Lecture Notes on Nonlinear and Adaptive Control of Advanced Aerospace Systems

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Chapter 1

Vehicle Models

Notation

Throughout the paper, the notation $\bullet_{\text{ref}}$ and $\bullet_{\text{cmd}}$ denotes either *exogenous* or *endogenous* reference trajectories, that is, signals which are function of externally-generated reference signals, state variables or both. The notation $\bullet^{\star}_{\text{ref}}$ denotes a constant setpoint, or a trim (i.e., equilibrium) condition corresponding to a given setpoint, whereas $\tilde{\bullet}$ denotes deviation from a setpoint or a reference trajectory, depending on the context. The notation $\hat{\bullet}$ denotes the estimate of a generic parameter or a function of estimated parameters; the ensuing estimation error is denoted by $\tilde{\bullet}$ as well. Externally-generated reference trajectories are assumed to be sufficiently smooth, though the order of differentiability shall not be explicitly stated. All vectors are assumed to be resolved in the vehicle body-fixed frame, $\mathcal{F}_b$, except when denoted by a left superscript. The set $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$ denotes the canonical basis in any frame. Matrices and vectors are expressed in boldface.

The aircraft considered in this study is a generic highly-maneuverable vehicle (GHMV) endowed with redundant control effectors, namely canards, ailerons (flaps), and ruddervators (tail effectors.) A sketch of the vehicle's geometry is shown in Figure 1.1. An axisymmetric engine provides thrust (controlled via the throttle), whereas the six aerodynamic control surfaces (right and left canard, right and left flap, and right and left tail, respectively) control the vehicle's attitude. Trust vectoring is not used in this study

Two different vehicle models have been used in this work: A comprehensive model of the vehicle dynamics obtained from wind-tunnel data and computational fluid dynamics modeling has been used for validation in computer simulation, whereas a reduced-complexity, control-oriented model has been employed for control design and stability analysis. In what follows, we give a short account of the two models. The relevant nomenclature is given in Table 1.1, to which the reader is referred for all the definitions of the variables.

1.1 Simulation Model (SM)

Four main coordinate frames are employed: A North-East-Down (NED) oriented Earth-fixed frame, $\mathscr{F}_e$, with origin at the Earth's center, a body-fixed frame, $\mathscr{F}_b$, a reference (or desired) frame, $\mathscr{F}_r$ (to be introduced later.) and the so-called *wind frame*, $\mathscr{F}_w$. For the purpose of this study, the Earth-fixed frame is assumed to be inertial, and the effect of the Earth rotation is ignored. A pictorial representation of the coordinate frames is

$\mathscr{F}_e, \mathscr{F}_b, \mathscr{F}_r, \mathscr{F}_w$	Coordinate frames (Earth-centered, body-fixed, reference, wind-axes)
${}^e\boldsymbol{p} = [x\ y\ z]^T$	Position of the center of $\mathscr{F}_b$ in $\mathscr{F}_e$
$\boldsymbol{R}_{ij} \in SO(3)$	Rotation matrix from $\mathscr{F}_j$ to $\mathscr{F}_i$
$\boldsymbol{\nu} = [u\ v\ w]^T$	Translational velocity of the vehicle in $\mathscr{F}_b$
$\boldsymbol{\omega} = [p\ q\ r]^T$	Angular velocity of the vehicle in $\mathscr{F}_b$
$\boldsymbol{S}(\boldsymbol{\omega}) \in so(3)$	Skew-symmetric operator, $\boldsymbol{S}(\boldsymbol{\omega})\boldsymbol{\nu} = \boldsymbol{\omega} \times \boldsymbol{\nu}$
$\boldsymbol{r}_g = [x_g\ y_g\ z_g]^T$	Center of gravity of the vehicle in $\mathscr{F}_b$
$\boldsymbol{\eta} = [\phi\ \theta\ \psi]^T$	Euler-angle parameterization of $\boldsymbol{R}_{eb}$
$\boldsymbol{\sigma} \in \mathbb{R}^3$	MRP parameterization of a rotation matrix
$\boldsymbol{F}_{\text{grav}} \in \mathbb{R}^3$	Gravity force
$\boldsymbol{F}_{A,\text{base}},\ \boldsymbol{M}_{A,\text{base}} \in \mathbb{R}^3$	Baseline aerodynamic force and moment
$\boldsymbol{F}_{A,\delta},\ \boldsymbol{M}_{A,\delta} \in \mathbb{R}^3$	Control aerodynamic force and moment
$\boldsymbol{F}_T \in \mathbb{R}^3$	Force due to engine thrust
T	Thrust
V_T, α, β	Airspeed, angle-of-attack, sideslip angle
χ, γ, μ	Bank angle, flight-path angle, heading angle
$h = -z$	Altitude
$\bar{q}, M_\infty$	Dynamic pressure, Mach number
m	Mass
$\boldsymbol{J} \in \mathbb{R}^{3\times 3}$	Inertia matrix, $\boldsymbol{J} = \boldsymbol{J}^T > \boldsymbol{0}$
δ_T	Trottle
$\boldsymbol{\delta} = [\delta_{\text{c},r}\ \delta_{\text{c},l}\ \delta_{\text{f},r}\ \delta_{\text{f},l}\ \delta_{\text{t},r}\ \delta_{\text{t},l}]^T$	Aerodynamic control surface deflections (right canard, left canard, right flap, left flap, right tail, left tail)

Table 1.1: Common nomenclature for the simulation and the control-design models

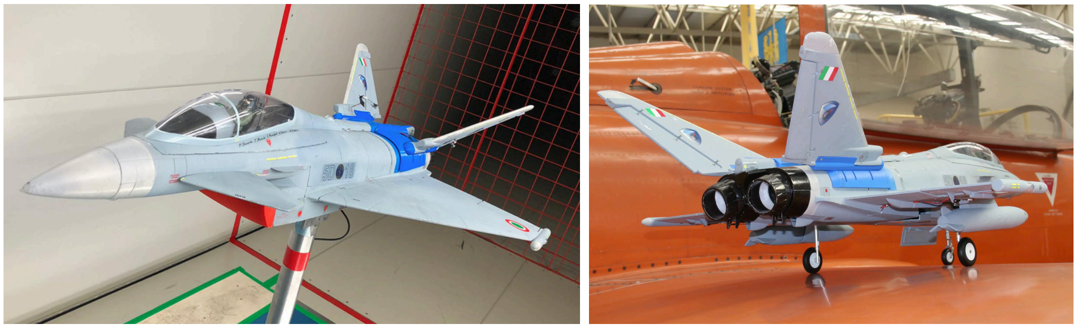
given in Figure 1.2. A standard roll-pitch-yaw angle parameterization for $\boldsymbol{R}_{eb}$ is employed for the only purpose of defining the output to be controlled for the vehicle attitude. The corresponding expression for $\boldsymbol{R}_{eb}$ reads as

$$\boldsymbol{R}_{eb}(\boldsymbol{\eta}) = \begin{pmatrix} \cos\theta\cos\psi & \sin\phi\sin\theta\cos\psi - \cos\phi\sin\psi & \cos\phi\sin\theta\cos\psi + \sin\phi\sin\psi \\ \cos\theta\sin\psi & \sin\phi\sin\theta\sin\psi + \cos\phi\cos\psi & \cos\phi\sin\theta\sin\psi - \sin\phi\cos\psi \\ -\sin\theta & \sin\phi\cos\theta & \cos\phi\cos\theta \end{pmatrix}$$

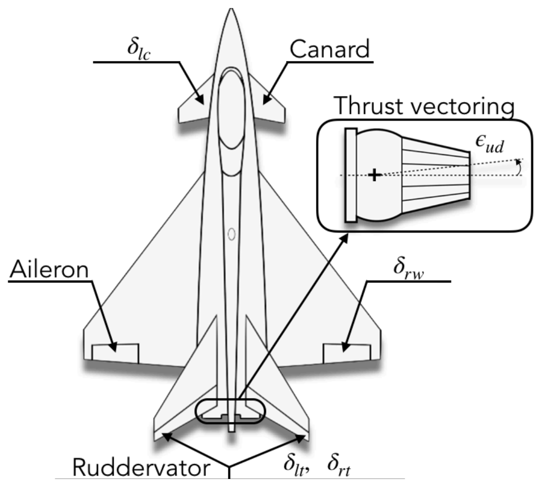
where $\boldsymbol{\eta} = [\phi\ \theta\ \psi]^T$ is the vector of Euler angles.

The wind-axis frame, $\mathscr{F}_w$, is used to express the vehicle velocity $\boldsymbol{\nu}$ in spherical coordinates via airspeed, angle-of-attack and sideslip, given respetively by [1]:

$$V_T = \sqrt{u^2 + v^2 + w^2}\,, \quad \alpha = \arctan(w/u)\,, \quad \beta = \arcsin(v/V_T)$$



(a) Prototype



(b) Schematics

Figure 1.1: Generic highly-maneuverable vehicle (GHMV) considered in this study.

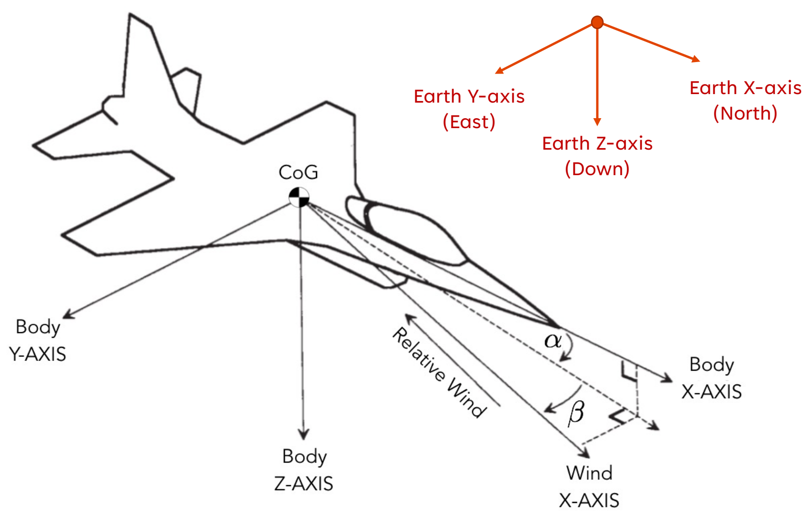


Figure 1.2: Earth-centered and body-fixed reference frames. Wind x-axis is also shown.

It is noted that the vehicle velocity in the wind frame is given by

$$^{w}\boldsymbol{\nu} = \begin{pmatrix} V_T \\ 0 \\ 0 \end{pmatrix}$$

and that the rotation matrix $\boldsymbol{R}_{bw}$ from wind axes to body axes reads as

$$\boldsymbol{R}_{bw}(\alpha, \beta) = \begin{pmatrix} \cos\beta\cos\alpha & -\sin\beta\cos\alpha & -\sin\alpha \\ \sin\beta & \cos\beta & 0 \\ \cos\beta\sin\alpha & -\sin\beta\sin\alpha & \cos\alpha \end{pmatrix}$$

Consequently, the relation between the component of the vehicle velocity resolved in the body frame and the wind-axis parameters is computed as

$$\boldsymbol{\nu} = \boldsymbol{R}_{bw}\,{}^{w}\boldsymbol{\nu}$$

which yields

$$u = V_T \cos\alpha\cos\beta\,, \quad v = V_T \sin\beta\,, \quad w = V_T \sin\alpha\cos\beta$$

The orientation of the wind frame with respect to the Earth-centered frame is typically expressed in term of an x-y-z rotation employing the bank angle χ, the flight-path angle γ, and the heading angle μ, yielding the rotation matrix (compare with the expression of $\boldsymbol{R}_{eb}$ as a function of the Euler angles $\boldsymbol{\eta}$):

$$\boldsymbol{R}_{ew}(\chi, \gamma, \mu) = \begin{pmatrix} \cos\gamma\cos\mu & \sin\chi\sin\gamma\cos\mu - \cos\chi\sin\mu & \cos\chi\sin\gamma\cos\mu + \sin\chi\sin\mu \\ \cos\gamma\sin\mu & \sin\chi\sin\gamma\sin\mu + \cos\chi\cos\mu & \cos\chi\sin\gamma\sin\mu - \sin\chi\cos\mu \\ -\sin\gamma & \sin\chi\cos\gamma & \cos\chi\cos\gamma \end{pmatrix}$$

In the sequel, we will not make use of the bank angle χ and the heading angle μ, as we shall not be concerned with navigation systems. However, the flight-path angle is used to determine the *climbing rate* of the vehicle. As a matter of fact, from the relation

$$^{e}\dot{\boldsymbol{p}} = \boldsymbol{R}_{ew}\,{}^{w}\boldsymbol{\nu}$$

one obtains

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} = \begin{pmatrix} V_T\cos\gamma\cos\mu \\ V_T\cos\gamma\sin\mu \\ -V_T\sin\gamma \end{pmatrix}$$

hence

$$\dot{h} = V_T \sin\gamma$$

is the required expression for the vehicle climb rate. Furthermore, using the relation

$$\boldsymbol{R}_{ew} = \boldsymbol{R}_{eb}\boldsymbol{R}_{bw}$$

one obtains the expression of the flight-path angle in terms of the Euler angles, the angle-of-attack and the slideslip angle as follows:

$$\sin\gamma = \cos\beta\cos\alpha\sin\theta - \cos\beta\sin\alpha\cos\theta\cos\phi - \sin\beta\sin\phi\cos\theta \tag{1.1}$$

In the simulation model, the aerodynamic forces and moments as well as the forces and moments due to the scramjet engine are provided by look-up tables. The forces and moments acting on the vehicle are summarized in Table 1.2 and illustrated in Figures 1.3–1.5, together with the definition of all attitude parameters used in this work.

L, D, C	Lift, drag and cross force
$\boldsymbol{F}_{A,\text{base}} = [X_A\ Y_A\ Z_A]^T$	Baseline aerodynamic force
$X_A = -D\cos\alpha\cos\beta + L\sin\alpha + C\sin\beta\cos\alpha$	Force along x-body axis
$Y_A = -D\sin\beta - C\cos\beta$	Force along y-body axis
$Z_A = -D\sin\alpha\cos\beta - L\cos\alpha + C\sin\beta\sin\alpha$	Force along z-body axis
$\boldsymbol{M}_{A,\text{base}} = [L_A\ M_A\ N_A]^T$	Baseline aerodynamic moment
$\boldsymbol{F}_{\text{grav}} = mg\boldsymbol{R}_{be}\boldsymbol{e}_3$	Gravity force
$\boldsymbol{F}_T = [X_T\ Y_T\ Z_T]^T$	Force due to engine thrust
$\boldsymbol{M}_T = [L_T\ M_T\ N_T]^T$	Moment due to engine thrust
$\boldsymbol{F}_{A,\delta} = [X_{A,\delta}\ Y_{A,\delta}\ Z_{A,\delta}]^T$	Aerodynamic force due to effectors
$\boldsymbol{M}_{A,\delta} = [L_{A,\delta}\ M_{A,\delta}\ N_{A,\delta}]^T$	Aerodynamic moments due to effectors

Table 1.2: Forces and moments for the simulation model

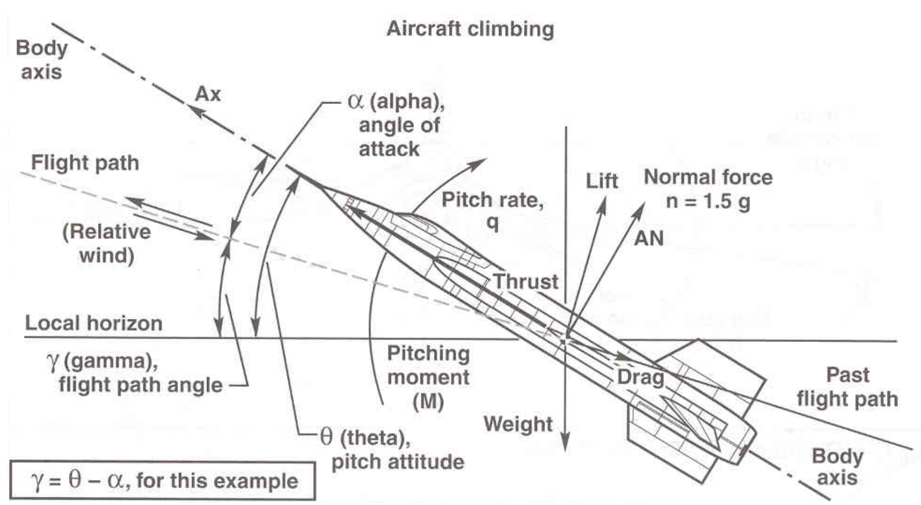


Figure 1.3: Longitudinal dynamics: Pitch angle, angle-of-attack, and flight-path angle; Pitching moment, lift, drag and weight.

