Game Theory and Network Systems

Summer School SIDRA 2021

Lecture 9

Optimal Targeting in Supermodular Games

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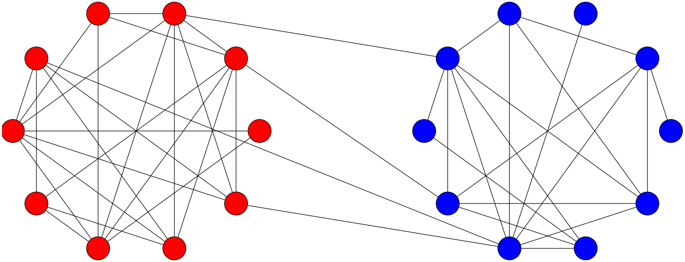
Network games

[Galeotti ea, "Network Games," Rev. of Economic Studies, 2010]:
"even the simplest games played on networks have multiple equilibria, which display a bewildering range of possible outcomes"

complexity vs structure

- multiple Nash equilibria, behavior of learning dynamics
- impact of network topology, strategic complements/substitutes
- tractable classes: potential, super-modular, quadratic games
- network intervention, targeting
- network formation

Network coordination games

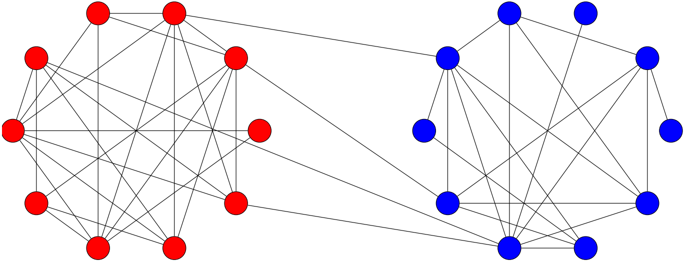


- (weighted, directed) graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, W)$
- node i player choosing $x_i \in \{0, 1\}$ so as to maximize utility

$$u_i(x) = \sum_{j \neq i} W_{ij} \left(r_i x_i x_j + (1 - r_i)(1 - x_i)(1 - x_j) \right)$$

where $r_i \in [0, 1]$ threshold

Network coordination games



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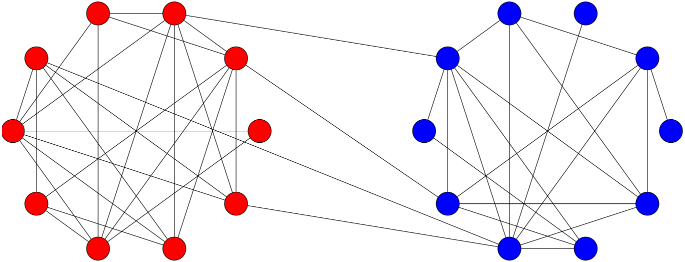
- best response: threshold rule

$$\mathcal{B}_i(x_{-i}) = \begin{cases} \{1\} & \text{if} \quad \sum_j P_{ij} x_j > r_i \\ \{0, 1\} & \text{if} \quad \sum_j P_{ij} x_j = r_i \\ \{0\} & \text{if} \quad \sum_j P_{ij} x_j < r_i \end{cases}$$

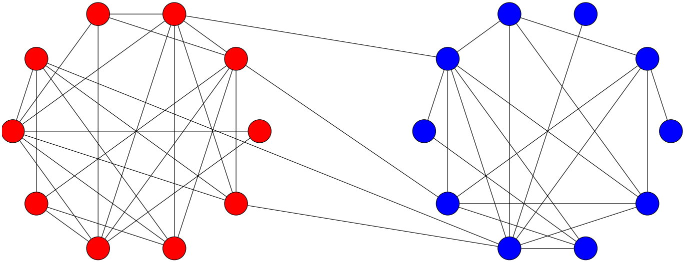
where P normalized weight matrix

Network coordination games



- [Morris,2000] Nash equilibria $\leftrightarrow$ network cohesiveness
- [Granovetter, 1978] sync best response dynamics, complete $\mathcal{G}$
- [Kempe, Kleinberg & Tardos, 2003] Optimal targeting (max contagion), submodularity in expectation with random thresholds
- ...

