Game Theory and Network Systems

Summer School SIDRA 2021

Lecture 8

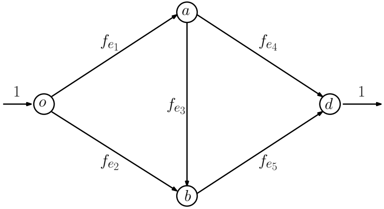
Interventions in Traffic Networks

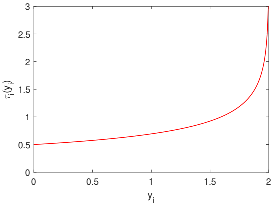
Giacomo Como and Fabio Fagnani
Department of Mathematical Sciences
Politecnico di Torino
Bertinoro, July 12-14, 2021

Today's lectures

- game theory models interactions among multiple selfish (conflicting) rational decision makers
- how can a system planner influence the outcome of a game?
- problem studied in mechanism design: applications to auctions, voting systems, matching mechanisms, ...
- today: pricing in congestion networks and optimal targeting in social networks

Continuous congestion game





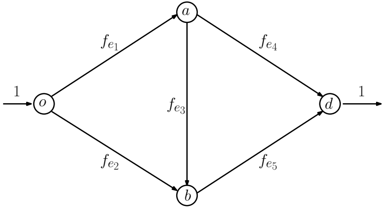
- Transportation network = directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$

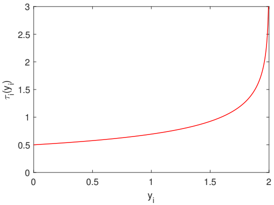
$$\mathcal{A} = \{o\text{-}d \text{ paths}\} \qquad H \in \mathbb{R}^{\mathcal{E}\times\mathcal{A}} \qquad H_{ea} = \begin{cases} 1 & \text{if} \quad e \text{ along } a \\ 0 & \text{if} \quad e \text{ not along } a \end{cases}$$

- nondecreasing convex link delay functions $\tau_e(y_e)$
- delay vector $\tau(y) = (\tau_e(y_e))_{e\in\mathcal{E}}$
- rewards = − total delay on different paths $r(z) = -H^T\tau(Hz)$
- average delay of population state $z \in \mathcal{Z} = \mathcal{P}(\mathcal{A})$

$$z^T r(z) = z^T H^T \tau(Hz) = \sum_e y_e \tau(y_e) \qquad y = Hz$$

System-Optimum Traffic Assignment





- node-link incidence matrix $B \in \{-1, 0, +1\}^{\mathcal{V} \times \mathcal{E}}$

$$B_{\theta(e),e} = +1\,, \quad B_{\kappa(e),e} = -1\,, \quad B_{ke} = 0 \quad \forall k \neq \theta(e), \kappa(e)$$

$$\theta(e) = \text{tail node of link } e \qquad \kappa(e) = \text{head node of link } e$$

- System-Optimum Traffic Assignment (SO-TAP):

$$\min_{\substack{y \geq 0 \\ By = \delta^{(o)} - \delta^{(d)}}} \sum_{e \in \mathcal{E}} y_e \tau_e(y_e)$$

- centralized convex optimization

Wardrop Equilibrium and the Price of Anarchy

- Wardrop equilibrium is flow vector $y^* = H^T z^*$ where z^* NE

$$z_a^* > 0 \qquad \Longrightarrow \qquad (H\tau(y^*))_a = \min_{b \in \mathcal{A}} \{(H\tau(y^*))_b\}$$