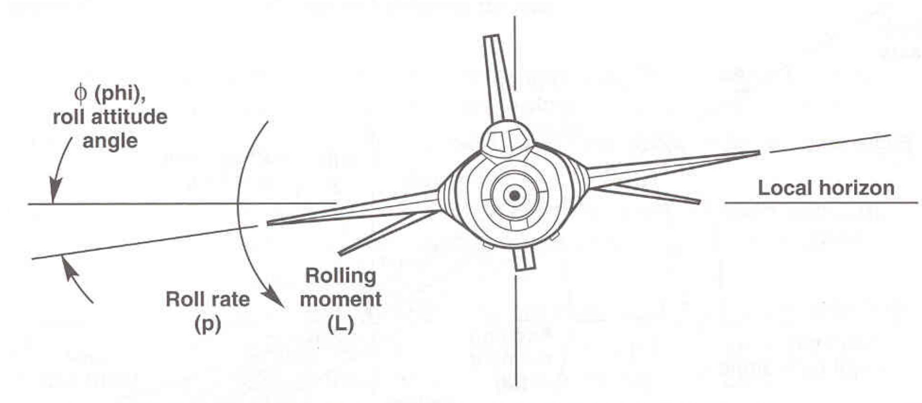


Figure 1.4: Lateral dynamics: Roll angle and rolling moment.

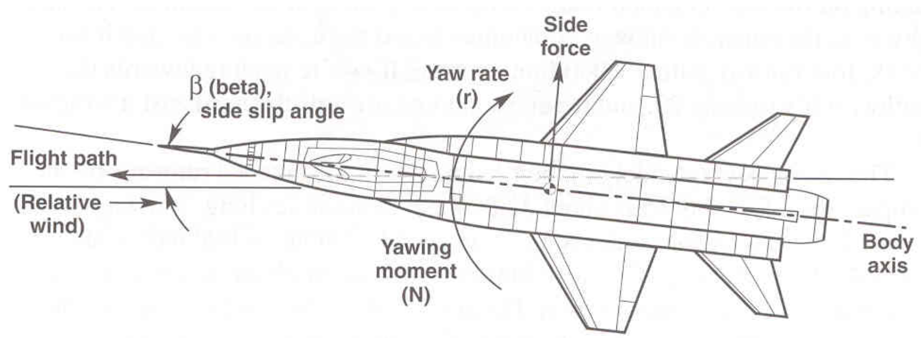


Figure 1.5: Lateral dynamics: Sideslip angle, yaw rate, side force and yawing moment.

Accordingly, the equations of motion of the simulation model read as

$$\begin{aligned}
{}^{e}\dot{\boldsymbol{p}} &= \boldsymbol{R}_{eb}\boldsymbol{\nu} \\
\dot{\boldsymbol{R}}_{eb} &= \boldsymbol{R}_{eb}\boldsymbol{S}(\boldsymbol{\omega}) \\
m\,\dot{\boldsymbol{\nu}} - m\boldsymbol{S}(\boldsymbol{r}_g)\dot{\boldsymbol{\omega}} &= m\boldsymbol{S}(\boldsymbol{\omega})\boldsymbol{S}(\boldsymbol{r}_g)\boldsymbol{\omega} - m\boldsymbol{S}(\boldsymbol{\omega})\boldsymbol{\nu} + \boldsymbol{F}_{\text{grav}} + \boldsymbol{F}_{A,\text{base}} + \boldsymbol{F}_{A,\delta} + \boldsymbol{F}_T \\
\boldsymbol{J}\dot{\boldsymbol{\omega}} + m\boldsymbol{S}(\boldsymbol{r}_g)\dot{\boldsymbol{\omega}} &= -\boldsymbol{S}(\boldsymbol{\omega})\boldsymbol{J}\boldsymbol{\omega} - m\boldsymbol{S}(\boldsymbol{r}_g)\boldsymbol{S}(\boldsymbol{\omega})\boldsymbol{\nu} + \boldsymbol{M}_{A,\text{base}} + \boldsymbol{M}_{A,\delta}
\end{aligned} \tag{1.2}$$

where

$$\boldsymbol{x} = ({}^{e}\boldsymbol{p}, \boldsymbol{R}_{eb}, \boldsymbol{\nu}, \boldsymbol{\omega}) \in \mathcal{X} := \mathbb{R}^3 \times SO(3) \times \mathbb{R}^3 \times \mathbb{R}^3$$

is the state,

$$\boldsymbol{u} = (\delta_T, \boldsymbol{\delta}) \in \mathcal{U} := \mathbb{R} \times \mathbb{R}^4$$

is the control input, and

$$\boldsymbol{y} = \left(V_T, v, \gamma, \dot{\psi}\right) \in \mathcal{Y} := \mathbb{R}_+ \times \mathbb{R} \times \mathbb{S} \times \mathbb{R}$$

is the regulated output with associated reference trajectory

$$\boldsymbol{y}_{\text{ref}} = \left(V_{T_{\text{ref}}}, 0, \gamma_{\text{ref}}, \dot{\psi}_{\text{ref}}\right) \in \mathcal{Y}$$

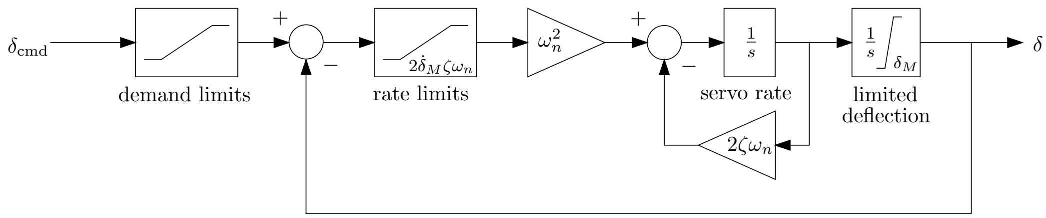


Figure 1.6: Constrained dynamics for the aerodynamic control effectors

Actuator dynamics of the form seen in Figure 1.6, given by

$$\dot{\boldsymbol{u}} = \boldsymbol{f}_{\rm act}(\boldsymbol{u}, \boldsymbol{u}_{\rm cmd}) \tag{1.3}$$

with command input $\boldsymbol{u}_{\rm cmd} = (\delta_{T,{\rm cmd}}, \boldsymbol{\delta}_{\rm cmd}) \in \mathcal{U}$, comprise magnitude, command and rate limiters for the aerodynamic effectors, as well as a magnitude limiter for the throttle input.

1.2 Control-design Model (CDM)

A dynamical model of reduced complexity has been adopted for control design and stability analysis. A curve-fitted analytical approximation of the forces and moments, obtained from look-up table data, have been adopted as given in Table 1.3 below.

$\boldsymbol{F}_{A,{\rm base}} = [X_A\ Y_A\ Z_A]^T$	Baseline aerodynamic force
$X_A = -D\cos\alpha + L\sin\alpha$	Force along x-body axis
$Y_A = \bar{q}SC_Y(\beta, M_\infty)$	Force along y-body axis
$Z_A = -D\sin\alpha - L\cos\alpha$	Force along z-body axis
$L = \bar{q}SC_L(\alpha, M_\infty)$	Lift
$D = \bar{q}SC_D(\alpha, M_\infty)$	Drag
$\boldsymbol{F}_{\rm grav} = mg\boldsymbol{R}_{be}\boldsymbol{e}_3$	Gravity force
$\boldsymbol{F}_T = [T\ 0\ 0]^T$	Force due to engine thrust
$\boldsymbol{M}_T = \boldsymbol{0}$	Moment due to engine thrust
$T = \bar{q}SC_T(\alpha, M_\infty)\delta_T$	Thrust
$\boldsymbol{M}_{A,{\rm base}} = [L_A\ M_A\ N_A]^T$	Baseline aerodynamic moment
$L_A = \bar{q}bSC_{L_A}(\alpha, \beta, M_\infty)$	Rolling moment (x-body axis)
$M_A = \bar{q}\bar{c}SC_{M_A}(\alpha, M_\infty)$	Pitching moment (y-body axis)
$N_A = \bar{q}bSC_{N_A}(\alpha, \beta, M_\infty)$	Yawing moment (z-body axis)
$\boldsymbol{F}_{A,\delta} = [0\ 0\ Z_{A,\delta}]^T$	Aerodynamic forces due to effectors
$\boldsymbol{F}_{A,\delta} = \bar{q}\boldsymbol{B}_1\boldsymbol{\delta},\ \ \boldsymbol{B}_1 \in \mathbb{R}^{3\times 6}$	Forces due to effectors
$\boldsymbol{M}_{A,\delta} = \bar{q}\boldsymbol{B}_2\boldsymbol{\delta},\ \ \boldsymbol{B}_2 \in \mathbb{R}^{3\times 6}$	Moments due to effectors
S, b, $\bar{c}$	Planform area, wingspan and chord

Table 1.3: Forces and moments for the control-design model

Further simplifications to the model are as follows:

- The center of mass is assumed at the origin of $\mathscr{F}_b$.
- The inertia matrix is diagonal, $\boldsymbol{J} = \mathrm{diag}(J_x, J_y, J_z)$.
- The side and normal forces due to thrust, i.e., Y_T and Z_T, are neglected.
- The moment due to thrust, i.e., $\boldsymbol{M}_T$, is neglected.
- The small body forces $X_{A,\delta}$ and $Y_{A,\delta}$ produced by the control effectors are neglected.
- The contribution of sideslip in the axial and normal baseline aerodynamic forces is neglected.

The equations of motion of the CDM, concisely written as

$$\begin{aligned} \dot{\boldsymbol{x}} &= \boldsymbol{f}_{\mathrm{CDM}}(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{\vartheta}) \\ \boldsymbol{y} &= \boldsymbol{h}(\boldsymbol{x}) \end{aligned} \tag{1.4}$$

have the following expression

$$\begin{aligned} {}^{e}\dot{\boldsymbol{p}} &= \boldsymbol{R}\boldsymbol{\nu} \\ \dot{\boldsymbol{R}} &= \boldsymbol{R}\boldsymbol{S}(\boldsymbol{\omega}) \\ m\,\dot{\boldsymbol{\nu}} &= -m\boldsymbol{S}(\boldsymbol{\omega})\boldsymbol{\nu} + mg\boldsymbol{R}^T\boldsymbol{e}_3 + \boldsymbol{F}_{A,\mathrm{base}} + \boldsymbol{F}_{A,\delta} + T\boldsymbol{e}_1 \\ \boldsymbol{J}\dot{\boldsymbol{\omega}} &= -\boldsymbol{S}(\boldsymbol{\omega})\boldsymbol{J}\boldsymbol{\omega} + \boldsymbol{M}_{A,\mathrm{base}} + \boldsymbol{M}_{A,\delta} \end{aligned} \tag{1.5}$$

where the simpler notation $\boldsymbol{R}$ will henceforth be used in place of $\boldsymbol{R}_{eb}$. The actuator dynamics The aerodynamic coefficients in Table 1.3 are given by

$$\begin{aligned} C_L(\alpha, M_\infty) &= C_L^0 + C_L^\alpha\alpha + C_L^{M_\infty\alpha}M_\infty\alpha + C_L^{\alpha^2}\alpha^2 + C_L^{M_\infty}M_\infty \\ C_D(\alpha, M_\infty) &= C_D^0 + C_D^\alpha\alpha + C_D^{\alpha^2}\alpha^2 + C_D^{M_\infty}M_\infty \\ C_Y(\beta, M_\infty) &= C_Y^\beta\beta + C_Y^{\beta M_\infty}\beta M_\infty \\ C_T(\alpha, M_\infty) &= C_T^0 + C_T^\alpha\alpha + C_T^{\alpha^2}\alpha^2 + C_T^{\alpha^3}\alpha^3 + C_T^{M_\infty}M_\infty \\ C_{L_A}(\alpha, \beta, M_\infty) &= C_{L_A}^\beta\beta + C_{L_A}^{M_\infty\beta}M_\infty\beta + C_{L_A}^{\alpha\beta}\alpha\beta \\ C_{M_A}(\alpha, M_\infty) &= C_{M_A}^0 + C_{M_A}^\alpha\alpha + C_{M_A}^{\alpha^2}\alpha^2 + C_{M_A}^{M_\infty}M_\infty + C_{M_A}^{M_\infty^2}M_\infty^2 + C_{M_A}^{M_\infty\alpha}M_\infty\alpha \\ C_{N_A}(\alpha, \beta, M_\infty) &= C_{N_A}^\beta\beta + C_{N_A}^{\beta^2}\beta^2 + C_{N_A}^{\beta^3}\beta^3 + C_{N_A}^{M_\infty\beta}M_\infty\beta + C_{N_A}^{\alpha\beta}\alpha\beta \end{aligned} \tag{1.6}$$

The vector $\boldsymbol{\vartheta}$ collects the uncertain parameters C_i^j of the aerodynamic coefficients (1.6). It is assumed that $\boldsymbol{\vartheta} \in \mathcal{P}$, where $\mathcal{P}$ is a known compact hypercube.

The aerodynamic force and moment due to the control effectors have the expressions

$$\boldsymbol{F}_{A,\delta} = \bar{q}\boldsymbol{B}_1\,\mathbf{diag}(\boldsymbol{\lambda})\boldsymbol{\delta}, \qquad \boldsymbol{M}_{A,\delta} = \bar{q}\boldsymbol{B}_2\,\mathbf{diag}(\boldsymbol{\lambda})\boldsymbol{\delta} \tag{1.7}$$

where the force control-effectiveness matrix

$$\boldsymbol{B}_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ -C_y^{\mathrm{c}} & C_y^{\mathrm{c}} & -C_y^{\mathrm{f}} & C_y^{\mathrm{f}} & C_y^{\mathrm{t}} & -C_y^{\mathrm{t}} \\ -C_z^{\mathrm{c}} & -C_z^{\mathrm{c}} & -C_z^{\mathrm{f}} & -C_z^{\mathrm{f}} & C_z^{\mathrm{t}} & C_z^{\mathrm{t}} \end{pmatrix}$$

and the moment control-effectiveness matrix

$$
\boldsymbol{B}_2 = \begin{pmatrix} -C_l^{\mathrm{c}} & C_l^{\mathrm{c}} & -C_l^{\mathrm{f}} & C_l^{\mathrm{f}} & C_l^{\mathrm{t}} & -C_l^{\mathrm{t}} \\ -C_m^{\mathrm{c}} & -C_m^{\mathrm{c}} & -C_m^{\mathrm{f}} & -C_m^{\mathrm{f}} & C_m^{\mathrm{t}} & C_m^{\mathrm{t}} \\ C_n^{\mathrm{c}} & -C_n^{\mathrm{c}} & C_n^{\mathrm{f}} & -C_n^{\mathrm{f}} & -C_n^{\mathrm{t}} & C_n^{\mathrm{t}} \end{pmatrix}
$$

are assumed to be known with sufficient accuracy, and the vector of *uncertain actuator effectiveness*

$$
\boldsymbol{\lambda} = [\lambda_1 \ \cdots \ \lambda_6]^T
$$

satisfies

$$
\boldsymbol{\lambda} \in \mathit{\Lambda} := \{\lambda_0 \leq \lambda_i \leq 1, \ i = 1, 2, \ldots, 6\}
$$

where $\lambda_0 \in (0, 1)$ is a given constant. For the control effectiveness matrix

$$
\boldsymbol{B} = \begin{pmatrix} \boldsymbol{B}_1 \\ \boldsymbol{B}_2 \end{pmatrix}
$$

it is assumed that $\operatorname{rank} \boldsymbol{B} = 4$ and $\operatorname{rank} \boldsymbol{B}_2 = 3$, which is consistent with the simplifications made on the control-design model. For convenience of the reader, the expression of the equations of the vehicle dynamics (i.e., the translational and angular velocity dynamics) are reported component-wise as follows:

$$
\begin{aligned}
\dot{u} &= -qw + rv + \tfrac{1}{m} X_A + \tfrac{1}{m} X_T - g \sin\theta \\
\dot{v} &= pw - ru + \tfrac{1}{m} Y_A + g \sin\phi \cos\theta \\
\dot{w} &= -pv + qu + \tfrac{1}{m} Z_A + \tfrac{1}{m} Z_{A,\delta} + g \cos\phi \cos\theta
\end{aligned} \tag{1.8}
$$

$$
\begin{aligned}
J_x \dot{p} &= (J_y - J_z)\, qr + L_A + L_{A,\delta} \\
J_y \dot{q} &= -\,(J_x - J_z)\, pr + M_A + M_{A,\delta} \\
J_z \dot{r} &= (J_x - J_y)\, pq + N_A + N_{A,\delta}
\end{aligned} \tag{1.9}
$$

