The student is asked to solve problems 1. and 2. and one between 3. and 4.

Problem 1. Consider the Cournot competition model with 2 firms,

$$u_i(x_1, x_2) = x_i p(x_1 + x_2) - c_i(x_i)$$

and a linear inverse demand function

$$p(x) = \max\{0, K - x\}$$

Compute the best response sets and the Nash equilibria for the case when

(a) the costs are linear $c_i(x) = c_i x$ but the marginal costs c_i for the two firms are possibly different

(b) the cost is the same but quadratic $c_i(x) = cx^2$

Consider now the extension to n firms and linear cost $c_i(x) = cx$ that is the same for all firms.

(c) Compute the best response sets and the Nash equilibria

(d) Discuss the behavior for $n \to +\infty$

Problem 2. Consider a game with player set $\mathcal{V} = \{1, 2, 3\}$, action set $\mathcal{A} = \{-1, +1\}$, and utility functions

$$u_1(x_1, x_2, x_3) = 2x_1x_2 + 2x_1x_3\,, \qquad u_2(x_1, x_2, x_3) = 2x_1x_2 - ax_2x_3\,,$$
$$u_3(x_1, x_2, x_3) = 2x_1x_3 - ax_2x_3\,,$$

where $a > 0$ is a parameter.

(a) Recognize that this game is pairwise graphical with respect to the 3-cycle C_3 writing down the utility functions of the constituent two-player games.

(b) Determine the best response functions for: (i) $a = 1$; (ii) $a = 2$; and (iii) $a = 3$.

(c) Determine the (pure strategy) Nash equilibria of the game for: (i) $a = 1$; (ii) $a = 2$; and (iii) $a = 3$.

(d) Determine whether the game is potential for arbitrary $a > 0$.

(e) For $a = 2$, consider the Noisy Best Response dynamics with parameter β and indicate with $\pi^{(\beta)}$ the equilibrium distribution. Determine $\lim_{\beta \to +\infty} \pi^{(\beta)}_{(1,1,1)}$.

Problem 3. Fix a weighted graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, W)$ with $n = |\mathcal{V}| \geq 2$, that is undirected ($W = W'$), without selfloops, and such that $w_i = \sum_j W_{ij} > 0$ for every node i. Consider the corresponding quadratic game on $\mathcal{G}$, $(\mathcal{V}, \mathbb{R}, \{u_i\})$ with utility functions

$$u_i(x) = -\frac{x_i^2}{2} + x_i + \delta \sum_{j \neq i} W_{ij} x_j x_i\,, \qquad i \in \mathcal{V}\,,$$

where $\delta \geq 0$ is a scalar parameter. We make the assumption that

$$\delta w_i < 1\,, \qquad \forall i \in \mathcal{V}\,. \tag{1}$$

and we recall that, in this case, the game has a unique Nash equilibrium given by

$$x^* = M\mathbb{1}$$

with $M = (I - \delta W)^{-1}$ and $\mathbb{1}$ the vector with all components equal to 1. Define

$$y = \sum_{j \in \mathcal{V}} x_j^*\,.$$

We want to study the problem of determining the "key player" $i \in \mathcal{V}$ whose removal will have a maximum reduction on y. Formally, for each $i \in \mathcal{V}$, let $\mathcal{G}^{(-i)} = (\mathcal{V} \setminus \{i\}, \mathcal{E}^{(-i)}, W^{(-i)})$ where $\mathcal{E}^{(-i)}$ is the subset of edges in $\mathcal{E}$ not touching node i and $W^{(-i)}$ is the matrix obtained from W by removing the i-th row and the i-th column.

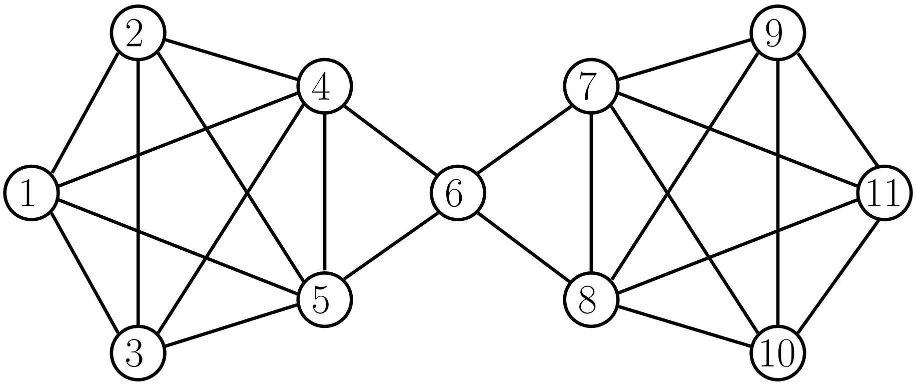

Figure 1: Graph of Prolem 3 (i).

(a) Prove that, under assumption (1), the quadratic game on the restricted graph $\mathcal{G}^{(-i)}$ admits a unique Nash equilibrium

$$x^{*(-i)} = M^{(-i)}\mathbb{1}\,.$$

(b) Prove that

$$M_{ij}M_{ik} = M_{ii}(M_{jk} - M_{jk}^{(-i)})$$

for every $k \neq i \neq j$.