Controlling network coordination games



- how to force the system to the most efficient Nash equilibrium?
- optimal targeting: selection of k nodes that if forced to play 1 have the largest possible impact on the system
- applications in marketing, election campaigns, diffusion of innovation in social networks, ...

Maximizing the Spread of Influence through a Social Network [Kempe, Kleinberg & Tardos, 2003]

- coordination game on given graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, W)$

$$u_i(x) = \sum_{j \neq i} W_{ij} \left(r_i x_i x_j + (1 - r_i)(1 - x_i)(1 - x_j) \right) \qquad x_i \in \{0, 1\}$$

- Problem: what is the set of nodes $\mathcal{S} \subseteq \mathcal{Z}$ with $|\mathcal{S}| = k$ such that, if the actions of players in $\mathcal{S}$ is frozen at 1, while the others start at 0 and iteratively do "irreversible" best response, then the final number of nodes converted to 1 is maximized?

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- Given $\mathcal{G}$, let

$$f(\mathcal{S}) = \mathbb{E}\left[\# \text{ nodes finally converted to play } 1\right]$$

where expectation is taken over thresholds r_i that are assumed independent and identically distributed uniformly on $[0, 1]$

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- Theorem: $f : \{0, 1\}^{\mathcal{V}} \to \mathbb{R}$ is submodular, i.e.,

$$\mathcal{S} \subseteq \mathcal{U} \qquad \Longrightarrow \qquad f(\mathcal{S} \cup \{i\}) - f(\mathcal{S}) \geq f(\mathcal{U} \cup \{i\}) - f(\mathcal{U}) \qquad \forall i \in \mathcal{V}$$

"decreasing differences"

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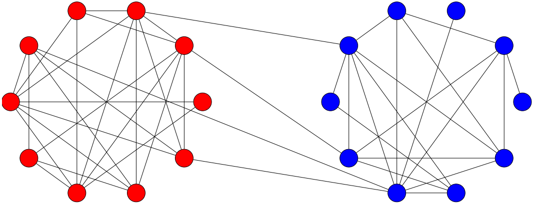
- Theorem [Nemhauser ea'78]: $f : \{0,1\}^{\mathcal{V}} \to \mathbb{R}$ monotone submodular. Let $\mathcal{S} \subseteq \mathcal{V}$ with $|\mathcal{S}| = k$ be obtained by selecting elements one at a time, each time choosing an element that provides the largest marginal increase in the function value. Let $\mathcal{S}^* \subseteq \mathcal{V}$ be a maximizer of $f(\mathcal{U})$ among all $\mathcal{U} \subseteq \mathcal{V}$ with $|\mathcal{U}| = k$. Then,

$$f(\mathcal{S}) \geq \left(1 - \frac{1}{e}\right) f(\mathcal{S}^*)$$

(while optimizing submodular functions is NP-hard in general)

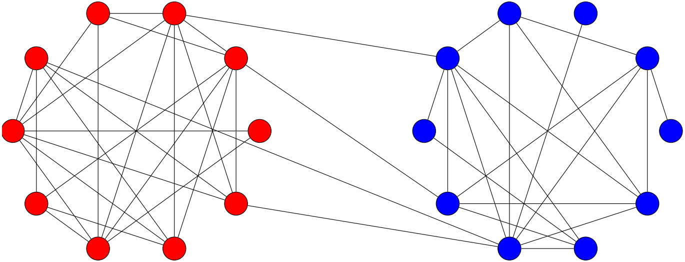
Maximizing the Spread of Influence through a Social Network [Kempe, Kleinberg & Tardos, 2003]



$$f(\mathcal{S}) = \mathbb{E}\,[\#\text{ nodes finally converted to play 1}]$$

- Theorem: $f : \{0,1\}^{\mathcal{V}} \to \mathbb{R}$ is submodular
- Corollary: Greedy optimal targeting with sub-optimality guarantees
- with given thresholds r_i, problem is no longer submodular