In the sequel we will make use of the relation between the angular rates and the time derivatives of the Euler angles, which reads as

$$
\dot{\boldsymbol{\eta}} = \boldsymbol{H}(\boldsymbol{\eta})\boldsymbol{\omega}, \qquad \boldsymbol{\omega} = \boldsymbol{H}^{-1}(\boldsymbol{\eta})\dot{\boldsymbol{\eta}}
$$

where the Jacobian of the transformation and its inverse read as

$$
\boldsymbol{H}(\boldsymbol{\eta}) = \begin{pmatrix} 1 & \sin\phi \tan\theta & \cos\phi \tan\theta \\ & \cos\phi & -\sin\phi \\ 0 & \sin\phi \sec\theta & \cos\phi \sec\theta \end{pmatrix}, \qquad \boldsymbol{H}^{-1}(\boldsymbol{\eta}) = \begin{pmatrix} 1 & 0 & -\sin\theta \\ 0 & \cos\phi & \sin\phi \cos\theta \\ 0 & -\sin\phi & \cos\phi \cos\theta \end{pmatrix}
$$

As a result, the component-wise expression of the body rates as a function of the Euler angles and their derivatives reads as

$$
\begin{aligned}
p &= \dot{\phi} - \sin\theta\, \dot{\psi} \\
q &= \cos\phi\, \dot{\theta} + \sin\phi \cos\theta\, \dot{\psi} \\
r &= -\sin\phi\, \dot{\theta} + \cos\phi \cos\theta\, \dot{\psi}
\end{aligned} \tag{1.10}
$$

Equations of motion in the wind axes

For further use, it is convenient to write explicitly the expression of the dynamics of the airspeed, V_T, which reads simply as

$$m\dot{V}_T = T\cos\alpha\cos\beta - D - mg\sin\gamma \tag{1.11}$$

Alternatively, the equations of motion can be written using the wind-axis parameters (V_T, α, β). To this aim, equation (1.11) shall be completed with the dynamics of the sideslip angle and the dynamics of the angle-of-attack, which read as

$$\begin{aligned} mV_T\dot{\beta} &= -T\cos\alpha\sin\beta - C - mV_T\left(r\cos\alpha - p\sin\alpha\right) + \\ &\quad + mg\left(\cos\alpha\sin\theta\sin\beta + \sin\phi\cos\theta\cos\beta - \sin\alpha\cos\phi\cos\theta\sin\beta\right) \\ mV_T\cos\beta\,\dot{\alpha} &= -T\sin\alpha - L + mV_T\left(q\cos\beta - p\cos\alpha\sin\beta - r\sin\alpha\sin\beta\right) + \\ &\quad + mg\left(\sin\alpha\sin\theta + \cos\alpha\cos\theta\cos\phi\right) \end{aligned} \tag{1.12}$$

The above equations are used to derive the *decoupled equations of motions* for the *longitudinal* and *lateral-directional* dynamics. To this end, to extract the longitudinal dynamics we consider the motion of the vehicle restricted to the vertical plane, and assume no sideslip and wing-level motion. Consequently, we first set $\beta = 0$ (no sideslip) and $\phi = 0$ (wing-level) in (1.1) to obtain:

$$\sin\gamma = \cos\alpha\sin\theta - \sin\alpha\cos\theta = \sin(\theta - \alpha)$$

that is,

$$\gamma = \theta - \alpha \tag{1.13}$$

which is the expression of the flight-path angle for the decoupled longitudinal dynamics. Then, by setting $\dot{\psi} = 0$ and $\dot{\phi} = 0$ in (1.10), one obtains

$$p = 0\,, \qquad q = \dot{\theta}\,, \qquad r = 0$$

and the equations of motion for the decoupled longitudinal dynamics[1]:

$$\begin{aligned} m\dot{V}_T &= T\cos\alpha - D(\alpha) - mg\sin(\theta - \alpha) \\ \dot{\alpha} &= q - \frac{1}{mV_T}\left[T\sin\alpha + L(\alpha) - mg\cos(\theta - \alpha)\right] \\ \dot{\theta} &= q \\ J_y\dot{q} &= M_A(\alpha) + M_{A,\delta} \end{aligned} \tag{1.14}$$

Equivalently, the above equations can be written in terms of the flight-path angle in substitution of the angle-of-attack as follows:

$$\begin{aligned} m\dot{V}_T &= T\cos\alpha - D(\alpha) - mg\sin\gamma \\ \dot{\gamma} &= \frac{1}{mV_T}\left[T\sin(\theta - \gamma) + L(\theta - \gamma) - mg\cos\gamma\right] \\ \dot{\theta} &= q \\ J_y\dot{q} &= M_A(\theta - \gamma) + M_{A,\delta} \end{aligned} \tag{1.15}$$

[1] It is noted that, to keep the equations consistent with standard textbooks (for example, [1]) here the contribution of aerodynamic surfaces to aerodynamic forces, $\boldsymbol{F}_{A,\delta}$, has been incorporated in the lift force, L.

which is advantageous as (V_T, γ) are output variables to be controlled.

The equations of motion for the decoupled lateral-directional dynamics are obtained by first setting $V_T = V_{T,0} =$ const and $\gamma = \gamma_0 =$ const, which yield the desired airspeed and the desired *climb rate* ($\gamma_0 = 0$ for level flight.) Then, the *trim* values $T = T_0 =$ const and $\alpha = \alpha_0 =$ const are determined from the equilibrium condition in (1.14):

$$\begin{aligned} 0 &= T_0 \cos\alpha_0 - D(\alpha_0) - mg \sin\gamma_0 \\ 0 &= T_0 \sin\alpha_0 + L(\alpha_0) - mg\cos\gamma_0 \end{aligned}$$

Finally, the trim value for the pitch angle is simply $\theta_0 = \gamma_0 + \alpha_0$, whereas the trim value for the aerodynamic pitch control moment, $M^0_{A,\delta}$, is computed from

$$0 = M_A(\alpha_0) + M^0_{A,\delta}$$

With the trim values for the longitudinal dynamics at hand, one obtain

$$\begin{aligned} mV_T\dot{\beta} = -T\cos\alpha_0 \sin\beta - C(\beta) - mV_{T,0}\,(r\cos\alpha_0 - p\sin\alpha_0) + \\ + mg\,(\cos\alpha_0 \sin\theta_0 \sin\beta + \sin\phi\cos\theta_0\cos\beta - \sin\alpha_0\cos\phi\cos\theta_0\sin\beta) \end{aligned} \tag{1.16}$$

and, for the rotational dynamics (recall that $q_0 = 0$)

$$\begin{aligned} \dot{\phi} &= p + \frac{\tan\theta_0}{\cos\phi} r \\ \dot{\psi} &= \frac{1}{\cos\theta_0\cos\phi} r \\ J_x\dot{p} &= L_A + L_{A,\delta} \\ J_z\dot{r} &= N_A + N_{A,\delta} \end{aligned} \tag{1.17}$$

Equations (1.14)–(1.16)–(1.17) are often linearized about the trim values[2] to obtain two linear systems that are completely decoupled. We shall not pursue this approach here. Rather, we will resort to the equations of motion (1.5) and (1.11) for the development of a control policy for the fully coupled nonlinear vehicle dynamics.

[2]In the next chapter, we will discover what is the trim value for ϕ corresponding to $\beta_0 = 0$.

Chapter 2

Problem Formulation

2.1 Control Objectives

For the system under consideration, given in (1.5) and reported below for the sake of convenience[1]

$$\begin{aligned}
{}^{e}\dot{\boldsymbol{p}} &= \boldsymbol{R}\boldsymbol{\nu} \\
\dot{\boldsymbol{R}} &= \boldsymbol{R}\boldsymbol{S}(\boldsymbol{\omega}) \\
m\,\dot{\boldsymbol{\nu}} &= -m\boldsymbol{S}(\boldsymbol{\omega})\boldsymbol{\nu} + mg\boldsymbol{R}^T\boldsymbol{e}_3 + \boldsymbol{F}_{A,\text{base}} + \boldsymbol{F}_{A,\delta} + T\boldsymbol{e}_1 \\
\boldsymbol{J}\dot{\boldsymbol{\omega}} &= -\boldsymbol{S}(\boldsymbol{\omega})\boldsymbol{J}\boldsymbol{\omega} + \boldsymbol{M}_{A,\text{base}} + \boldsymbol{M}_{A,\delta}
\end{aligned}$$

the output to be regulated comprises airspeed, lateral velocity (equivalently, sideslip angle), flight-path angle and yaw angle

$$\boldsymbol{y} = \left(V_T, v, \gamma, \dot{\psi}\right) \in \mathcal{Y} := \mathbb{R}_+ \times \mathbb{R} \times \mathbb{S} \times \mathbb{R}$$

The class of reference trajectories

$$\boldsymbol{y}_{\text{ref}}(t) = \left(V_{T_{\text{ref}}}(t), 0, \gamma_{\text{ref}}(t), \dot{\psi}_{\text{ref}}(t)\right) \in \mathcal{Y}$$

considered in this work are assumed to satisfy the following assumption:

Assumption 2.1.1 *The trajectory $\boldsymbol{y}_{\text{ref}}(\cdot) : \mathbb{R}_{\geq 0} \to \mathcal{Y}$ is a smooth signal satisfying:*

$$\lim_{t\to\infty} \|\boldsymbol{y}_{\text{ref}}(t) - \boldsymbol{y}^\star_{\text{ref}}\| = 0$$

where $\boldsymbol{y}^\star_{\text{ref}} = [V^\star_{T,\text{ref}}\ 0\ \gamma^\star_{\text{ref}}, \dot{\psi}^\star_{\text{ref}}]$ is a constant setpoint corresponding to a desired trim condition *of the vehicle.*

A trim condition for the vehicle corresponds to an equilibrium point $(\phi^\star, \theta^\star, \boldsymbol{\nu}^\star, \boldsymbol{\omega}^\star)$ of the translational and rotational dynamics (1.8)-(1.9), where translational and angular accelerations vanish (note that the right-hand side of (1.8) does not depend on ψ, whereas the right-hand side of (1.9) does not depend on $\boldsymbol{\eta}$ altogether.) The desired trim condition entails flying at constant airspeed and constant climb rate (possibly zero), with zero lateral velocity and constant turn rate (possibly zero).

[1] Recall that $\boldsymbol{R} = \boldsymbol{R}_{eb}$ and that (if not otherwise noted) all vectors are expressed in the body-fixed frame.

With these definitions at hand, the problem addressed in this work is stated as the design, for the system (1.4)–(1.3), of a dynamic state-feedback controller of the form

$$\begin{aligned}\dot{\boldsymbol{x}}_c &= \boldsymbol{F}_c(\boldsymbol{x}_c, \boldsymbol{x}, \boldsymbol{u}, \boldsymbol{y}_{\mathrm{ref}})\\ \boldsymbol{u}_{\mathrm{cmd}} &= \boldsymbol{H}_c(\boldsymbol{x}_c, \boldsymbol{x}, \boldsymbol{u}, \boldsymbol{y}_{\mathrm{ref}})\end{aligned} \tag{2.1}$$

with initial condition $\boldsymbol{x}_c(0)$ in a given set $\mathcal{K}_c$, such that:

1. Assuming $\boldsymbol{u} = \boldsymbol{u}_{\mathrm{cmd}}$, for any initial condition $\boldsymbol{x}(0)$, any reference trajectory $\boldsymbol{y}_{\mathrm{ref}}(\cdot)$ and any $\boldsymbol{x}_c(0) \in \mathcal{K}_c$, the trajectories of the closed-loop system (1.5) are bounded[2] and satisfy
$$\lim_{t\to\infty} \|\tilde{\boldsymbol{y}}(t)\| = 0$$
for all $\boldsymbol{\vartheta} \in \mathcal{P}$, where $\tilde{\boldsymbol{y}} := \boldsymbol{y} - \boldsymbol{y}_{\mathrm{ref}}$ denotes the output tracking error.

2. The controller is robust against the mismatch between $\boldsymbol{u}$ and $\boldsymbol{u}_{\mathrm{cmd}}$ given by (1.3), in a sense to be specified.

2.2 System Inversion

2.2.1 Right-inverse at trim

The *right inverse at trim* of system (1.5) is defined as the collection of all trajectories $(\boldsymbol{x}^\star_{\mathrm{ref}}(\cdot), \boldsymbol{u}^\star_{\mathrm{ref}}(\cdot)) \in \mathcal{X} \times \mathcal{U}$ in both the state and the input spaces that are compatible with the desired trim condition, that is, such that

$$\begin{aligned}\dot{\boldsymbol{x}}^\star_{\mathrm{ref}}(t) &= \boldsymbol{f}_{\mathrm{CDM}}(\boldsymbol{x}^\star_{\mathrm{ref}}(t), \boldsymbol{u}^\star_{\mathrm{ref}}(t), \boldsymbol{\vartheta})\\ \boldsymbol{y}^\star_{\mathrm{ref}} &= \boldsymbol{h}(\boldsymbol{x}^\star_{\mathrm{ref}}(t))\end{aligned} \tag{2.2}$$

for all $t \geq 0$.

Rotational Dynamics

To determine the right-inverse at trim for the rotational dynamics, we start from assigning the yaw trim trajectory by direct integration of the trim yaw rate, that is

$$\psi^\star_{\mathrm{ref}}(t) := \psi(0) + \dot{\psi}^\star_{\mathrm{ref}}\, t \tag{2.3}$$

where the initial condition is arbitrary, but conveniently selected here as the initial yaw angle of the vehicle. As at trim necessarily $\dot{\boldsymbol{\omega}}^\star_{\mathrm{ref}} = 0$, one obtains from (1.10)

$$\begin{aligned}p^\star_{\mathrm{ref}} &= \dot{\phi}^\star_{\mathrm{ref}}(t) - \sin\theta^\star_{\mathrm{ref}}(t)\,\dot{\psi}^\star_{\mathrm{ref}}\\ q^\star_{\mathrm{ref}} &= \cos\phi^\star_{\mathrm{ref}}(t)\,\dot{\theta}^\star_{\mathrm{ref}}(t) + \sin\phi^\star_{\mathrm{ref}}(t)\cos\theta^\star_{\mathrm{ref}}(t)\,\dot{\psi}^\star_{\mathrm{ref}}\\ r^\star_{\mathrm{ref}} &= -\sin\phi^\star_{\mathrm{ref}}(t)\,\dot{\theta}^\star_{\mathrm{ref}}(t) + \cos\phi^\star_{\mathrm{ref}}(t)\cos\theta^\star_{\mathrm{ref}}(t)\,\dot{\psi}^\star_{\mathrm{ref}}\end{aligned} \tag{2.4}$$

The condition $v^\star_{\mathrm{ref}} = 0$ yields directly $\beta^\star_{\mathrm{ref}} = 0$ from the relation $v = V_T \sin\beta$. As a consequence, the expression of the flight-path angle (1.1) is greatly simplified at trim:

$$\sin\gamma^\star_{\mathrm{ref}} = \cos\alpha^\star_{\mathrm{ref}}(t)\sin\theta^\star_{\mathrm{ref}}(t) - \sin\alpha^\star_{\mathrm{ref}}(t)\cos\theta^\star_{\mathrm{ref}}(t)\cos\phi^\star_{\mathrm{ref}}(t) \tag{2.5}$$

[2] Clearly, we shall not require the trajectories $(x(t), y(t))$ to be bounded, but only that they exist for all $t \geq 0$.

