Game Theory and Network Systems

Summer School SIDRA 2021

Lecture 7

Evolutionary Game Theory

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Population Games

- continuous population of unitary total mass
- players choose actions from finite action set $\mathcal{A}$
- $\mathcal{Z} = \mathcal{P}(\mathcal{A}) = \{z \in \mathbb{R}_+^{\mathcal{A}} : \sum_a z_a = 1\}$ simplex of mixed strategies
- $z \in \mathcal{Z}$ population "strategy" (state)
- $z_a =$ fraction of player playing action a
- reward functions

$$r_a : \mathcal{Z} \to \mathbb{R} \qquad \forall a \in \mathcal{A}$$

- $r_a(z) =$ utility of players playing a in population state z
- reward vector field $r : \mathcal{Z} \to \mathbb{R}^{\mathcal{A}}$; $r(z) = (r_a(z))_{a \in \mathcal{A}}$
- monomorphic state $z = \delta^a$ vs polymorphic state

Special case 1: 2-player symm. game, random matching

- finite symmetric 2-player game with utilities $u_i(a,b) = U_{ab}$
- linear rewards

$$r_a(z) = \sum_{b \in \mathcal{A}} z_b u_i(a,b) = (Uz)_a \qquad r(z) = Uz$$

- Interpretation: every agent is randomly matched with one other agent, and they play a symmetric 2-player game. The payoffs of the game give the "fitness" level of the players.
- Example: (modified) Rock-Scissor-Paper $\mathcal{A} = \{R, S, P\}$

$$r(z) = Uz \qquad\qquad U = \begin{pmatrix} \gamma & +1 & -1 \\ -1 & \gamma & +1 \\ +1 & -1 & \gamma \end{pmatrix}$$

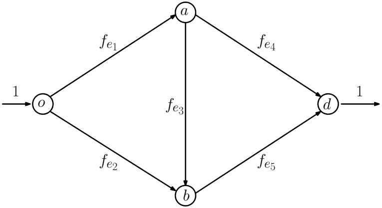
Canonical (Biological) Example: Hawk-Dove game

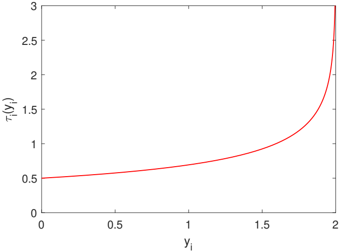
- $\mathcal{A} = \{H, D\}$

$$r(z) = Uz \qquad\qquad U = \begin{pmatrix} (v-c)/2 & v \\ 0 & v/2 \end{pmatrix}$$

- Interpretation: There is a resource of value v to be shared. If a player plays "Hawk," it is aggressive and will try to take the whole resource for itself. If the other player is playing "Dove," it will succeed in doing so. If both players are playing "Hawk," then they fight and they share the resource but lose c in the process. If they are both playing "Dove," then they just share the resource.

Special case 2: Continuous congestion game





- (convex) nondecreasing delay functions $\tau_e(f_e)$. E.g.,

$$\tau_e(f_e) = \frac{l_e}{1 - f_e/c_e}, \qquad e \in \mathcal{E}$$

$$r_a(z) = -c_a(z_a)$$

Nash equilibria

- Definition: $z^* \in \mathcal{Z}$ is a Nash equilibrium (NE) if

$$(z^*)^T r(z^*) \geq z^T r(z^*) \qquad \forall z \in \mathcal{Z}$$

$z^* \in \mathcal{Z}$ is a strict NE if

$$(z^*)^T r(z^*) > z^T r(z^*) \qquad \forall z \in \mathcal{Z} \setminus \{z^*\}$$

- Lemma: z^* strict NE $\Longrightarrow z^* = \delta^a$ monomorphic

Hawk-Dove game (cont'd)

- $\mathcal{A} = \{H, D\}$

$$r(z) = Uz \qquad U = \begin{pmatrix} (v-c)/2 & v \\ 0 & v/2 \end{pmatrix}$$

- $v > c > 0 \implies z^* = (1,0) = \delta^H$ strict NE
- $v = c > 0 \implies z^* = (1,0) = \delta^H$ non-strict monomorphic NE
- $0 < v < c \implies$ unique polymorphic NE

$$z^* = (v/c, (c-v)/c) = (1-\varepsilon)\delta^H + \varepsilon\delta^D \qquad \varepsilon = \frac{c-v}{c}$$

Evolutionary Stability

- Definition 1: $z^* \in \mathcal{Z}$ is an evolutionary stable strategy (ESS) if $\exists \bar{\varepsilon} > 0$ such that

$$(z^*)^T r((1-\varepsilon)z^* + \varepsilon z) > z^T r((1-\varepsilon)z^* + \varepsilon z)$$

for every $z \in \mathcal{Z} \setminus \{z^*\}$ and every $0 < \varepsilon < \bar{\varepsilon}$

- Interpretation: z^* is ESS if it cannot be invaded by any $z \neq z^*$, i.e., starting with a population playing z^*, and replacing fraction ε of it with mutant population playing z, the new players do worse (have lower fitness) than the original ones.