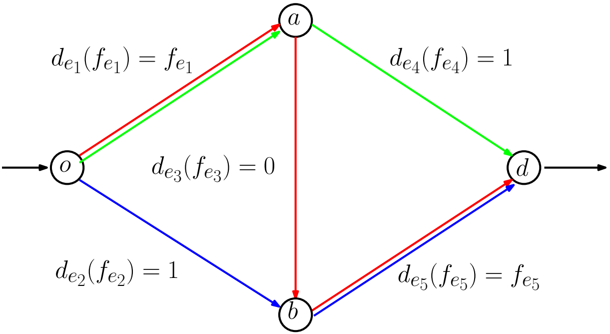
- price of anarchy is

$$\text{PoA} = \frac{\displaystyle\sum_{e \in \mathcal{E}} y_e^* \tau_e(y_e^*)}{\displaystyle\min_{\substack{y \geq 0 \\ By = \delta^{(o)} - \delta^{(d)}}} \sum_{e \in \mathcal{E}} y_e \tau_e(y_e)},$$

total delay at the Wardrop equilibrium / total delay at system optimum

- Observe: PoA≥ 1

Example - Braess paradox

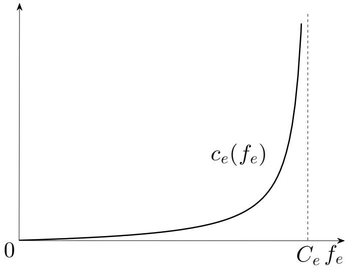


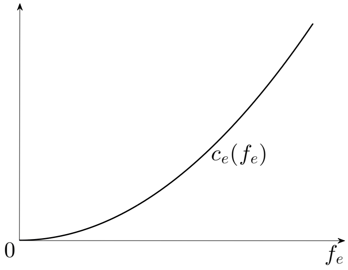
- Three paths $z_{p^{(1)}}$, $z_{p^{(2)}}$, $z_{p^{(3)}}$
- Wardrop equilibrium: $z_{p^{(1)}} = z_{p^{(2)}} = 0$, $z_{p^{(3)}} = 1$
- Social optimum: $z_{p^{(1)}} = z_{p^{(2)}} = 1/2$, $z_{p^{(3)}} = 0$

Convex Network Flow Optimization

$$M(\nu) := \min_{\substack{y \geq 0 \\ By = \nu}} \sum_{e \in \mathcal{E}} \psi_e(y_e)$$

- $\nu \in \mathbb{R}^{\mathcal{V}}$ s.t. $\sum_e \nu_e = 0$ exogenous flow vector
- convex nondecreasing cost functions $\psi_e(y_e)$ on every link $e \in \mathcal{E}$





Convex Network Flow Optimization (cont'd)

$$M(\nu) := \min_{\substack{y \geq 0 \\ By = \nu}} \sum_{e \in \mathcal{E}} \psi_e(y_e)$$

- Lagrange multipliers λ_i for every node $i \in \mathcal{V}$
- Lagrangian function

$$\begin{aligned} L(f, \lambda, \nu) &= \sum_{e \in \mathcal{E}} \psi_e(f_e) + \sum_{i \in \mathcal{V}} \lambda_i \left(\sum_{e \in \mathcal{E}: \kappa(e) = i} f_e - \sum_{e \in \mathcal{E}: \theta(e) = i} f_e + \nu_i \right) \\ &= \sum_{e \in \mathcal{E}} \left(\psi_e(f_e) - f_e(\lambda_{\theta(e)} - \lambda_{\kappa(e)}) \right) + \sum_{i \in \mathcal{V}} \lambda_i \cdot \nu_i \end{aligned}$$

- Dual function:

$$D(\lambda, \nu) := \min_{y \geq 0} L(f, \lambda, \nu)$$

Complementary slackness conditions

Lemma: For convex nondecreasing cost $\psi_e(y_e)$ and Lagrange multipliers $(\lambda_{\theta(e)}, \lambda_{\kappa(e)})$ in $\mathbb{R}^2$

$$y_e^* \in \operatorname*{argmin}_{y_e \geq 0} \left\{ \psi_e(y_e) - (\lambda_{\theta(e)} - \lambda_{\kappa(e)}) y_e \right\} \qquad (1)$$

if and only if

$$\psi_e'(y_e^*) \geq \lambda_{\theta(e)} - \lambda_{\kappa(e)}\,, \qquad y_e^* \left(\psi_e'(y_e^*) - (\lambda_{\theta(e)} - \lambda_{\kappa(e)}) \right) = 0$$