Note that in (2.4) and (2.5), $\dot{\psi}^\star_{\text{ref}}$ and $\gamma^\star_{\text{ref}}$ are assigned. The fact that $p^\star_{\text{ref}}, q^\star_{\text{ref}}, r^\star_{\text{ref}}$ are constant suggests to look for a *constant solution* $(\phi^\star_{\text{ref}}, \theta^\star_{\text{ref}}, \alpha^\star_{\text{ref}})$ of the equation (2.5). Under this assumption, the system of differential equations (2.4) yields directly the definition of the body rates at trim

$$\begin{aligned} p^\star_{\text{ref}} &= -\sin\theta^\star_{\text{ref}}\,\dot{\psi}^\star_{\text{ref}} \\ q^\star_{\text{ref}} &= \sin\phi^\star_{\text{ref}}\cos\theta^\star_{\text{ref}}\,\dot{\psi}^\star_{\text{ref}} \\ r^\star_{\text{ref}} &= \cos\phi^\star_{\text{ref}}\cos\theta^\star_{\text{ref}}\,\dot{\psi}^\star_{\text{ref}} \end{aligned} \tag{2.6}$$

whereas (2.5) reads as

$$\sin\gamma^\star_{\text{ref}} = \cos\alpha^\star_{\text{ref}}\sin\theta^\star_{\text{ref}} - \sin\alpha^\star_{\text{ref}}\cos\theta^\star_{\text{ref}}\cos\phi^\star_{\text{ref}} \tag{2.7}$$

Using the identity

$$a\sin x + b\cos x = \sqrt{a^2+b^2}\,\sin\left(x + \arctan\left(\frac{b}{a}\right)\right)$$

one obtains from (2.7)

$$\sin\gamma^\star_{\text{ref}} = \sqrt{(\cos\alpha^\star_{\text{ref}})^2 + (\sin\alpha^\star_{\text{ref}})^2(\cos\phi^\star_{\text{ref}})^2}\,\sin\left(\theta^\star_{\text{ref}} - \arctan\left(\tan\alpha^\star_{\text{ref}}\cos\phi^\star_{\text{ref}}\right)\right)$$

hence the expression for $\theta^\star_{\text{ref}}$ as a function of $\gamma^\star_{\text{ref}}$, $\phi^\star_{\text{ref}}$ and $\alpha^\star_{\text{ref}}$:

$$\theta^\star_{\text{ref}} = \arcsin\left(\frac{\sin\gamma^\star_{\text{ref}}}{\sqrt{(\cos\alpha^\star_{\text{ref}})^2 + (\sin\alpha^\star_{\text{ref}})^2(\cos\phi^\star_{\text{ref}})^2}}\right) + \arctan\left(\tan\alpha^\star_{\text{ref}}\cos\phi^\star_{\text{ref}}\right) \tag{2.8}$$

The trim reference for the roll angle is obtained from the equation of the lateral velocity

$$\dot{v} = pw - ru + \tfrac{1}{m}Y_{A,\text{base}} + g\sin\phi\cos\theta$$

As $\beta = 0$ implies $Y_{A,\text{base}} = 0$, the above equation at trim reads as

$$p^\star_{\text{ref}}w^\star_{\text{ref}} - r^\star_{\text{ref}}u^\star_{\text{ref}} + g\sin\phi^\star_{\text{ref}}\cos\theta^\star_{\text{ref}} = 0$$

Using (2.6) and the identities

$$u^\star_{\text{ref}} = V^\star_{T,\text{ref}}\cos\alpha^\star_{\text{ref}}\,, \quad w^\star_{\text{ref}} = V^\star_{T,\text{ref}}\sin\alpha^\star_{\text{ref}}$$

one obtains the equation for the *coordinated turn*

$$\sin\phi^\star_{\text{ref}} = \mathcal{G}^\star_{\text{ref}}\left(\sin\alpha^\star_{\text{ref}}\tan\theta^\star_{\text{ref}} + \cos\alpha^\star_{\text{ref}}\cos\phi^\star_{\text{ref}}\right) \tag{2.9}$$

where

$$\mathcal{G}^\star_{\text{ref}} := \frac{V^\star_{T,\text{ref}}\dot{\psi}^\star_{\text{ref}}}{g}$$

is the *centripetal acceleration* at trim. Similarly to the previous case for $\gamma^\star_{\text{ref}}$, equation (2.9) can be used to obtain the expression of $\phi^\star_{\text{ref}}$ as a function of $\theta^\star_{\text{ref}}$ and $\alpha^\star_{\text{ref}}$:

$$\phi^\star_{\text{ref}} = \arcsin\left(\frac{\mathcal{G}^\star_{\text{ref}}}{\sqrt{1+\mathcal{G}^{\star 2}_{\text{ref}}(\cos\alpha^\star_{\text{ref}})^2}}\sin\alpha^\star_{\text{ref}}\tan\theta^\star_{\text{ref}}\right) + \arctan\mathcal{G}^\star_{\text{ref}}\cos\alpha^\star_{\text{ref}} \tag{2.10}$$

Note that the combination of (2.8) and (2.10) (equivalently, (2.7) and (2.9)) define $\theta^\star_{\text{ref}}$ and $\phi^\star_{\text{ref}}$ as a function of the free parameter $\alpha^\star_{\text{ref}}$.

Surprisingly enough, equations (2.7) and (2.9) can be solved for $\phi^\star_{\text{ref}}$, yielding the explicit expression (independent of $\theta^\star_{\text{ref}}$)

$$\tan\phi^\star_{\text{ref}} = \frac{\mathcal{G}^\star_{\text{ref}}}{\cos\alpha^\star_{\text{ref}}} \frac{1-(\sin\gamma^\star_{\text{ref}})^2 + \sin\gamma^\star_{\text{ref}}\tan\alpha^\star_{\text{ref}}\sqrt{(1+(\mathcal{G}^\star_{\text{ref}})^2)(1-(\sin\gamma^\star_{\text{ref}})^2)}}{1-(\sin\gamma^\star_{\text{ref}})^2(1+(1+(\mathcal{G}^\star_{\text{ref}})^2(\tan\alpha^\star_{\text{ref}})^2} \tag{2.11}$$

whereas $\theta^\star_{\text{ref}}$ can be obtained from equation (2.8). The following should be noted:

- At *level flight*, $\gamma^\star_{\text{ref}} = 0$ hence

$$\phi^\star_{\text{ref}} = \arctan\left(\frac{\mathcal{G}^\star_{\text{ref}}}{\cos\alpha^\star_{\text{ref}}}\right), \qquad \theta^\star_{\text{ref}} = \arctan\left(\tan\alpha^\star_{\text{ref}}\cos\phi^\star_{\text{ref}}\right)$$

- When $\dot\psi^\star_{\text{ref}} = 0$, then $\mathcal{G}^\star_{\text{ref}} = 0$, hence

$$\phi^\star_{\text{ref}} = 0, \qquad \theta^\star_{\text{ref}} = \gamma^\star_{\text{ref}} + \alpha^\star_{\text{ref}}$$

- When $\gamma^\star_{\text{ref}} = 0$ and $\dot\psi^\star_{\text{ref}} = 0$

$$\phi^\star_{\text{ref}} = 0, \qquad \theta^\star_{\text{ref}} = \alpha^\star_{\text{ref}}$$

To summarize, the trim value for the Euler angles, $\boldsymbol{\eta}^\star_{\text{ref}}$, is determined from equation (2.11), equation (2.8) and equation (2.3), whereas the trim value for $\boldsymbol{\omega}^\star_{\text{ref}}$ is expressed by equation (2.6). Finally, the trim value for the aerodynamic control moments, $\boldsymbol{M}^\star_{A,\delta}$ is determined from the rotational dynamics (1.9) as folllows

$$\begin{aligned}
L^\star_{A,\delta} &= -\left(J_y - J_z\right) q^\star_{\text{ref}} r^\star_{\text{ref}} \\
M^\star_{A,\delta} &= \left(J_x - J_z\right) p^\star_{\text{ref}} r^\star_{\text{ref}} - M^\star_{A,\text{base}}(\alpha^\star_{\text{ref}}) \\
N^\star_{A,\delta} &= -\left(J_x - J_y\right) p^\star_{\text{ref}} q^\star_{\text{ref}}
\end{aligned}$$

once it is noticed that when $\beta = 0$, $L_{A,\text{base}}$ and $N_{A,\text{base}}$ vanish and $M_{A,\text{base}}$ is a function of α only. It is stressed that the trim condition for the rotational dynamics is parameterized in terms of $\alpha^\star_{\text{ref}}$, as $\gamma^\star_{\text{ref}}$ and $\dot\psi^\star_{\text{ref}}$ are assigned.

Translational Dynamics

As derived in the previous section, the trim value for the translational velocity is determined as a function of $\alpha^\star_{\text{ref}}$ as follows (recall that $V^\star_{T,\text{ref}}$ is assigned):

$$u^\star_{\text{ref}} = V^\star_{T,\text{ref}}\cos\alpha^\star_{\text{ref}}\,, \quad v^\star_{\text{ref}} = 0\,, \qquad w^\star_{\text{ref}} = V^\star_{T,\text{ref}}\sin\alpha^\star_{\text{ref}}$$

The trim value for the thrust, $T^\star_{\text{ref}}$, is obtained directly from equation (1.11), which at trim reads as

$$0 = T^\star_{\text{ref}}\cos\alpha^\star_{\text{ref}} - D^\star_{\text{ref}}(\alpha^\star_{\text{ref}}) - mg\sin\gamma^\star_{\text{ref}}$$

Finally, the equation for the vertical dynamics in (1.8)

$$\dot w = -pv + qu + \tfrac{1}{m}Z_A + \tfrac{1}{m}Z_{A,\delta} + g\cos\phi\cos\theta$$

is used to determine $\alpha^\star_{\text{ref}}$. At trim, the above equation reads as

$$0 = q^\star_{\text{ref}} V^\star_{T,\text{ref}} \cos\alpha^\star_{\text{ref}} + \tfrac{1}{m} Z^\star_A(\alpha^\star_{\text{ref}}) + \tfrac{1}{m} Z^\star_{A,\delta} + g\cos\phi^\star_{\text{ref}}\cos\theta^\star_{\text{ref}} \tag{2.12}$$

where

$$Z^\star_A(\alpha^\star_{\text{ref}}) = -D^\star_{\text{ref}}(\alpha^\star_{\text{ref}})\sin\alpha^\star_{\text{ref}} - L^\star_{\text{ref}}(\alpha^\star_{\text{ref}})\cos\alpha^\star_{\text{ref}}$$

Assume that $Z_{A,\delta} = 0$, that is, the aerodynamic control surfaces do not provide control authority to the vertical dynamic (this happens, for instance, if canard effectors are not present and the vehicle has only a minimal or traditional control suite.) If this the case, then $\alpha^\star_{\text{ref}}$ is constrained to be the equilibrium value for which the right-hand side of the above equation vanishes, that is

$$q^\star_{\text{ref}} V^\star_{T,\text{ref}} \cos\alpha^\star_{\text{ref}} + \tfrac{1}{m} Z^\star_A(\alpha^\star_{\text{ref}}) + g\cos\phi^\star_{\text{ref}}\cos\theta^\star_{\text{ref}} = 0$$

On the other hand, if the aerodynamic control force $Z_{A,\delta}$ is available, then $\alpha^\star_{\text{ref}}$ can be assigned independently and $Z^\star_{A,\delta}$ determined in such a way that (2.12) holds. Note that the possibility of imposing an arbitrary value for $\alpha^\star_{\text{ref}}$ entails selecting the forward and vertical velocity at trim, while maintaining the desired setpoint $\boldsymbol{y}^\star_{\text{ref}}$. This possibility is clearly enabled solely by the availability of the extra input $Z_{A,\delta}$, which corresponds to a property known as *weak input redundancy* [2, 3].

Remark 2.2.1.1 *It is important to notice that the inverse model at trim can not be determined a priori, due to uncertainty in the value of the parameter vector $\boldsymbol{\vartheta}$, and require the use of adaptive control techniques or integral control.*

2.2.2 General approximate right-inverse

For a time-varying reference trajectory $\boldsymbol{y}_{\text{ref}}(t)$, $t \geq 0$, satisfying Assumption 2.1.1, we define an approximate inverse by using the relations obtained in the previous section, which pertain to a trim condition. Consequently, given time-varying references $V^\star_{T,\text{ref}}(t)$, $\gamma^\star_{\text{ref}}(t)$, $\dot\psi^\star_{\text{ref}}(t)$ and $\alpha^\star_{\text{ref}}(t)$, we let $\boldsymbol{x}_{\text{ref}}(t)$ be defined by

$$\begin{aligned}
\phi_{\text{ref}}(t) &:= \arctan\left(\frac{\mathcal{G}_{\text{ref}}(t)}{\cos\alpha_{\text{ref}}(t)}\,\frac{1-\sin\gamma_{\text{ref}}(t)^2 + \sin\gamma_{\text{ref}}(t)\tan\alpha_{\text{ref}}(t)\sqrt{(1+(\mathcal{G}_{\text{ref}}(t))^2)(1-\sin\gamma_{\text{ref}}(t)^2)}}{1-\sin\gamma_{\text{ref}}(t)^2(1+(1+(\mathcal{G}_{\text{ref}}(t))^2\tan\alpha_{\text{ref}}(t)^2}\right)\\
\theta_{\text{ref}}(t) &:= \arcsin\left(\frac{\sin\gamma_{\text{ref}}(t)}{\sqrt{\cos\alpha_{\text{ref}}(t)^2 + \sin\alpha_{\text{ref}}(t)^2\cos\phi_{\text{ref}}(t)^2}}\right) + \arctan\left(\tan\alpha_{\text{ref}}(t)\cos\phi_{\text{ref}}(t)\right)\\
\phi_{\text{ref}}(t) &:= \psi(0) + \int_0^t \dot\psi_{\text{ref}}(s)\mathrm{d}s\\
u_{\text{ref}}(t) &:= V_{T,\text{ref}}(t)\cos\alpha_{\text{ref}}(t)\\
v_{\text{ref}}(t) &:= 0\\
w_{\text{ref}}(t) &:= V_{T,\text{ref}}(t)\sin\alpha_{\text{ref}}(t)\\
p_{\text{ref}}(t) &:= -\sin\theta_{\text{ref}}(t)\,\dot\psi_{\text{ref}}(t)\\
q_{\text{ref}}(t) &:= \sin\phi_{\text{ref}}(t)\cos\theta_{\text{ref}}(t)\,\dot\psi_{\text{ref}}(t)\\
r_{\text{ref}}(t) &:= \cos\phi_{\text{ref}}(t)\cos\theta_{\text{ref}}(t)\,\dot\psi_{\text{ref}}(t)
\end{aligned} \tag{2.13}$$

Note that, as we seek to assign an arbitrary reference $\alpha_{\text{ref}}(t)$ using the aerodynamic control $Z_{A,\delta}$, the approximate right inverse (2.13) can be explicitly computed, as it only depends on kinematic variables and known parameters. On the other hand, $\boldsymbol{u}_{\text{ref}}(t)$ can not be computed explicitly, as it depends on the unknown aerodynamic coefficients. As mentioned, this limitation will be circumvented in the design of the controller by means of adaptive and robust control techniques.

It must be noted that the Euler angles reference, $\boldsymbol{\eta}_{\text{ref}}(t) = (\phi_{\text{ref}}(t), \theta_{\text{ref}}(t), \psi_{\text{ref}}(t))$ defines the orientation of a desired reference frame, $\mathscr{F}_r$, with respect to the Earth-centered frame, $\mathscr{F}_e$. Consequently, $\boldsymbol{\eta}_{\text{ref}}(t)$ is the Euler angle parameterization of the rotation matrix $\boldsymbol{R}_{\text{ref}} := \boldsymbol{R}_{er}$, obeying the differential equation

$$\begin{aligned} \dot{\boldsymbol{R}}_{\text{ref}} &= \boldsymbol{R}_{\text{ref}} \boldsymbol{S}({}^r\boldsymbol{\omega}_{\text{ref}}) \\ \boldsymbol{R}_{\text{ref}}(0) &= \boldsymbol{R}_{\text{ref},0} \end{aligned} \tag{2.14}$$

where the angular velocity reference, *resolved in the desired reference frame* $\mathscr{F}_r$, is precisely ${}^r\boldsymbol{\omega}_{\text{ref}}(t) = (p_{\text{ref}}(t), q_{\text{ref}}(t), r_{\text{ref}}(t))$. Consequently, ${}^r\boldsymbol{\omega}_{\text{ref}}$ must be resolved in the body-fixed frame, $\mathscr{F}_b$, when performing computations involving the angular velocity of the vehicle, $\boldsymbol{\omega}$.