Optimal targeting in super-modular games



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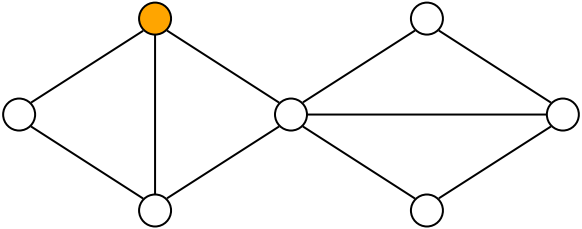
- strategic complements, super-modular game

$$x_{-i} \geq y_{-i} \quad \Longrightarrow \quad u_i(1, x_{-i}) - u_i(0, x_{-i}) \geq u_i(1, y_{-i}) - u_i(0, y_{-i})$$

- how to force the system to maximal Nash equilibrium $x^* = \mathbb{1}$?

Optimal targeting in super-modular games



- Control set $\mathcal{S} \subseteq \mathcal{V} \quad \Longrightarrow \quad$ game with $x_{\mathcal{S}}$ frozen to 1
- Monotone Best Response (MBR) Path:

$$\mathcal{S} = \mathcal{U}_0 \subset \mathcal{U}_1 \subset \ldots \subset \mathcal{U}_l = \mathcal{V}$$

$$\mathcal{U}_k = \mathcal{U}_{k-1} \cup \{i_k\} \qquad u_{i_k}(\mathbb{1}_{\mathcal{U}_k}) \geq u_{i_k}(\mathbb{1}_{\mathcal{U}_{k-1}}) \qquad k = 1, \ldots, l$$

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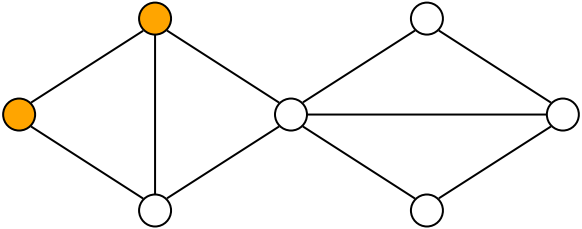

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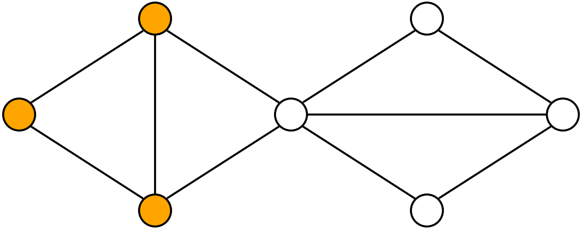

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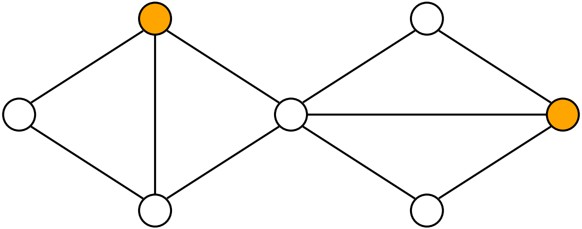

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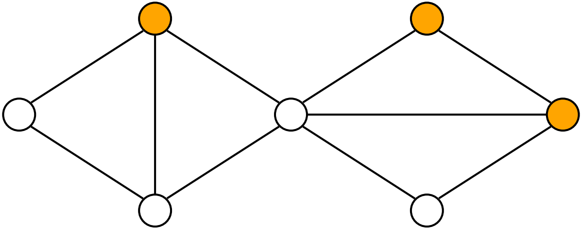
- Sufficient control set: $\mathcal{S} \subseteq \mathcal{V}$ s.t. $\exists$ MBR path from $\mathcal{S}$ to $\mathcal{V}$
- Lemma: For super-modular games $\exists$ MBR path $\Leftrightarrow$ $\exists$ BR path