Moreover, if $\psi_e(\,\cdot\,)$ is strictly convex and

$$\lambda_{\theta(e)} - \lambda_{\kappa(e)} < \sup\{\psi_e'(y_e) :\, y_e \geq 0\}\,,$$

then the minimizer in (1) is unique and given by

$$y_e^* = \begin{cases} 0 & \text{if} \quad \lambda_{\theta(e)} - \lambda_{\kappa(e)} \leq \psi_e'(0) \\ (\psi_e')^{-1}(\lambda_{\theta(e)} - \lambda_{\kappa(e)}) & \text{if} \quad \lambda_{\theta(e)} - \lambda_{\kappa(e)} > \psi_e'(0)\,. \end{cases}$$

Complementary slackness conditions

Proof:

Lagrangian techniques and duality

Proposition: f^* in $\mathbb{R}_+^{\mathcal{E}}$ s.t. $By^* = \nu$ and λ^* in $\mathbb{R}^{\mathcal{V}}$ satisfy complementary slackness conditions on every link e in $\mathcal{E}$ $\iff$

(a) f^* is optimal solution of

$$M(\nu) := \min_{\substack{y \geq 0 \\ By = \nu}} \sum_{e \in \mathcal{E}} \psi_e(y_e)$$

(b) λ^* is an optimal solution of

$$M^*(\nu) = \max_{\lambda \in \mathbb{R}^{\mathcal{E}}} D(\lambda, \nu)$$

(c) $M(\nu) = M^*(\nu)$

Lagrangian techniques and duality

- note that

$$D(\lambda, \nu) = \sum_e \psi_e^*(\lambda_{\theta(e)} - \lambda_{\kappa(e)}) - \sum_i \lambda_i \cdot \nu_i$$

where the dual costs

$$\psi_e^*(p) = \sup_{y \geq 0} \{py - \psi_e(y)\} \qquad e \in \mathcal{E}$$

represent the maximum profit that a link operator can make if it charges p per unit of flow and pays $\psi_e(y)$ to transport y units

Lagrangian techniques and duality

Theorem: $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, nonincreasing convex costs $\psi_e(y_e)$

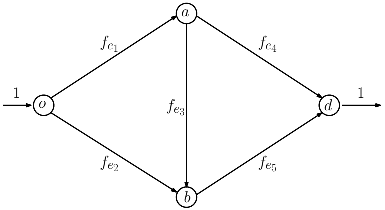
(i) flow vector f^* optimal $\Leftrightarrow$ satisfies complementary slackness for Lagrange multipliers $\lambda = \lambda^*$ that solve dual cost maximiz.

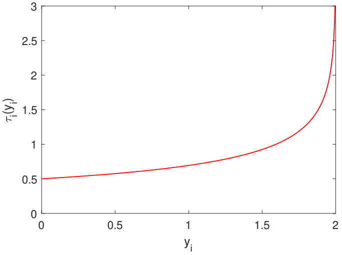
$$M^*(\nu) = \max_{\lambda \in \mathbb{R}^{\mathcal{V}}} D(\lambda, \nu)$$

(ii) if optimal cost $M(\nu)$ is differentiable in ν, then

$$\frac{\partial}{\partial \nu_i} M(\nu) - \frac{\partial}{\partial \nu_j} M(\nu) = \lambda_i^* - \lambda_j^*$$

User optimum traffic assignment





- nondecreasing delay functions $\tau_e(y_e)$
- corresponding convex nondecreasing cost

$$\psi_e(y_e) = \int_0^{y_e} \tau_e(s)\mathrm{d}s$$

- User Optimum Traffic Assignment (UO-TAP):

$$\min_{\substack{y \geq 0 \\ By = \nu}} \sum_{e \in \mathcal{E}} \int_0^{y_e} \tau_e(s)\mathrm{d}s$$