2.3 System Decomposition

To give a more useful formulation of the control problem towards the definition of the overall control strategy, we perform a decomposition of the equations of motion into a suitable structure. To this end, recall that the vehicle attitude reference is represented by the orientation of the reference frame $\mathscr{F}_r$ with respect to the Earth frame $\mathscr{F}_e$, described in turn by the rotation matrix $\boldsymbol{R}_{\text{ref}} := \boldsymbol{R}_{er}$ obeying the kinematic equation (2.14) The *attitude error* is selected as the orientation of $\mathscr{F}_b$ in $\mathscr{F}_r$, represented by the rotation matrix

$$\tilde{\boldsymbol{R}} := \boldsymbol{R}_{rb} = \boldsymbol{R}_{\text{ref}}^T \boldsymbol{R} \tag{2.15}$$

obeying the attitude error kinematics

$$\dot{\tilde{\boldsymbol{R}}} = \tilde{\boldsymbol{R}} \boldsymbol{S}(\tilde{\boldsymbol{\omega}}) \tag{2.16}$$

where $\tilde{\boldsymbol{\omega}} := \boldsymbol{\omega} - \boldsymbol{\omega}_{\text{ref}} = \boldsymbol{\omega} - \tilde{\boldsymbol{R}}^{T\,r}\boldsymbol{\omega}_{\text{ref}}$ is the tracking error for the angular velocity, resolved in the body-fixed frame.

With the given definitions at hand, the equations of motion of the CDM are written in terms of the attitude tracking error $(\tilde{\boldsymbol{R}}, \tilde{\boldsymbol{\omega}})$ as follows[3]

$$m\dot{\boldsymbol{\nu}} = -m\boldsymbol{S}(\boldsymbol{\omega}_{\text{ref}})\boldsymbol{\nu} + mg\boldsymbol{R}_{\text{ref}}^T\boldsymbol{e}_3 + \boldsymbol{F}_{A,\text{base}} + \boldsymbol{F}_{A,\delta} + T\boldsymbol{e}_1 + \boldsymbol{\Delta}(\tilde{\boldsymbol{R}}, \tilde{\boldsymbol{\omega}}, \boldsymbol{\nu}) \tag{2.17}$$

$$\dot{\tilde{\boldsymbol{R}}} = \tilde{\boldsymbol{R}} \boldsymbol{S}(\tilde{\boldsymbol{\omega}}) \tag{2.18}$$

$$\boldsymbol{J}\dot{\tilde{\boldsymbol{\omega}}} = -\boldsymbol{S}(\boldsymbol{\omega})\boldsymbol{J}\boldsymbol{\omega} + \boldsymbol{M}_{A,\text{base}} + \boldsymbol{M}_{A,\delta} - \boldsymbol{J}\dot{\boldsymbol{\omega}}_{\text{ref}} \tag{2.19}$$

In (2.17), the perturbation term

$$\boldsymbol{\Delta}(\tilde{\boldsymbol{R}}, \tilde{\boldsymbol{\omega}}, \boldsymbol{\nu}) := mg\big(\tilde{\boldsymbol{R}}^T - \boldsymbol{I}_3\big)\boldsymbol{R}_{\text{ref}}^T\boldsymbol{e}_3 - m\boldsymbol{S}(\tilde{\boldsymbol{\omega}})\boldsymbol{\nu} \tag{2.20}$$

[3]Note that we have dropped the kinematic equation ${}^e\dot{\boldsymbol{p}} = \boldsymbol{R}\boldsymbol{\nu}$ from the CDM, as it is inessential to the control problem considered here.

satisfies

$$\boldsymbol{\Delta}(\boldsymbol{I}_3, \boldsymbol{0}, \boldsymbol{\nu}) = \boldsymbol{0} \qquad \forall \boldsymbol{\nu} \in \mathbb{R}^3 \tag{2.21}$$

where $\boldsymbol{I}_3 \in \mathbb{R}^{3\times 3}$ is the identity matrix. In addition, since $\|\boldsymbol{\nu}\| = V_T$ and $\|\boldsymbol{R}_{\text{ref}}\| = 1$, the perturbation term $\boldsymbol{\Delta}(\tilde{\boldsymbol{R}}, \tilde{\boldsymbol{\omega}}, \boldsymbol{\nu})$ is a bounded function of $\boldsymbol{\nu}$ for any fixed $(\tilde{\boldsymbol{R}}, \tilde{\boldsymbol{\omega}})$ whenever V_T is bounded.

For system (2.17)–(2.19), define the auxiliary *attitude tracking output* $\boldsymbol{y}_{\text{att}} := \tilde{\boldsymbol{R}} - \boldsymbol{I}_3$. The *zero dynamics* of (2.17)–(2.19) with respect to $\boldsymbol{y}_{\text{att}}$ are given by

$$m\dot{\boldsymbol{\nu}} = -m\boldsymbol{S}(\boldsymbol{\omega}_{\text{ref}})\boldsymbol{\nu} + mg\boldsymbol{R}_{\text{ref}}^T\boldsymbol{e}_3 + \boldsymbol{F}_{A,\text{base}} + \boldsymbol{F}_{A,\delta} + T\boldsymbol{e}_1 \tag{2.22}$$

whereas (2.17) constitutes the *internal dynamics* of the system with respect to $\boldsymbol{y}_{\text{att}}$. The control objectives listed in Section 2.1 can then be conveniently decomposed into the design of a control policy for the zero dynamics (2.22), which involves the selection of the thrust T (equivalently, the throttle input, δ_T) and the aerodynamic control force $\boldsymbol{F}_{A,\delta}$, and the design of a control law ensuring boundedness of trajectories and asymptotic regulation of $\boldsymbol{y}_{\text{att}}$ using the aerodynamic control moments, $\boldsymbol{M}_{A,\delta}$. Specifically, the control goals are restated as follows:

1. **Airspeed Control:** Find the control input command $\delta_{T,\text{cmd}}(\cdot)$ such that when $\delta_T = \delta_{T,\text{cmd}}$ the trajectories of system (2.17) satisfy

   $$\lim_{t\to\infty} |V_T(t) - V_{T,\text{ref}}(t)| = 0$$

   robustly with respect to the perturbation $\boldsymbol{\Delta}$ and the uncertain parameters $\boldsymbol{\vartheta} \in \mathcal{P}$. Furthermore, the control shall provide robustness against the mismatch $\tilde{\delta}_T := \delta_T - \delta_{T,\text{cmd}}$.

2. **Lateral/Vertical Velocity Control:** Find the control command $\boldsymbol{F}_{A,\text{cmd}}(\cdot)$ for the aerodynamic control force $\boldsymbol{F}_{A,\delta}$ such that when $\boldsymbol{F}_{A,\delta} = \boldsymbol{F}_{A,\text{cmd}}$ the trajectories of system (2.17) are bounded and satisfy

   $$\lim_{t\to\infty} |v(t)| = 0\,, \qquad \lim_{t\to\infty} |w(t) - w_{\text{ref}}(t)| = 0$$

   when the perturbation $\boldsymbol{\Delta}$ vanishes, robustly with respect to $\boldsymbol{\vartheta} \in \mathcal{P}$.

3. **Attitude Control:** Find the control command $\boldsymbol{M}_{A,\text{cmd}}(\cdot)$ for the aerodynamic control moment $\boldsymbol{M}_{A,\delta}$ such that when $\boldsymbol{M}_{A,\delta} = \boldsymbol{M}_{A,\text{cmd}}$ the trajectories of (2.18)–(2.19) are bounded and satisfy $\lim_{t\to\infty} \|\boldsymbol{y}_{\text{att}}(t)\| = 0$, robustly with respect to $\boldsymbol{\vartheta} \in \mathcal{P}$.

4. **Dynamic Control Allocation;** The aerodynamic control forces and moments shall be allocated dynamically across the available aerodynamic actuators to solve for $\boldsymbol{\delta}_{\text{cmd}}$ the overdetermined system

   $$\bar{q}\boldsymbol{B}\,\mathbf{diag}(\boldsymbol{\lambda})\boldsymbol{\delta}_{\text{cmd}} = \begin{pmatrix} \boldsymbol{F}_{A,\text{cmd}} \\ \boldsymbol{M}_{A,\text{cmd}} \end{pmatrix}$$

   robustly with respect to the uncertain actuator effectiveness $\boldsymbol{\lambda} \in \Lambda$, and to minimize the mismatch between $\boldsymbol{\delta}$ and $\boldsymbol{\delta}_{\text{cmd}}$. Furthermore, the reference trajectory for the angle-of-attack α_{ref} shall be selected to provide additional robustness against the constrained actuator dynamics (1.3).

Both airspeed and attitude control fall within the scope of the *inner-loop controller* presented in the next chapter, whereas the lateral/vertical velocity control falls within the scope of the *outer-loop controller*, presented in Chapter 4.

Chapter 3

Inner-loop Control

3.1 Model-Recovery Anti-Windup for Adaptive Controllers

Consider the prototypical smooth SISO nonlinear system

$$
\begin{aligned}
\dot{\boldsymbol{x}}_1 &= \boldsymbol{f}(\boldsymbol{x}_1) + \boldsymbol{g}(\boldsymbol{x}_1)x_2 \\
\dot{x}_2 &= \boldsymbol{\vartheta}^T\boldsymbol{\phi}_1(t,\boldsymbol{x}) + \boldsymbol{\vartheta}^T\boldsymbol{\phi}_2(t,\boldsymbol{x})\mathrm{sat}(u) \\
y &= x_2
\end{aligned}
\tag{3.1}
$$

with state $\boldsymbol{x} = (\boldsymbol{x}_1, x_2) \in \mathbb{R}^{n-1} \times \mathbb{R}$, control input $u \in \mathbb{R}$, regulated output $y \in \mathbb{R}$, and plant parameter vector $\boldsymbol{\vartheta} \in \mathbb{R}^p$, where $\mathrm{sat}(\cdot)$ is a (possibly, asymmetric) saturation function[1].

Standing assumptions:

1. All vector fields in (3.1) are assumed to be smooth with respect to their arguments.
2. The regressors $\boldsymbol{\phi}_1(t,\boldsymbol{x})$ and $\boldsymbol{\phi}_2(t,\boldsymbol{x})$ are assumed to be bounded functions of t for any fixed $\boldsymbol{x} \in \mathbb{R}^n$.
3. The value of the parameter vector $\boldsymbol{\vartheta}$ is not known a priori, but it is assumed that $\boldsymbol{\vartheta} \in \Theta$, where $\Theta \subset \mathbb{R}^p$ is a known compact and convex set.
4. The so-called *high-frequency gain* satisfies $\boldsymbol{\vartheta}^T\boldsymbol{\phi}_2(t,\boldsymbol{x}) \geq b_0 > 0$ for all $t \in \mathbb{R}$, all $\boldsymbol{x} \in \mathbb{R}^n$ and all $\boldsymbol{\vartheta} \in \Theta$, where b_0 is a known constant.
5. The $\boldsymbol{x}_1$-dynamics of (3.1) are assumed to be **input-to-state stable** with respect to the input x_2.

Let a reference trajectory $y_{\mathrm{ref}}(t) \in \mathbb{R}$, $t \geq 0$ be given with its first derivative, which are assumed to be smooth and bounded. The *control objective* for system (3.1) is stated as follows: Find a (possibly dynamic) state-feedback controller

$$\dot{\boldsymbol{\xi}} = \boldsymbol{f}_c(\boldsymbol{\xi}, \boldsymbol{x}, y_{\mathrm{ref}}) \tag{3.2}$$

$$u = \boldsymbol{h}_c(\boldsymbol{\xi}, \boldsymbol{x}, y_{\mathrm{ref}}) \tag{3.3}$$

[1]Static saturation functions are considered here merely for notational simplicity. They can be replaced by more general operators modeling dynamic effects such as actuator dynamics, rate limiters, and combinations thereof.

so that all forward trajectories of the closed-loop system (3.1)-(3.2) are bounded and satisfy

$$\lim_{t\to\infty} |e(t)| = 0$$

where $e := y - y_{\mathrm{ref}}$ is the *tracking error.*

The following result is well known, but stated explicitly here, together with a sketch of the proof, to facilitate the discussion:

Proposition 3.1.1 *Consider the adaptive controller*

$$\begin{aligned}
\dot{\hat{\boldsymbol{\vartheta}}} &= \operatorname*{Proj}_{\hat{\boldsymbol{\vartheta}}\in\Theta}\{\boldsymbol{\Gamma}\boldsymbol{\phi}(t,\boldsymbol{x},u)e\}\ , \quad \hat{\boldsymbol{\vartheta}}(0)\in \operatorname{int}\Theta \\
u &= \frac{1}{\hat{\boldsymbol{\vartheta}}^T\boldsymbol{\phi}_2(t,\boldsymbol{x})}\left[-\hat{\boldsymbol{\vartheta}}^T\boldsymbol{\phi}_1(t,\boldsymbol{x}) + \dot{y}_{\mathrm{ref}} - ke\right]
\end{aligned} \tag{3.4}$$

where $\boldsymbol{\Gamma} = \boldsymbol{\Gamma}^T > \mathbf{0}$*,* $k > 0$ *are controller gains,* $\mathrm{Proj}(\cdot)$ *is a smooth projection operator, and*

$$\boldsymbol{\phi}(t,\boldsymbol{x},u) := \boldsymbol{\phi}_1(t,\boldsymbol{x}) + \boldsymbol{\phi}_2(t,\boldsymbol{x})\mathrm{sat}(u)$$

For all initial conditions $(\boldsymbol{x}(0),\hat{\boldsymbol{\vartheta}}(0))$ *and signals* $y_{\mathrm{ref}}(\cdot)$ *such that* $\mathrm{sat}(u(t)) = u(t)$ *for all* $t \geq 0$*, the controller* (3.4) *ensures boundedness of all forward trajectories of the closed-loop system and regulation of the tracking error* $e(t)$*.*

Proof. Using the coordinates $(\boldsymbol{x}_1, e, \tilde{\boldsymbol{\vartheta}})$, where $\tilde{\boldsymbol{\vartheta}} := \boldsymbol{\vartheta} - \hat{\boldsymbol{\vartheta}}$, the closed-loop system reads as

$$\begin{aligned}
\dot{\boldsymbol{x}}_1 &= \boldsymbol{f}(\boldsymbol{x}_1) + \boldsymbol{g}(\boldsymbol{x}_1)\left[e + y_{\mathrm{ref}}(t)\right] \\
\dot{e} &= -ke + \tilde{\boldsymbol{\vartheta}}^T\boldsymbol{\phi}\left(t,\boldsymbol{x},u\right) - \tilde{\boldsymbol{\vartheta}}^T\boldsymbol{\phi}_2(t,\boldsymbol{x})\mathrm{dz}(u) \\
\dot{\tilde{\boldsymbol{\vartheta}}} &= -\operatorname*{Proj}_{\hat{\boldsymbol{\vartheta}}\in\Theta}\{\boldsymbol{\Gamma}\boldsymbol{\phi}(t,\boldsymbol{x},u)e\}
\end{aligned} \tag{3.5}$$

where $\mathrm{dz}(u) := u - \mathrm{sat}(u)$ is the dead-zone function and $\boldsymbol{x} = \mathrm{col}(\boldsymbol{x}_1, e + y_{\mathrm{ref}})$. Define the Lyapunov function candidate $V_2(e,\tilde{\boldsymbol{\vartheta}}) := \frac{1}{2}e^2 + \frac{1}{2}\tilde{\boldsymbol{\vartheta}}^T\boldsymbol{\Gamma}^{-1}\tilde{\boldsymbol{\vartheta}}$, which is obviously a positive definite and radially unbounded function of the partial state $(e,\tilde{\boldsymbol{\vartheta}})$. In particular, there exist constants $\lambda_i > 0$, $i = 1,2$, such that

$$\lambda_1 |e|^2 + \lambda_2\|\tilde{\boldsymbol{\vartheta}}\|^2 \leq V_2(e,\tilde{\boldsymbol{\vartheta}}) \quad \text{for all } (e,\tilde{\boldsymbol{\vartheta}}) \in \mathbb{R}\times\mathbb{R}^p$$

The derivative of V_2 along the trajectories of the overall closed-loop system reads as

$$\dot{V}_2 = -ke^2 - e\,\hat{\boldsymbol{\vartheta}}^T\boldsymbol{\phi}_2(t,\boldsymbol{x})\mathrm{dz}(u)$$

