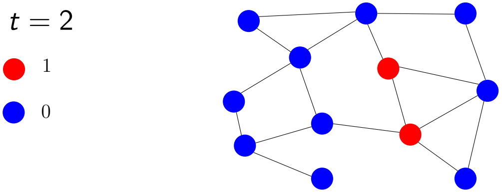
Example: $c = 2.5$

$x(t+1) = f^{+}(x(t))$
$x(0) = \mathbb{1}$

$t = 2$



Partnership model: examples

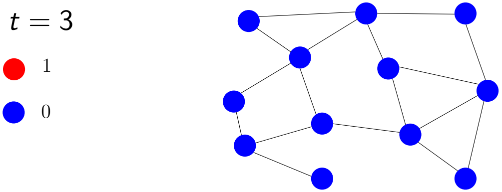
$$u_i(x_i, x_{-i}) = x_i(-c_i + \sum_j W_{ij} x_j), \qquad x_i \in \mathcal{A} = \{0, 1\}$$

- The minimal Nash equilibrium is always $\underline{x} = 0$
- The maximal Nash equilibrium can be found with the iterated dominance technique.

Example: $c = 2.5$

$x(t+1) = f^+(x(t))$
$x(0) = \mathbb{1}$

$t = 3$



Partnership model: examples

$$u_i(x_i, x_{-i}) = x_i(-c_i + \sum_j W_{ij} x_j), \qquad x_i \in \mathcal{A} = \{0, 1\}$$

- The minimal Nash equilibrium is always $\underline{x} = 0$
- The maximal Nash equilibrium can be found with the iterated dominance technique.

Example: $c = 2.5$

$x(t+1) = f^+(x(t))$ $\qquad t = 3$

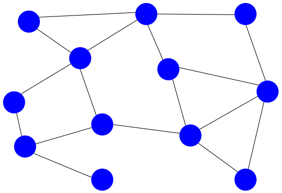
$x(0) = \mathbb{1}$

$\bar{x}^* = x(3) = \underline{x}^*$

Unique Nash!

● 1

● 0



Comparative statics

Theorem

Consider a supermodular game $(\mathcal{V}, \{\mathcal{A}_i\}, \{u_i\})$ where utilities $u_i(x_i, x_{-i}, c)$ depend on a vector c in such a way that

$$u_i(b_i, x_{-i}, c) - u_i(a_i, x_{-i}, c) \leq u_i(b_i, x_{-i}, c') - u_i(a_i, x_{-i}, c')$$

if $a_i \leq b_i$ and, componentwise, $c \leq c'$
Then, indicated with $\underline{x}^*(c)$ and $\bar{x}^*(c)$, respectively, the minimum and maximum of Nash equilibria as functions of c, we have that they are both increasing in c.

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