Optimal targeting in super-modular games



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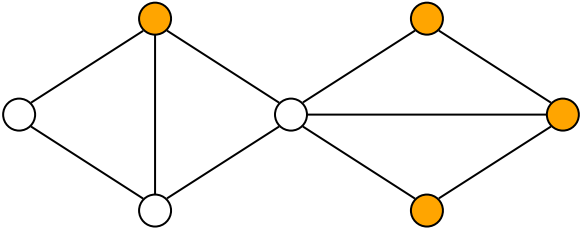

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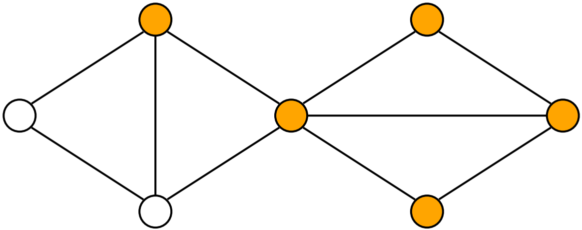

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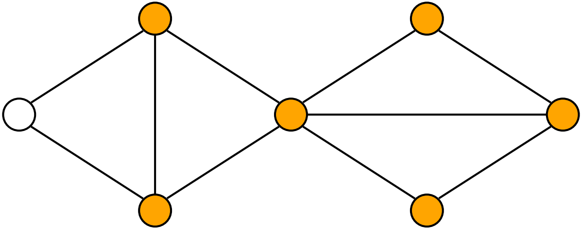

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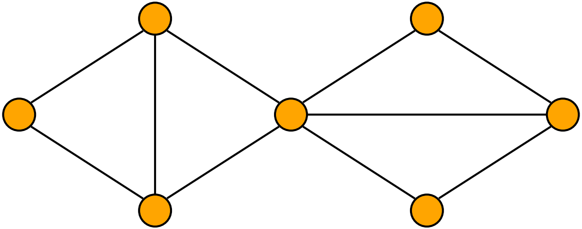

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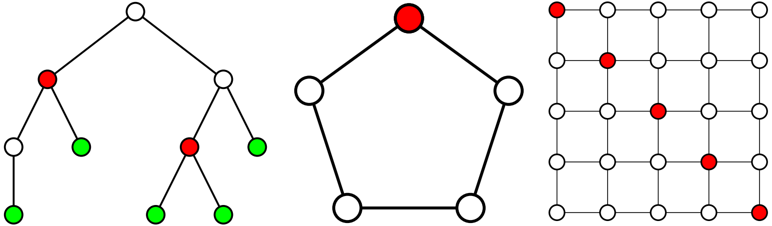
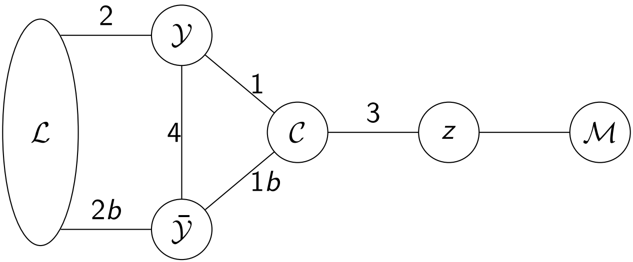


- sufficient control set: $\mathcal{S} \subseteq \mathcal{V}$ s.t. $\exists$ MIP from $\mathcal{S}$ to $\mathcal{V}$
- Minimal Sufficient Control (MSC) Problem:
  Find sufficient control set $\mathcal{S} \subseteq \mathcal{V}$ of minimal size $|\mathcal{S}|$
- c.f. [Kempe, Kleinberg & Tardos, 2003]: choose k nodes for max contagion, submodularity in expectation with random thresholds

Complexity of optimal targeting

- problem MSC: find sufficient control set $\mathcal{S} \subseteq \mathcal{V}$ of minimal size
- Theorem [Durand, Como & Fagnani, 2020]:
  MSC is NP-hard on general graph
- Proof: 3-SAT can be reduced to MSC



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Distributed optimal targeting

- Idea: start from maximal Nash $x^* = \mathbb{1}$, try to follow a MBR path backwards by moving according to "worst response". Add noise.