Fix the reference trajectory $y_{\mathrm{ref}}(\cdot)$, and the initial condition $(\boldsymbol{x}_1(0), x_2(0), \hat{\boldsymbol{\vartheta}}(0)) \in \mathbb{R}^{n-1}\times\mathbb{R}\times\operatorname{int}\Theta$, and denote by $[0,T_{\max})$ the maximal interval of existence and uniqueness of the trajectories of (3.5). Assume that $\mathrm{dz}(u(t)) = 0$ for all $t \in [0,T_{\max})$. Then, along the forward trajectory of the closed-loop system

$$\frac{\mathrm{d}}{\mathrm{d}t}V_2(e(t),\tilde{\boldsymbol{\vartheta}}(t)) \leq -k\,|e(t)|^2 \leq 0 \quad \text{for all } t\in[0,T_{\max})$$

which implies that

$$\lambda_1\,|e(t)|^2 + \lambda_2\|\tilde{\boldsymbol{\vartheta}}(t)\|^2 \leq V_2(e(t), \tilde{\boldsymbol{\vartheta}}(t)) \leq V_2(e(0), \tilde{\boldsymbol{\vartheta}}(0)) \quad \text{for all } t \in [0, T_{\max})$$

Consequently

$$\limsup_{t\to T_{\max}} |e(t)| < \infty, \quad \limsup_{t\to T_{\max}} \|\tilde{\boldsymbol{\vartheta}}(t)\| < \infty$$

hence, the trajectories of the $(e, \tilde{\boldsymbol{\vartheta}})$-subsystem remain bounded within the maximal interval of existence and uniqueness. Note that since $y_{\mathrm{ref}}(\cdot)$ is a bounded signal, the first inequality above implies that

$$\limsup_{t\to T_{\max}} |x_2(t)| < \infty$$

To show that, indeed, $T_{\max} = +\infty$ we show that the trajectories of the $\boldsymbol{x}_1$-subsystem are bounded as well over $[0, T_{\max})$. Recall that by assumption the $\boldsymbol{x}_1$-subsystem is ISS. Consequently, there exist class-$\mathcal{K}$ functions $\gamma_0(\cdot)$, $\gamma_1(\cdot)$ such that the solutions of the system

$$\dot{\boldsymbol{x}}_1 = \boldsymbol{f}(\boldsymbol{x}_1) + \boldsymbol{g}(\boldsymbol{x}_1)v, \tag{3.6}$$

with external input $v(t) \in \mathbb{R}$ (defined for all $t \geq 0$) satisfy

$$\|\boldsymbol{x}_1(\cdot)\|_\infty \leq \max\{\gamma_0(\|\boldsymbol{x}_1(0)\|),\, \gamma_1(\|v(\cdot)\|_\infty)\}$$

Now, let $v(\cdot) = x_{2\tau}(\cdot)$, $\tau \in [0, T_{\max})$, be the **truncated signal** of $x_2(\cdot)$, defined as follows[2]

$$x_{2\tau}(t) = \begin{cases} x_2(t) & t \in [0, \tau) \\ 0 & t \geq \tau \end{cases}$$

In correspondence to this choice, the solution of (3.6) satisfies, for all $\tau \in [0, T_{\max})$,

$$\|\boldsymbol{x}_{1\tau}(\cdot)\|_\infty \leq \max\{\gamma_0(\|\boldsymbol{x}_1(0)\|),\, \gamma_1(\|x_{2\tau}(\cdot)\|_\infty)\}$$

hence

$$\limsup_{\tau\to T_{\max}} \|\boldsymbol{x}_{1\tau}(\cdot)\|_\infty \leq \max\left\{\gamma_0(\|\boldsymbol{x}_1(0)\|),\, \gamma_1\left(\limsup_{\tau\to T_{\max}} \|x_{2\tau}(\cdot)\|_\infty\right)\right\} < \infty$$

As a result, the overall forward trajectory is bounded on its maximal interval of existence and uniqueness, hence $T_{\max} = +\infty$. Finally, convergence of the tracking error is established via the *La Salle/Yoshizawa Theorem* applied to the $(e, \tilde{\boldsymbol{\vartheta}})$-subsystem via the Lyapunov function candidate $V_2(e, \tilde{\boldsymbol{\vartheta}})$. □

When the saturation is active, a perturbation is introduced in the Lyapunov equation, which now reads as

$$\dot{V} = -ke^2 - e\,\hat{\boldsymbol{\vartheta}}^T \boldsymbol{\phi}_2(t, \boldsymbol{x})\mathrm{dz}(u) \tag{3.7}$$

This perturbation couples the x_2-dynamics and the adaptation mechanism, and has an effect similar to integrator wind-up [4], with a possible destabilization of the closed-loop

[2] Note that for $\tau \in [0, T_{\max})$, $x_{2\tau}(t)$ is defined for all $t \geq 0$, whereas $x_2(t)$ is defined only on $[0, T_{\max})$.

system [5]. Even if boundedness of trajectories is maintained, the performance degradation may be severe, as the trajectory of the estimates may take a long time to recover (typically, when the saturation is active, $\hat{\boldsymbol{\vartheta}}(t)$ converges to or evolves on the boundary of the set Θ.) Approaches reported in the literature to mitigate the effect of input saturation in adaptive control loops include error augmentation [6, 7], modifications of the reference model [8–10] and command filtering [11, 12], all with suitable adjustments of the update law.

The modification proposed hereafter follows the paradigm of Model Recovery Anti-Windup (MRAW) (see [4] and references therein.) The MRAW strategy attempts to recover the behavior of the unconstrained closed-loop system when it is possible to do so. To begin, the following *adaptive observer* is added to the plant dynamics

$$\begin{aligned} \dot{\hat{x}}_2 &= \hat{\boldsymbol{\vartheta}}^T \boldsymbol{\phi}_1(t, \boldsymbol{x}) + \hat{\boldsymbol{\vartheta}}^T \boldsymbol{\phi}_2(t, \boldsymbol{x}) \mathrm{sat}(u) + k\tilde{x}_2 \\ \hat{x}_2(0) &= y_{\mathrm{ref}}(0) \end{aligned} \tag{3.8}$$

where $\tilde{x}_2 := x_2 - \hat{x}_2$ is the observation error. The observation error is used in place of the tracking error in the definition of the update law

$$\dot{\hat{\boldsymbol{\vartheta}}} = \underset{\hat{\boldsymbol{\vartheta}} \in \Theta}{\mathrm{Proj}} \left\{ \boldsymbol{\Gamma} \boldsymbol{\phi}(t, \boldsymbol{x}, u) \tilde{x}_2 \right\}, \quad \hat{\boldsymbol{\vartheta}}(0) \in \mathrm{int}\, \Theta \tag{3.9}$$

The tracking error is replaced by the *observed tracking error* $\hat{e} = \hat{x}_2 - y_{\mathrm{ref}}$, and the control input is chosen as

$$u = \frac{1}{\hat{\boldsymbol{\vartheta}}^T \boldsymbol{\phi}_2(t, \boldsymbol{x})} \left[-\hat{\boldsymbol{\vartheta}}^T \boldsymbol{\phi}_1(t, \boldsymbol{x}) + \dot{y}_{\mathrm{ref}} - k_s \hat{e} - k\tilde{x}_2 \right] \tag{3.10}$$

where $k_s > 0$ is a controller gain. Note that, since $\hat{e} = x_2 - \tilde{x}_2 - y_{\mathrm{ref}} = e - \tilde{x}_2$, choosing $k_s = k$ one recovers the same control input as in the original design.

Proposition 3.1.2 *The anti-windup modification* (3.8)–(3.10) *guarantees the following properties for the closed-loop system:*

1. *For all initial conditions* $(\boldsymbol{x}(0), \hat{\boldsymbol{\vartheta}}(0))$ *and signals* $y_{\mathrm{ref}}(\cdot)$ *such that, for the original closed-loop system* (3.5), $\mathrm{sat}(u(t)) = u(t)$ *for all* $t \geq 0$, *the anti-windup controller is inactive, and the behavior of the unconstrained controller is replicated.*

2. *For all initial conditions* $(\boldsymbol{x}(0), \hat{\boldsymbol{\vartheta}}(0))$ *and signals* $y_{\mathrm{ref}}(\cdot)$ *of such that the forward trajectory* $\hat{x}_2(\cdot)$ *is bounded*[3], *the adaptive controller* (3.8)–(3.10) *is "well behaved", in the sense that* $\hat{\boldsymbol{\vartheta}}(t)$, $t \geq 0$, *is bounded and* $\lim_{t\to\infty} \tilde{x}_2(t) = 0$.

3. *Under the same assumptions in 2),* $\lim_{t\to\infty} \mathrm{dz}(u(t)) = 0$ *implies* $\lim_{t\to\infty} \hat{e}(t) = 0$. *Consequently,* $\lim_{t\to\infty} e(t) = 0$

[3]Under a global Lipschitz condition for $\boldsymbol{\phi}(t, \boldsymbol{x}, u)$, boundedness of $\hat{x}_2(t)$, $t \geq 0$, can be relaxed to existence over the semi-infinite interval.

Proof. Using the coordinates $(\boldsymbol{x}_1, \tilde{x}_2, \tilde{\boldsymbol{\vartheta}}, \hat{e})$, the closed-loop system, after easy manipulations, reads as

$$
\begin{aligned}
\dot{\boldsymbol{x}}_1 &= \boldsymbol{f}(\boldsymbol{x}_1) + \boldsymbol{g}(\boldsymbol{x}_1)\left[\tilde{x}_2 + y_{\mathrm{ref}}(t) + \hat{e}\right] \\
\dot{\tilde{x}}_2 &= -k\tilde{x}_2 + \tilde{\boldsymbol{\vartheta}}^T \boldsymbol{\phi}\left(t, \boldsymbol{x}, u\right), \quad \tilde{x}_2(0) = e(0) \\
\dot{\tilde{\boldsymbol{\vartheta}}} &= -\operatorname*{Proj}_{\hat{\boldsymbol{\vartheta}} \in \Theta}\left\{\boldsymbol{\Gamma}\boldsymbol{\phi}\left(t, \boldsymbol{x}, u\right)\tilde{x}_2\right\} \\
\dot{\hat{e}} &= -k_s\hat{e} - \hat{\boldsymbol{\vartheta}}^T \boldsymbol{\phi}_2(t, \boldsymbol{x}, u)\mathrm{dz}(u), \quad \hat{e}(0) = 0
\end{aligned} \tag{3.11}
$$

where the initial conditions $(\tilde{x}_2(0), \hat{e}(0))$ have been reported explicitly, and $\boldsymbol{x} = \mathrm{col}(\boldsymbol{x}_1, \tilde{x}_2 + y_{\mathrm{ref}} + \hat{e})$. It is easy to see that if $\mathrm{dz}(u(t)) = 0$ for all $t \geq 0$, then $\hat{e}(t) = 0$ for all $t \geq 0$ as well, hence $\tilde{x}_2(t) = e(t)$ for all $t \geq 0$. As a result, the first three equations in (3.11) and (3.5) coincide (with same initial conditions), yielding the behavior of the original *unconstrained* closed-loop system. For the second property, boundedness of $(\boldsymbol{x}_1(t), \tilde{x}_2(t), \tilde{\boldsymbol{\vartheta}}(t))$, $t \geq 0$, follows from the same arguments as in Prop. 3.1.1. Boundedness of the trajectory $\hat{e}(t)$ for $t \geq 0$ implies that $\tilde{x}_2(t)$ is uniformly continuous over the semi-infinite interval; this, together with square-integrability, implies $\lim_{t\to\infty} \tilde{x}_2(t) = 0$ by virtue of Barbălat's Lemma (alternatively, by La Salle/Yoshizawa Theorem.) Finally, the assumptions establish the third property as a consequence of exponential stability of the $\hat{e}$-dynamics when $\mathrm{dz}(u) = 0$. □

3.2 Adaptive Airspeed Control

The airspeed dynamics have the following expression [1]

$$\dot{V}_T = a_T \cos\alpha \cos\beta - a_D - g\sin\gamma \tag{3.12}$$

where

$$a_T := \frac{\bar{q}S}{m} C_T(\alpha, M_\infty)\delta_T\,, \quad a_D := \frac{\bar{q}S}{m} C_D(\alpha, M_\infty)$$

denote acceleration due to thrust and drag, respectively, and

$$\gamma = \arcsin\left(\cos\alpha\cos\beta\sin\theta - \sin\alpha\cos\beta\cos\phi\cos\theta - \sin\beta\sin\phi\cos\theta\right)$$

is the flight-path angle. For simplicity, the actuator for the throttle is modeled as a static asymmetric saturation function, $\delta_T = \mathrm{sat}\left(\delta_{T,\mathrm{cmd}}\right)$. Recalling the expression of the aerodynamic coefficients (1.6), the right-hand side of (3.12) admits a linear parameterization of the form

$$\dot{V}_T = \boldsymbol{\vartheta}_V^T \boldsymbol{\phi}_1(t, V_T) + \boldsymbol{\vartheta}_V^T \boldsymbol{\phi}_2(t, V_T)\mathrm{sat}(\delta_{T,\mathrm{cmd}}) - g\sin\gamma(t) \tag{3.13}$$

where $\boldsymbol{\vartheta}_V = [C_T^0 \;\; C_T^\alpha \;\; \cdots \;\; C_D^{\alpha^2} \;\; C_D^{M_\infty}]^T$ and the regressors $\boldsymbol{\phi}_1(\cdot)$, $\boldsymbol{\phi}_2(\cdot)$ have obvious form. Note that in (3.13) we have regarded $\gamma(t)$, $\alpha(t)$ and $\beta(t)$ as exogenous signals. Similarly, Mach number and dynamic pressure are functions of airspeed and of altitude, following their definitions

$$M_\infty := \frac{V_T}{c(h)}\,, \quad \bar{q} := \tfrac{1}{2}\rho(h)V_T^2$$

where $c(h)$ and $\rho(h)$ denote, respectively, local speed of sound and air density, which are both bounded functions of altitude.

System (3.13) is in the form (3.1), required for the application of the adaptive controller with MRAW modification (3.8)–(3.10). Clearly, in this case the state $\boldsymbol{x}_1$ is void and x_2 is identified with V_T. Furthermore, we let $y_{\mathrm{ref}} = V_{T,\mathrm{ref}}$ be the reference output to be tracked, identify e with the tracking error $\tilde{V}_T := V_T - V_{T,\mathrm{ref}}$, identify $\hat{\boldsymbol{\vartheta}}$ with the parameter estimate $\hat{\boldsymbol{\vartheta}}_V$, identify $\hat{x}_2$ with the state of the observer $\hat{V}_T$, identify $\tilde{x}_2$ with the observation error $\tilde{V}_{T,\mathrm{obsv}} := V_T - \hat{V}_T$, let $\hat{e} = \hat{V}_T - V_{T,\mathrm{ref}}$ be the observed tracking error, and $\Theta = \mathcal{P}$ the parameter set. With these definitions at hand, the following result is immediate consequence of Proposition 3.1.2:

Proposition 3.2.1 *Assume that $\gamma(t)$ exists for all $t \geq 0$ and that there exist positive constants $\alpha_{\max} < \pi/2$ and $\beta_{\max} < \pi/2$ such that, for all $t \geq 0$,*

$$|\alpha(t)| \leq \alpha_{\max}\,, \qquad |\beta(t)| \leq \beta_{\max}$$

Then, the adaptive airspeed controller[4]

$$\begin{aligned}
\dot{\hat{V}}_T &= \hat{\boldsymbol{\vartheta}}_V^T \boldsymbol{\phi}_1(t, V_T) + \hat{\boldsymbol{\vartheta}}_V^T \boldsymbol{\phi}_2(t, V_T)\,\mathrm{sat}(\delta_{T,\mathrm{cmd}}) - g \sin\gamma(t) + k_V \tilde{V}_{T,\mathrm{obsv}} \\
\dot{\hat{\boldsymbol{\vartheta}}}_V &= \underset{\hat{\boldsymbol{\vartheta}}_V \in \mathcal{P}}{\mathrm{Proj}} \left\{ \boldsymbol{\Gamma}_V \left[\boldsymbol{\phi}_1(t, V_T) + \boldsymbol{\phi}_2(t, V_T)\,\mathrm{sat}(\delta_{T,\mathrm{cmd}}) \right] \tilde{V}_T \right\}, \quad \hat{\boldsymbol{\vartheta}}_V(0) \in \mathrm{int}\,\mathcal{P} \\
u &= \frac{1}{\hat{\boldsymbol{\vartheta}_V}^T \boldsymbol{\phi}_2(t, V_T)} \left[-\hat{\boldsymbol{\vartheta}}_V^T \boldsymbol{\phi}_1(t, V_T) + \dot{V}_{T,\mathrm{ref}} - k_V \tilde{V}_T + g \sin\gamma(t) \right]
\end{aligned} \tag{3.14}$$

with $\boldsymbol{\Gamma}_V = \boldsymbol{\Gamma}_V^T > \mathbf{0}$ and $k_V > 0$, ensures that the results of Proposition 3.1.2 hold for the airspeed dynamics (3.13)[5]*.*

Remark 3.2.1.1 *The assumption that the angle-of-attack and sideslip angle evolve away from singularities, albeit customary, is made here to allow the derivation of the airspeed controller as a separate entity from the rest of the controller. It is the task of the overall architecture to achieve this desired property for the closed-loop system.*

3.3 Adaptive Attitude Control

In this section, we address the problem of regulating the attitude tracking output, $\boldsymbol{y}_{\mathrm{att}} = \tilde{\boldsymbol{R}} - \boldsymbol{I}$. Recall from the previous section that the attitude tracking error is defined as the orientation of $\mathscr{F}_b$ in $\mathscr{F}_r$, represented by the rotation matrix

$$\tilde{\boldsymbol{R}} := \boldsymbol{R}_{rb} = \boldsymbol{R}_{\mathrm{ref}}^T \boldsymbol{R}$$

with associated angular velocity error resolved in the body frame

$$\tilde{\boldsymbol{\omega}} := \boldsymbol{\omega} - \boldsymbol{\omega}_{\mathrm{ref}} = \boldsymbol{\omega} - \tilde{\boldsymbol{R}}^T{}^r\boldsymbol{\omega}_{\mathrm{ref}}$$

[4]Compensation of the acceleration of gravity must be included in the feedforward component of the control signal.