(c) Prove that a node i^* in $\mathcal{V}$ maximizes

$$y - y^{(-i)}\,, \qquad \text{where } y^{(-i)} = \sum_{j \neq i} x_j^{*(-i)}$$

if and only if i^* maximizes the expression

$$z_i^2/M_{ii}$$

where $z = M\mathbb{1}$.

(d) In the special case when W is a the adjacency matrix of the graph depicted in Figure 1, for values $\delta = 0.1$ and $\delta = 0.2$, compute the quantities z_i and M_{ii} for every node i and individuate a "key player".

Problem 4. Consider the following stylized model of traffic congestion. There is a finite set of parallel routes $\mathcal{R} := \{1, 2, \ldots, k\}$ connecting one city (origin) to another (destination). There is a population of commuters: each commuter uses one route. The delay incurred by anycommuter usingroute $i \in \mathcal{R}$ is given by $\tau_i(z_i)$, where z_i is the fraction of commuters using route i, and

$$\tau_i : [0, 1] \to (0, +\infty)$$

is a differentiable increasing latency function. A *Wardrop equilibrium* is a probability vector $z^* \in \mathcal{P}(\mathcal{R})$ such that, for every $i \in \mathcal{R}$, one has that

$$z_i^* > 0 \qquad \Longrightarrow \qquad \tau_i(z_i^*) \leq \tau_j(z_j^*)\,, \quad \forall j \in \mathcal{R}\,.$$

(i.e., the delay on every route used by a nonzero fraction of commuters is not higher than the delay on any other route: the interpretation is that, in such a situation, no commuter has an incentive to modify his/her route choice.) Define

$$\Phi(z) := \sum_{i \in \mathcal{R}} \int_0^{z_i} \tau_i(s) \mathrm{d}s\,, \qquad z \in \mathcal{P}(\mathcal{R})\,. \tag{2}$$

(a) Show that $\Phi(z)$ is strictly convex on $\mathcal{P}(\mathcal{R})$.

(b) Show that z^* is a minimum of $\Phi(z)$ on $\mathcal{P}(\mathcal{R})$ if and only if z^* is a Wardrop equilibrium. *(hint: you are looking at the minimization of a convex function with linear constraints)*

(c) Use (a) and (b) to conclude that there exists a unique Wardrop equilibrium $z^* \in \mathcal{P}(\mathcal{R})$.

Now, let

$$g_i(z) := \frac{\exp(-\beta \tau_i(z_i^n))}{\sum_{r \in \mathcal{R}} \exp(-\beta \tau_r(z_r^n))}\,, \qquad r \in \mathcal{R}\,,$$

where $\beta > 0$ is a parameter whose inverse $1/\beta$ is a measure of noise. Let $z(t)$ be the solution of the Cauchy problem associated to the ODE

$$\dot{z} = g(z) - z\,, \tag{3}$$

with a given initial condition $z(0) \in \mathcal{P}(\mathcal{R})$. Define

$$V_\beta(z) := \Phi(z) - \frac{1}{\beta} \mathrm{H}(z)\,,$$

where $\Phi(z)$ is as in (2) and $\mathrm{H}(z) := -\sum_i z_i \log z_i$ is the entropy of z. (Convention: $0 \log 0 = 0$.)

(e) Show that the entropy function is strictly concave, so that $V_\beta(z)$ is strictly convex.

Let z^β be the unique minimum of $V_\beta(z)$ on $\mathcal{P}(\mathcal{R})$.

(f) Show that $z^\beta \to z^*$ as $\beta \to \infty$.

(g) Prove that $z_i^\beta > 0$ for all $i \in \mathcal{R}$.

(h) Use (g) to prove that $g(z^\beta) = z^\beta$, i.e., z^β is an equilibrium for (3).

(i) Prove that, for every initial condition $z(0) \in \mathcal{P}(\mathcal{R})$, the solution of the initial value problem associated to (3) satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t} V_\beta(z(t)) \leq 0\,,$$

with equality of and only if $z(t) = z^\beta$. Conclude that z^β is a globally attractive equilibrium for the dynamical system (3) on $\mathcal{P}(\mathcal{R})$.

(j) Prove that by using dynamic feedback tolls $\omega_i(z_i) = z_i \cdot \tau(z_i)$, the dynamical system (3) admits a globally asymptotically stable equilibrium $\overline{z}^\beta$ such that $\overline{z}^\beta \overset{\beta \to +\infty}{\longrightarrow} z^\circ$, where

$$z^\circ = \underset{z \in \mathcal{P}(\mathcal{R})}{\operatorname{argmin}} \sum_{i \in \mathcal{R}} z_i \tau_i(z_i)$$

is the social optimum traffic assignment problem.

(k) (optional) Generalize the results to the case when the routes from the origin to the destination are not parallel and there is a link-route incidence matrix H.