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- Z_t^ε Markov chain on $\{0,1\}^{\mathcal{V}}$, transition probabilities

$$P^\epsilon_{x,y} = \begin{cases} 1/n & \text{if} \quad y = x - \delta_i \text{ and } u_i(y) \leq u_i(x) \\ \varepsilon/n & \text{if} \quad y = x + \delta_i \text{ and } u_i(y) \geq u_i(x) \\ 0 & \text{if} \quad \text{otherwise} \end{cases}$$

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- Theorem [Durand, Como & Fagnani, 2020]

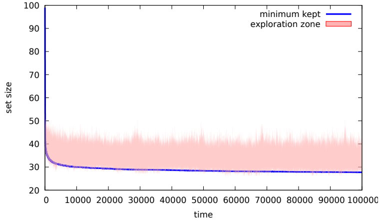
Z_t^ε reversible Markov chain with stationary distribution

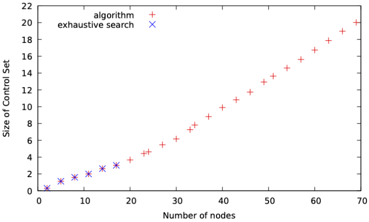
$$\mu^\varepsilon_x \begin{cases} \propto \varepsilon^{||x||_1} & \text{if} \quad \mathrm{supp}(x) \text{ sufficient control set} \\ = 0 & \text{if} \quad \mathrm{supp}(x) \text{ otherwise} \end{cases}$$

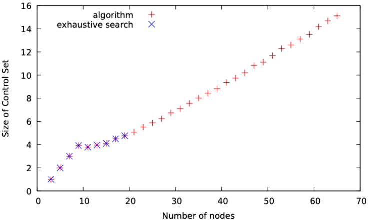
- Corollary:

μ^ε concentrates on sufficient control sets of minimal size as $\varepsilon \downarrow 0$

Distributed optimal targeting: simulations

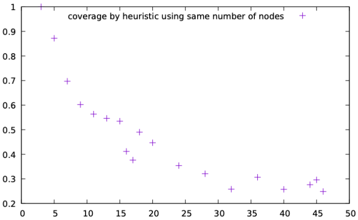


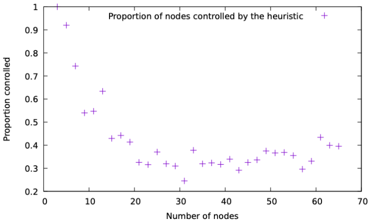


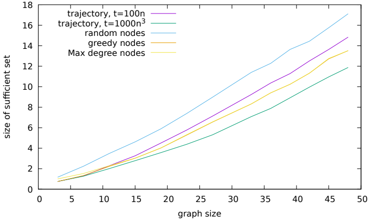


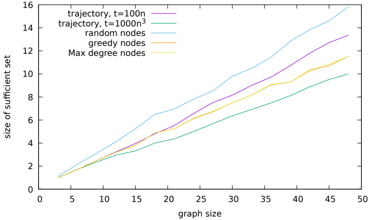
Erdos-Renyi with $p = 0.4$ (dense) and $p = \frac{4}{n} \log n$ (sparse), $n = 100$

Distributed optimal targeting vs heuristics









Conclusion

Optimal targeting in super-modular games:

- NP-hard problem
- distributed approximation algorithm
- different interventions and costs?
- richer action spaces?
- games with strategic substitutes? E.g, public good