Wardrop equilibrium

- Proposition: Wardrop equilibrium = solution of UO-TAP

$$M(\nu) := \min_{\substack{y \geq 0 \\ By = \delta^{(o)} - \delta^{(d)}}} \sum_{e \in \mathcal{E}} \int_0^{y_e} \tau_e(s) \mathrm{d}s$$

- Proof:

Toll design

- Vector of tolls: $\omega = (\omega_e)_{e \in \mathcal{E}}$
- Perceived cost $= \omega_e + \tau_e(y_e)$ on link e
- Wardrop equilibrium with tolls

$$y^{(\omega)} = Hz^*$$

where z^* NE of population game $(\mathcal{A}, r(z) = H^T(\tau(Hz) + \omega))$, i.e.,

$$z_a^* > 0 \qquad \Longrightarrow \qquad (H(\omega + \tau(y^{(\omega)})))_a = \min_{b \in \mathcal{A}} \{(H(\omega + \tau(y^{(\omega)})))_b\}$$

- Can we find ω s.t. PoA(ω)$= 1$?

Toll design

- Theorem: For strictly increasing $\tau_e(y_e)$, convex $y_e \cdot \tau_e(y_e)$

Wardrop equilibrium with tolls ω satisfies

$$y^{(\omega)} = \underset{\substack{f \in \mathbb{R}_+^{\mathcal{E}} \\ By = \delta^{(o)} - \delta^{(d)}}}{\operatorname{argmin}} \sum_{e \in \mathcal{E}} \left(\int_0^{y_e} \tau_e(s) + \omega_e y_e \right)$$

- Corollary: With marginal cost tolls

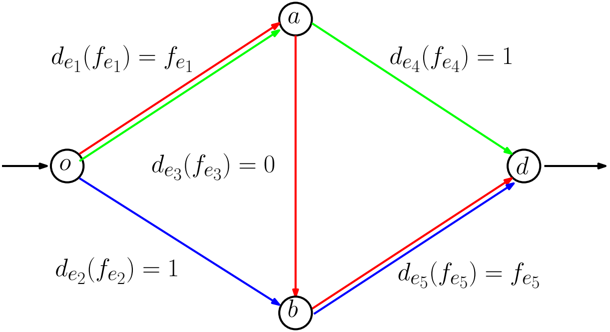
$$\omega_e^\circ = y_e^\circ \tau_e'(y_e^\circ)$$

computed at social optimum $f^\circ = \underset{By = \delta^{(o)} - \delta^{(d)}}{\operatorname{argmin}} \sum_{e \in \mathcal{E}} y_e \cdot \tau_e(y_e)$

$$f^{(\omega^\circ)} = f^\circ$$

- marginal cost tolls $\Leftrightarrow$ "internalize negative externality"

Example – Toll design



- Tolls $\omega_{e_1} = \omega_{e_5} = 1/2$, $\omega_{e_2} = \omega_{e_3} = \omega_{e_4} = 0$
- Wardrop equilibrium with tolls: $z_{p^{(1)}} = z_{p^{(2)}} = 1/2$, $z_{p^{(3)}} = 1$
- Social optimum: $z_{p^{(1)}} = z_{p^{(2)}} = 1/2$, $z_{p^{(3)}} = 0$

Toll design (cont'd)

- can be extended to multiple origin-destination
- computing fixed marginal cost tolls $\omega_e^\circ = y_e^\circ \tau_e'(y_e^\circ)$ requires computing SO-TAP solution y°, which in turn depends on exogenous net flows ν
- more robust solution: decentralized feedback marginal cost tolls

$$\omega_e(y_e) = y_e \tau_e'(y_e)$$

- convergence to SO guaranteed also with learning/imitation dynamics coupled with physical traffic flow dynamics, see G. Como and R. Maggistro, "Distributed Dynamic Pricing of Multiscale Transportation Networks," *IEEE TAC*, 2021.
- Challenge: heterogeneous preferences: game no longer potential