Evolutionary Stability

- Definition 1: $z^* \in \mathcal{Z}$ is an evolutionary stable strategy (ESS) if $\exists \overline{\varepsilon} > 0$ such that

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- Definition 1*: $z^* \in \mathcal{Z}$ is ESS if

$$(z^* - z)^T r(z) > 0$$

for all $z \neq z^*$ in a neighborhood of z^* in $\mathcal{Z}$

- Definition 1 $\Leftrightarrow$ Definition 1* (neighborhood $\{||z - z^*||_1 < \overline{\varepsilon}\}$)

Evolutionary Stability

- Definition 2: z^* is an evolutionary stable strategy (ESS) if

$$(z^*)^T r(z^*) \geq z^T r(z^*)$$

for all $z \in \mathcal{Z}$, and there exists a neighborhood of z^* where

$$z \neq z^* \quad (z^*)^T r(z^*) = z^T r(z^*) \qquad \Longrightarrow \qquad (z^*)^T r(z) > z^T r(z)$$

- Interpretation: An evolutionarily stable strategy z^* is a Nash equilibrium. If z^* is not a strict Nash equilibrium, then any other strategy z that is a best response to z^* must be worse against itself than against z^*.

Evolutionary Stability

- Proposition: Definition 1 and Definition 2 are equivalent
- Proof:

Evolutionary Stability

- Theorem: Population game with finite action $\mathcal{A}$, rewards $r(z)$.

(i) z^* ESS $\implies$ z^* isolated NE

(ii) z^* strict NE $\implies$ z^* ESS

Hawk-Dove game (cont'd)

- $\mathcal{A} = \{H, D\}$

$$r(z) = Uz \qquad\qquad U = \begin{pmatrix} (v-c)/2 & v \\ 0 & v/2 \end{pmatrix}$$

- $v > c > 0 \Longrightarrow z^* = (1, 0)$ strict NE $\Rightarrow$ monomorphic ESS

Hawk-Dove game (cont'd)

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- $v > c > 0 \Longrightarrow z^* = (1, 0)$ strict NE $\Rightarrow$ monomorphic ESS
- $v = c > 0 \Longrightarrow z^* = (1, 0)$ non-strict monomorphic NE

Is z^* an ESS?

Hawk-Dove game (cont'd)

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$$r(z) = Uz \qquad U = \begin{pmatrix} (v-c)/2 & v \\ 0 & v/2 \end{pmatrix}$$

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Is z^* an ESS?

- $0 < v < c \implies$ unique polymorphic NE

$$z^* = (v/c, (c-v)/c) = (1-\varepsilon)\delta^H + \varepsilon\delta^D \qquad \varepsilon = \frac{c-v}{c}$$

Is z^* an ESS?

(Modified) Rock-Scissor-Paper

- $\mathcal{A} = \{R, S, P\}$

$$r(z) = Uz \qquad U = \begin{pmatrix} \gamma & +1 & -1 \\ -1 & \gamma & +1 \\ +1 & -1 & \gamma \end{pmatrix}$$

- $0 \leq \gamma < 1 \implies$ unique NE is polymorphic

$$z^* = (1/3, 1/3, 1/3)$$

Is z^* an ESS?

Replicator Dynamics

- Population game with finite action $\mathcal{A}$, rewards $r(z)$
- Replicator Dynamics

$$\dot{z}_a = z_a \left(r_a(z) - \sum_b z_b r_b(z) \right) \qquad \forall a \in \mathcal{A}$$

- Interpretation: per capita rate of growth of fraction of a-players, i.e., $(\dot{\log z_a}) = \dot{z}_a / z_a$ is the difference between the reward $r_a(z)$ and the average reward

$$\bar{r}(z) = z^T r(z) = \sum_b z_b r_b(z)$$

Replicator Dynamics

- Proposition: z^* NE $\Longrightarrow$ z^* equilibrium point for replicator dynamics
- Converse is NOT true. E.g., every pure strategy δ^a is an equilibrium point
- Definition: $z^* \in \mathcal{Z}$ is a restricted NE if $z_a^* > 0 \Rightarrow r_a(z^*) = \bar{r}(z^*)$
- Proposition: If z^* equilibrium point for replicator dynamics then z^* is a restricted NE

Replicator Dynamics

- Theorem: Population game with finite action $\mathcal{A}$, rewards $r(z)$.

z^* ESS $\implies$ z^* locally asymptotically stable eq. pt. for replicator

Replicator Dynamics

- Theorem: Population game with finite action $\mathcal{A}$, rewards $r(z)$.

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- Proof: Let
$$V(z) = \sum_a z_a^* \log \frac{z_a^*}{z_a}$$

(vwith convention $0 \log 0 = 0$ and $0/0 = 1$). By Jensen's inequality

$$-V(z) = \sum_a z_a^* \log \frac{z_a}{z_a^*} \leq \log \sum_a z_a^* \frac{z_a}{z_a^*} = 0$$

with $=$ iff $z = z^*$. So, $V(z)$ is positive definite w.r.t. z^*. Since z^* is ESS, Definition 1* implies that, in neighborhood of z^*

$$\begin{aligned} \dot{V}(z) &= \sum_a \frac{\partial V}{\partial z_a}(z)\dot{z}_a = -\sum_a \frac{z_a^*}{z_a} z_a \left(r_a(z) - \sum_b z_b r_b(z) \right) \\ &= -(z^* - z)^T r(z) \leq 0 \end{aligned}$$

with $=$ iff $z = z^*$. Hence $V(z)$ strict Lyapunov function w.r.t z^*. ■

Nonconvergence of Replicator Dynamics

- two-player symmetric zero-sum game (e.g., Rock-Scissor-Paper)
- associated population game has reward vector

$$r(z) = Uz \text{ with } U^T = -U$$

- Proposition: Population game with $r(z) = Uz$ and $U^T = -U$.

Then, Replicator dynamics does NOT converge from any $z(0) \neq z^*$

Proof: Let $V(z) = \sum_a z_a^* \log \frac{z_a^*}{z_a}$. Then,

$$\dot{V}(z) = \sum_a z_a r_a(z) = z^T U z = z^T U^T z = -z^T U z = 0$$

Potential Population Games