[5]Note that, for the sake of simplicity, we have selected the same value for injection gain of the observer and the stabilizing gain of the controller, that is, $k = k_s = k_V$.

Consequently, the attitude error dynamics are written as

$$\begin{aligned}\dot{\tilde{\boldsymbol{R}}} &= \tilde{\boldsymbol{R}}\boldsymbol{S}(\tilde{\boldsymbol{\omega}}) \\ \boldsymbol{J}\dot{\tilde{\boldsymbol{\omega}}} &= -\boldsymbol{S}(\boldsymbol{\omega})\boldsymbol{J}\boldsymbol{\omega} + \boldsymbol{M}_{A,\text{base}} + \boldsymbol{M}_{A,\delta} - \boldsymbol{M}_{\text{ref}}\end{aligned} \tag{3.15}$$

where

$$\boldsymbol{M}_{\text{ref}} := \boldsymbol{J}\tilde{\boldsymbol{R}}^{T\,r}\dot{\boldsymbol{\omega}}_{\text{ref}} - \boldsymbol{J}\boldsymbol{S}(\tilde{\boldsymbol{\omega}})\tilde{\boldsymbol{R}}^{T\,r}\boldsymbol{\omega}_{\text{ref}}$$

is the term that accounts for the time derivative of the angular velocity reference resolved in the body frame. To avoid dealing directly with rotation matrices, $\tilde{\boldsymbol{R}}$ is parameterized by means of the *Modified Rodrigues Parameters* (MRP) $\boldsymbol{\sigma} \in \mathbb{R}^3$, defined as

$$\boldsymbol{\sigma} := \tan\left(\frac{\varphi}{4}\right)\vec{\boldsymbol{\lambda}}$$

where $(\varphi, \vec{\boldsymbol{\lambda}}) \in [0, 2\pi) \times \mathbb{S}^3$ is the angle-axis parameterization of $\tilde{\boldsymbol{R}}$. Among all the minimal parameterization of $SO(3)$, MRPs yield the largest domain of non-singularity of the representation [13]. It is noted that $\boldsymbol{\sigma} = \boldsymbol{0}$ corresponds to $\tilde{\boldsymbol{R}} = \boldsymbol{I}$. The propagation equation of the MRPs

$$\dot{\boldsymbol{\sigma}} = \tfrac{1}{2}\boldsymbol{G}(\boldsymbol{\sigma})\tilde{\boldsymbol{\omega}} \tag{3.16}$$

where

$$\boldsymbol{G}(\boldsymbol{\sigma}) := \frac{1 - \boldsymbol{\sigma}^T\boldsymbol{\sigma}}{2}\boldsymbol{I} + \boldsymbol{S}(\boldsymbol{\sigma}) + \boldsymbol{S}^2(\boldsymbol{\sigma})$$

is then used in lieu of the first equation in (3.15). System (3.16) is known to be a *lossless system* with respect to the input/output pair $(\boldsymbol{\sigma}, \tilde{\boldsymbol{\omega}})$, with positive definite, proper and locally quadratic storage function $V(\boldsymbol{\sigma}) = 2\ln(1 + \boldsymbol{\sigma}^T\boldsymbol{\sigma})$ [13]. This property is exploited by choosing the *augmented angular velocity error*

$$\boldsymbol{\omega}_{\text{err}} = \boldsymbol{K}_\sigma\boldsymbol{\sigma} + \tilde{\boldsymbol{\omega}} \tag{3.17}$$

where $\boldsymbol{K}_\sigma \in \mathbb{R}^{3\times3}$, $\boldsymbol{K}_\sigma = \boldsymbol{K}_\sigma^T > \boldsymbol{0}$, is a gain matrix to be selected. and $\boldsymbol{\omega}_{\text{aux}}$ is an auxiliary stabilizing term. Using (3.17), one obtains

$$\dot{\boldsymbol{\sigma}} = -\tfrac{1}{2}\boldsymbol{G}(\boldsymbol{\sigma})\boldsymbol{K}_\sigma\boldsymbol{\sigma} + \tfrac{1}{2}\boldsymbol{G}(\boldsymbol{\sigma})\boldsymbol{\omega}_{\text{err}} \tag{3.18}$$

Proposition 3.3.1 *System* (3.18) *is* input-to-state stable *with respect to the input* $\boldsymbol{\omega}_{\text{err}}$. *In particular, its state satisfies the asymptotic bound (see [14])*

$$\|\boldsymbol{\sigma}\|_a \leq \frac{1}{\lambda_{\min}(\boldsymbol{K}_\sigma)}\|\boldsymbol{\omega}_{\text{err}}\|_a \tag{3.19}$$

where $\lambda_{\min}(\boldsymbol{K}_\sigma)$ *denotes the smallest eigenvalue of* $\boldsymbol{K}_\sigma$, *and*

$$\|\boldsymbol{\sigma}\|_a := \limsup_{t\to\infty}\|\boldsymbol{\sigma}(t)\|$$

is the asymptotic norm *of* $\boldsymbol{\sigma}(\cdot)$.

Proof. Easy computations show that the derivative of the Lyapunov function candidate $V(\boldsymbol{\sigma}) = 2\ln(1+\boldsymbol{\sigma}^T\boldsymbol{\sigma})$ along trajectories of (3.18) satisfies

$$\dot{V} \leq -\|\boldsymbol{\sigma}\| \Big[\lambda_{\min}(\boldsymbol{K}_\sigma)\|\boldsymbol{\sigma}\| - \|\boldsymbol{\omega}_{\text{err}}\|\Big]$$

where $\|\cdot\|$ denotes the Euclidean norm. The result follows directly from [14, Lemma 3.3], by setting $\alpha_1(s) = \alpha_2(s) = 2\ln(1+s^2)$ and $\chi(s) = \frac{s}{\lambda_{\min}(\boldsymbol{K}_\sigma)}$. □

Next, we devote our attention to controlling the angular velocity error dynamics, which in the new coordinates $\boldsymbol{\omega}_{\text{err}}$ read as

$$\boldsymbol{J}\dot{\boldsymbol{\omega}}_{\text{err}} = -\boldsymbol{S}(\boldsymbol{\omega})\boldsymbol{J}\boldsymbol{\omega} + \boldsymbol{M}_{A,\text{base}} + \boldsymbol{M}_{A,\delta} - \boldsymbol{M}_{\text{ref}} + \tfrac{1}{2}\boldsymbol{K}_\sigma\boldsymbol{G}(\boldsymbol{\sigma})\tilde{\boldsymbol{\omega}} \tag{3.20}$$

The baseline aerodynamic moment admits a linear parameterization in terms of the unknown aerodynamic coefficients, as follows

$$\boldsymbol{M}_{A,\text{base}} = \boldsymbol{\Psi}_1(t)\boldsymbol{\vartheta}_{\text{base}} \tag{3.21}$$

where $\boldsymbol{\vartheta}_{\text{base}} = [C_{L_A}^{\beta}\ C_{L_A}^{\beta M_\infty}\ \cdots\ C_{N_A}^{\alpha\beta}]^T$, and $\boldsymbol{\Psi}_1(t)$ is a suitably defined matrix-valued function. Note that in (3.21) we have denoted as a time variability the dependence of the regressors on $\alpha(t)$, $\beta(t)$ and $V_T(t)$ via $M_\infty(t)$ and $\bar{q}(t)$. Conversely, the control aerodynamic moment has the expression

$$\boldsymbol{M}_{A,\delta} = \bar{q}\boldsymbol{B}_2\,\mathbf{diag}(\boldsymbol{\lambda})\boldsymbol{\delta} \tag{3.22}$$

where the moment control-effectiveness matrix

$$\boldsymbol{B}_2 = \begin{pmatrix} -C_l^{\text{c}} & C_l^{\text{c}} & -C_l^{\text{f}} & C_l^{\text{f}} & C_l^{\text{t}} & -C_l^{\text{t}} \\ -C_m^{\text{c}} & -C_m^{\text{c}} & -C_m^{\text{f}} & -C_m^{\text{f}} & C_m^{\text{t}} & C_m^{\text{t}} \\ C_n^{\text{c}} & -C_n^{\text{c}} & C_n^{\text{f}} & -C_n^{\text{f}} & -C_n^{\text{t}} & C_n^{\text{t}} \end{pmatrix}$$

is assumed to be known with sufficient accuracy, and the vector of uncertain actuator effectiveness

$$\boldsymbol{\lambda} = [\lambda_1\ \cdots\ \lambda_6]^T$$

satisfies

$$\boldsymbol{\lambda} \in \Lambda := \{\lambda_0 \leq \lambda_i \leq 1,\ i = 1,2,\ldots,6\}$$

where $\lambda_0 \in (0,1)$ is a given constant.

To apply the results of Proposition 3.1.2, the following adaptive observer of the angular velocity dynamics is introduced

$$\begin{aligned} \boldsymbol{J}\dot{\hat{\boldsymbol{\omega}}} &= -\boldsymbol{S}(\boldsymbol{\omega})\boldsymbol{J}\boldsymbol{\omega} + \hat{\boldsymbol{M}}_{A,\text{base}} + \hat{\boldsymbol{M}}_{A,\delta} + \boldsymbol{K}_\omega(\boldsymbol{\omega} - \hat{\boldsymbol{\omega}}) \\ \hat{\boldsymbol{\omega}}(0) &= \boldsymbol{\omega}_{\text{ref}}(0) \end{aligned} \tag{3.23}$$

where $\boldsymbol{K}_\omega = \boldsymbol{K}_\omega^T > \mathbf{0}$ is the observer gain, and

$$\begin{aligned} \hat{\boldsymbol{M}}_{A,\text{base}} &:= \boldsymbol{\Psi}_1(t)\hat{\boldsymbol{\vartheta}}_{\text{base}} \\ \hat{\boldsymbol{M}}_{A,\delta} &:= \bar{q}\boldsymbol{B}_2\,\mathbf{diag}(\hat{\boldsymbol{\lambda}})\boldsymbol{\delta} = \bar{q}\boldsymbol{B}_2\,\mathbf{diag}(\boldsymbol{\delta})\hat{\boldsymbol{\lambda}} =: \boldsymbol{\Psi}_2(t,\boldsymbol{\delta})\hat{\boldsymbol{\lambda}} \end{aligned}$$

are the estimated aerodynamic moments. The dynamics of the observer error $\tilde{\boldsymbol{\omega}}_{\text{obs}} := \boldsymbol{\omega} - \hat{\boldsymbol{\omega}}$ read as

$$\boldsymbol{J}\dot{\tilde{\boldsymbol{\omega}}}_{\text{obs}} = -\boldsymbol{K}_{\omega}\tilde{\boldsymbol{\omega}}_{\text{obs}} + \boldsymbol{\Psi}_1(t)\tilde{\boldsymbol{\vartheta}}_{\text{base}} + \boldsymbol{\Psi}_2(t,\boldsymbol{\delta})\tilde{\boldsymbol{\lambda}} \tag{3.24}$$

where $\tilde{\boldsymbol{\vartheta}}_{\text{base}} := \boldsymbol{\vartheta}_{\text{base}} - \hat{\boldsymbol{\vartheta}}_{\text{base}}$ and $\tilde{\boldsymbol{\lambda}} := \boldsymbol{\lambda} - \hat{\boldsymbol{\lambda}}$ are estimation errors. Using the Lyapunov function candidate

$$W_{\omega}(\tilde{\boldsymbol{\omega}}_{\text{obs}}, \tilde{\boldsymbol{\vartheta}}_{\text{base}}, \tilde{\boldsymbol{\lambda}}) := \tfrac{1}{2}\tilde{\boldsymbol{\omega}}_{\text{obs}}^T\boldsymbol{J}\tilde{\boldsymbol{\omega}}_{\text{obs}} + \tfrac{1}{2}\tilde{\boldsymbol{\vartheta}}_{\text{base}}^T\boldsymbol{\Gamma}_{\text{base}}^{-1}\tilde{\boldsymbol{\vartheta}}_{\text{base}} + \tfrac{1}{2}\tilde{\boldsymbol{\lambda}}^T\boldsymbol{\Gamma}_{\lambda}^{-1}\tilde{\boldsymbol{\lambda}} \tag{3.25}$$

the update laws for the estimates $\hat{\boldsymbol{\vartheta}}_{\text{base}}$ and $\hat{\boldsymbol{\lambda}}$ are readily found to be

$$\begin{aligned}
\dot{\hat{\boldsymbol{\vartheta}}}_{\text{base}} &= \operatorname*{Proj}_{\hat{\boldsymbol{\vartheta}}_{\text{base}}\in\mathcal{P}}\left\{\boldsymbol{\Gamma}_{\text{base}}\boldsymbol{\Psi}_1(t)^T\tilde{\boldsymbol{\omega}}_{\text{obs}}\right\}, \quad \hat{\boldsymbol{\vartheta}}_{\text{base}}(0) \in \operatorname{int}\mathcal{P} \\
\dot{\hat{\boldsymbol{\lambda}}} &= \operatorname*{Proj}_{\hat{\boldsymbol{\lambda}}\in\Lambda}\left\{\boldsymbol{\Gamma}_{\lambda}\boldsymbol{\Psi}_2(t,\boldsymbol{\delta})^T\tilde{\boldsymbol{\omega}}_{\text{obs}}\right\}, \quad \hat{\boldsymbol{\lambda}}(0) = \mathbf{1}
\end{aligned} \tag{3.26}$$

where $\boldsymbol{\Gamma}_{\text{base}} = \boldsymbol{\Gamma}_{\text{base}}^T > \mathbf{0}$ and $\boldsymbol{\Gamma}_{\lambda} = \boldsymbol{\Gamma}_{\lambda}^T > \mathbf{0}$ are gain matrices, and $\mathbf{1} = [1\ 1\ \cdots\ 1]^T$ (note that all actuators are initially assumed to be "healthy.")
For the dynamics of the adaptive observer (3.24)–(3.26) in error coordinates, namely

$$\begin{aligned}
\boldsymbol{J}\dot{\tilde{\boldsymbol{\omega}}}_{\text{obs}} &= -\boldsymbol{K}_{\omega}\tilde{\boldsymbol{\omega}}_{\text{obs}} + \boldsymbol{\Psi}_1(t)\tilde{\boldsymbol{\vartheta}}_{\text{base}} + \boldsymbol{\Psi}_2(t,\boldsymbol{\delta})\tilde{\boldsymbol{\lambda}} \\
\dot{\tilde{\boldsymbol{\vartheta}}}_{\text{base}} &= -\operatorname*{Proj}_{\hat{\boldsymbol{\vartheta}}_{\text{base}}\in\mathcal{P}}\left\{\boldsymbol{\Gamma}_{\text{base}}\boldsymbol{\Psi}_1(t)^T\tilde{\boldsymbol{\omega}}_{\text{obs}}\right\} \\
\dot{\tilde{\boldsymbol{\lambda}}} &= -\operatorname*{Proj}_{\hat{\boldsymbol{\lambda}}\in\Lambda}\left\{\boldsymbol{\Gamma}_{\lambda}\boldsymbol{\Psi}_2(t,\boldsymbol{\delta})^T\tilde{\boldsymbol{\omega}}_{\text{obs}}\right\}
\end{aligned} \tag{3.27}$$

the following result holds:

Proposition 3.3.2 *Assume that there exist positive constants $\alpha_{\max} < \pi/2$ and $\beta_{\max} < \pi/2$ such that, for all $t \geq 0$,*

$$|\alpha(t)| \leq \alpha_{\max}, \qquad |\beta(t)| \leq \beta_{\max}$$

and that the forward trajectory $V_T(t)$, $t \geq 0$, is bounded. Then, all forward trajectories of system (3.27) *(originating from initial conditions as in* (3.23) *and* (3.26)*) are bounded and satisfy*

$$\|\tilde{\boldsymbol{\omega}}_{\text{obs}}\|_a = 0$$

Proof. The proof follows directly from application of La Salle/Yoshizawa Theorem using the Lyapunov function candidate (3.25). □

We are now left with the problem of controlling the observer dynamics (3.23). To begin, define the *estimated tracking error* for the angular velocity as

$$\hat{\boldsymbol{e}}_{\omega} := \hat{\boldsymbol{\omega}} - \boldsymbol{\omega}_{\text{ref}}$$

and note that

$$\begin{aligned}
\boldsymbol{\omega}_{\text{err}} &= \boldsymbol{K}_{\sigma}\boldsymbol{\sigma} + \tilde{\boldsymbol{\omega}} = \boldsymbol{K}_{\sigma}\boldsymbol{\sigma} + \boldsymbol{\omega} - \boldsymbol{\omega}_{\text{ref}} \\
&= \boldsymbol{K}_{\sigma}\boldsymbol{\sigma} + \boldsymbol{\omega} - \hat{\boldsymbol{\omega}} + \hat{\boldsymbol{\omega}} - \boldsymbol{\omega}_{\text{ref}} \\
&= \boldsymbol{K}_{\sigma}\boldsymbol{\sigma} + \hat{\boldsymbol{e}}_{\omega} + \tilde{\boldsymbol{\omega}}_{\text{obs}} \\
&= \hat{\boldsymbol{\omega}}_{\text{err}} + \tilde{\boldsymbol{\omega}}_{\text{obs}}
\end{aligned} \tag{3.28}$$

where we have defined the *estimated augmented error*, $\hat{\boldsymbol{\omega}}_{\text{err}}$, as[6]

$$\hat{\boldsymbol{\omega}}_{\text{err}} := \boldsymbol{K}_\sigma \boldsymbol{\sigma} + \hat{e}_\omega$$

Accordingly, the dynamics of $\hat{\boldsymbol{\omega}}_{\text{err}}$ reads as

$$\boldsymbol{J}\dot{\hat{\boldsymbol{\omega}}}_{\text{err}} = -\boldsymbol{S}(\boldsymbol{\omega})\boldsymbol{J}\boldsymbol{\omega} + \hat{\boldsymbol{M}}_{A,\text{base}} + \hat{\boldsymbol{M}}_{A,\delta} - \boldsymbol{M}_{\text{ref}} + \tfrac{1}{2}\boldsymbol{K}_\sigma \boldsymbol{G}(\boldsymbol{\sigma})\tilde{\boldsymbol{\omega}} + \boldsymbol{K}_\omega \tilde{\boldsymbol{\omega}}_{\text{obs}}$$

Let

$$\begin{aligned}\hat{\boldsymbol{M}}_{A,\delta} &= \bar{q}\boldsymbol{B}_2\,\mathbf{diag}(\hat{\boldsymbol{\lambda}})\boldsymbol{\delta}\\ &= \bar{q}\boldsymbol{B}_2\,\mathbf{diag}(\hat{\boldsymbol{\lambda}})\boldsymbol{\delta}_{\text{cmd}} + \bar{q}\boldsymbol{B}_2\,\mathbf{diag}(\hat{\boldsymbol{\lambda}})\tilde{\boldsymbol{\delta}}\\ &= \hat{\boldsymbol{M}}_{A,\text{cmd}} + \bar{q}\boldsymbol{B}_2\,\mathbf{diag}(\hat{\boldsymbol{\lambda}})\tilde{\boldsymbol{\delta}}\end{aligned}$$

where

$$\hat{\boldsymbol{M}}_{A,\text{cmd}} := \bar{q}\boldsymbol{B}_2\,\mathbf{diag}(\hat{\boldsymbol{\lambda}})\boldsymbol{\delta}_{\text{cmd}}$$

is the estimated commanded moment and

$$\tilde{\boldsymbol{\delta}} := \boldsymbol{\delta} - \boldsymbol{\delta}_{\text{cmd}}$$

is the mismatch between the commanded deflections of the aerodynamic surfaces and the actual deflections. The control[7]

$$\hat{\boldsymbol{M}}_{A,\text{cmd}} = \boldsymbol{S}(\boldsymbol{\omega})\boldsymbol{J}\boldsymbol{\omega} - \hat{\boldsymbol{M}}_{A,\text{base}} + \boldsymbol{M}_{\text{ref}} - \tfrac{1}{2}\boldsymbol{K}_\sigma \boldsymbol{G}(\boldsymbol{\sigma})\tilde{\boldsymbol{\omega}} - \boldsymbol{K}_\omega \boldsymbol{\omega}_{\text{err}} \tag{3.29}$$

yields

$$\boldsymbol{J}\dot{\hat{\boldsymbol{\omega}}}_{\text{err}} = -\boldsymbol{K}_\omega \hat{\boldsymbol{\omega}}_{\text{err}} + \bar{q}\boldsymbol{B}_2\,\mathbf{diag}(\hat{\boldsymbol{\lambda}})\tilde{\boldsymbol{\delta}} \tag{3.30}$$

As the dynamic pressure depends on the vehicle airspeed, and the estimate $\hat{\lambda}$ vary within the compact set Λ, the following holds:

Proposition 3.3.3 *Let the assumptions of Proposition 3.3.2 hold. Assume, in addition, that $\tilde{\boldsymbol{\delta}}(t)$, $t \geq 0$, is bounded*[8]*. Then, the system* (3.30) *is ISS with respect to $\tilde{\delta}_b$ as input, with as asymptotic bound of the form*

$$\|\hat{\boldsymbol{\omega}}_{\text{err}}\|_a \leq \frac{\mu_\omega}{\lambda_{\min}(\boldsymbol{K}_\omega)}\|\tilde{\boldsymbol{\delta}}_{\text{err}}\|_a \tag{3.31}$$

where μ_ω is a suitable constant.

Proof. The proof follows immediately from [14, Lemma 3.3] and the fact that, letting $\bar{q}_{\max} := \sup_{t\geq 0}\bar{q}(t)$ and using the induced matrix 2-norm, one obtains

$$\|\bar{q}(t)\boldsymbol{B}_2\,\mathbf{diag}(\hat{\boldsymbol{\lambda}})\| \leq \bar{q}_{\max}\|\boldsymbol{B}_2\|\|\mathbf{diag}(\hat{\boldsymbol{\lambda}})\| \leq \bar{q}_{\max}\|\boldsymbol{B}_2\| =: \mu_\omega$$

□

Reverting back to the MRP parameterization of the attitude error, owing to Proposition 3.3.1 and to the last identity in (3.28), one obtains the final result of this section:

[6] Compare with identity (3.17).

[7] Note that, for the sake of simplicity, we have used the same matrix, $\boldsymbol{K}_\omega$, for the output injection gain of the observer and for the feedback gain of the stabilizer.

[8] The extra assumption of boundedness of $\tilde{\boldsymbol{\delta}}$ is needed due to the fact that the very definition of the command $\boldsymbol{\delta}_{\text{cmd}}$ depends on $\tilde{\boldsymbol{\delta}}$.

Proposition 3.3.4 *Assume that the assumptions of Proposition 3.3.3 hold. Then, system* (3.18) *is* input-to-state stable *with respect to the input* $\tilde{\boldsymbol{\delta}}$. *In particular, its state satisfies the asymptotic bound*

$$\|\boldsymbol{\sigma}\|_a \leq \frac{2\mu_\omega}{\lambda_{\min}\left(\boldsymbol{K}_\sigma\right)\lambda_{\min}\left(\boldsymbol{K}_\omega\right)}\|\tilde{\boldsymbol{\delta}}\|_a \tag{3.32}$$

Proof. The proof follows directly from the bounds established in Propositions 3.19 and 3.31 by noticing that

$$\|\boldsymbol{\omega}_{\text{err}}\| = \|\hat{\boldsymbol{\omega}}_{\text{err}} + \tilde{\boldsymbol{\omega}}\| \leq 2\max\{\|\hat{\boldsymbol{\omega}}_{\text{err}}\|, \|\tilde{\boldsymbol{\omega}}\|\}$$

and that, according to Proposition 3.3.2, $\|\tilde{\boldsymbol{\omega}}\|_a = 0$. □

We are left to determine how to efficiently solve for $\boldsymbol{\delta}_{\text{cmd}}$ the overdetermined system of equation

$$\bar{q}\boldsymbol{B}_2\,\mathbf{diag}(\hat{\boldsymbol{\lambda}})\boldsymbol{\delta}_{\text{cmd}} = \hat{\boldsymbol{M}}_{A,\text{cmd}}$$

with $\hat{\boldsymbol{M}}_{A,\text{cmd}}$ given in (3.29). This will be dealt with in the sequel.

Chapter 4

Outer-loop Control

4.1 Control of the Lateral Velocity

The equation of the dynamics of the lateral velocity, introduced in (1.8), reads as

$$\dot{v} = pw - ru + \tfrac{1}{m} Y_A(\beta) + g \sin\phi \cos\theta \tag{4.1}$$

Using the attitude error $(\tilde{\boldsymbol{R}}, \tilde{\boldsymbol{\omega}})$ defined in (2.16) and the perturbation term $\boldsymbol{\Delta}$ defined in (2.20), the lateral dynamics is written as part of the internal dynamics of the system with respect to the *attitude tracking error* $\boldsymbol{y}_{\text{att}} := \tilde{\boldsymbol{R}} - \boldsymbol{I}_3$ as follows

$$\dot{v} = p_{\text{ref}} w - r_{\text{ref}} u + g \sin(\phi_{\text{ref}}) \cos(\theta_{\text{ref}}) + \tfrac{1}{m} Y_A(\beta) + d_{v,1}(\boldsymbol{\sigma}, \tilde{\boldsymbol{\omega}}, \boldsymbol{\nu}) \tag{4.2}$$

where

$$d_{v,1}(\boldsymbol{\sigma}, \tilde{\boldsymbol{\omega}}, \boldsymbol{\nu}) := \boldsymbol{e}_2^T \boldsymbol{\Delta}(\tilde{\boldsymbol{R}}, \tilde{\boldsymbol{\omega}}, \boldsymbol{\nu})$$

Recall that $d_{v,1}(\boldsymbol{0}, \boldsymbol{0}, \boldsymbol{\nu}) = 0$ for all $\boldsymbol{\nu} \in \mathbb{R}^3$, and that $d_{v,1}(\boldsymbol{0}, \boldsymbol{0}, \boldsymbol{\nu})$ is a bounded function of $\boldsymbol{\nu}$ for any fixed $(\boldsymbol{\sigma}, \tilde{\boldsymbol{\omega}})$ whenever V_T is bounded. The first two term in the right-hand side of (4.2) are expanded as follows, making use of the approximation $\cos\beta \approx 1$:

$$\begin{aligned}
p_{\text{ref}} w - r_{\text{ref}} u &= p_{\text{ref}} V_T \sin\alpha - r_{\text{ref}} V_T \cos\alpha \\
&= p_{\text{ref}} V_{T,\text{ref}} \sin\alpha - r_{\text{ref}} V_{T,\text{ref}} \cos\alpha + (p_{\text{ref}} \sin\alpha - r_{\text{ref}} \cos\alpha) \tilde{V}_T \\
&= p_{\text{ref}} V_{T,\text{ref}} \sin\alpha_{\text{ref}} - r_{\text{ref}} V_{T,\text{ref}} \cos\alpha_{\text{ref}} + (p_{\text{ref}} \sin\alpha - r_{\text{ref}} \cos\alpha) \tilde{V}_T \\
&\quad + V_{T,\text{ref}} \left[p_{\text{ref}} (\sin\alpha - \sin\alpha_{\text{ref}}) - r_{\text{ref}} (\cos\alpha - \cos\alpha_{\text{ref}}) \right] \\
&= p_{\text{ref}} w_{\text{ref}} - r_{\text{ref}} u_{\text{ref}} + d_{v,2}(\tilde{V}_T) + d_{v,3}(\alpha)
\end{aligned}$$

where:

- $d_{v,2}(\tilde{V}_T) := (p_{\text{ref}} \sin\alpha - r_{\text{ref}} \cos\alpha) \tilde{V}_T$ is a perturbation that vanishes at $\tilde{V}_T = 0$ and is bounded for all fixed $\tilde{V}_T$;
- $d_{v,3}(\alpha) := V_{T,\text{ref}} \left[p_{\text{ref}} (\sin\alpha - \sin\alpha_{\text{ref}}) - r_{\text{ref}} (\cos\alpha - \cos\alpha_{\text{ref}}) \right]$ is a bounded perturbation that vanishes at $\alpha = \alpha_{\text{ref}}$.

Since $\alpha = \alpha_{\text{ref}}$ if and only if $V_T = V_{T,\text{ref}}$ and $w = w_{\text{ref}}$, the perturbation $d_{v,3}(\alpha)$ vanishes when $\tilde{V}_T = 0$ and $\tilde{w} := w - w_{\text{ref}} = 0$. Consequently, we shall adopt the more informative notation $d_{v,3}(\alpha) = d_{v,3}(\tilde{V}_T, \tilde{w})$.

As a result, the dynamics of the lateral velocity is written as follows

$$\begin{aligned}\dot{v} &= p_{\text{ref}} w_{\text{ref}} - r_{\text{ref}} u_{\text{ref}} + g \sin(\phi_{\text{ref}}) \cos(\theta_{\text{ref}}) + \tfrac{1}{m} Y_A(\beta) + d_{v,1}(\boldsymbol{\sigma}, \tilde{\boldsymbol{\omega}}, \boldsymbol{\nu}) \\ &\quad + d_{v,2}(\tilde{V}_T) + d_{v,3}(\tilde{V}_T, \tilde{w})\end{aligned}$$

Note that, at trim,

$$p^\star_{\text{ref}} w^\star_{\text{ref}} - r^\star_{\text{ref}} u^\star_{\text{ref}} + g \sin(\phi^\star_{\text{ref}}) \cos(\theta^\star_{\text{ref}}) = 0$$

due to the selection of $\phi^\star_{\text{ref}}$ as the roll angle for coordinated turn. As a result,

$$d_{v,\text{ref}}(t) := p_{\text{ref}}(t) w_{\text{ref}}(t) - r_{\text{ref}}(t) u_{\text{ref}}(t) + g \sin(\phi_{\text{ref}}(t)) \cos(\theta_{\text{ref}}(t))$$

satisfies, by assumption,

$$\lim_{t \to \infty} |d_{v,\text{ref}}(t)| = 0$$

It should be noted that the side force $Y_A(\beta) = \bar{q} S C_Y(\beta, M_\infty)$ satisfies the following property (recall that $\beta = \arcsin(v/V_T)$):

Property 4.1.0.1 *There exists a class-$\mathcal{K}$ function $\kappa(\cdot) : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ such that*

$$\tfrac{1}{m} Y_A(\beta) v < -\kappa(|v|) \quad \textit{for all } v \in \mathbb{R} \textit{ such that } |v| \leq V_T$$

Furthermore, there exist numbers $\varrho > 0$, $\bar{\kappa} > 0$ such that $\kappa(s) \geq \kappa s$ for all $s \in [0, \varrho]$.

Property 4.1.0.1 establishes the fact that the side force provides dissipation for the lateral velocity dynamics (4.2). Finally, defining $\tilde{\boldsymbol{x}} := \boldsymbol{x} - \boldsymbol{x}_{\text{ref}}$ and letting

$$d_v(t, \tilde{\boldsymbol{x}}) := d_{v,1}(\boldsymbol{\sigma}, \tilde{\boldsymbol{\omega}}, \boldsymbol{\nu}) + d_{v,2}(\tilde{V}_T) + d_{v,3}(\tilde{V}_T, \tilde{w}) + d_{v,\text{ref}}(t)$$

be the overall state-dependent perturbation (note that $d_{v,\text{ref}}(t)$ depends only on exogenous reference signals), the lateral dynamics is written as

$$\dot{v} = \tfrac{1}{m} Y_A(\beta) + d_v(t, \tilde{\boldsymbol{x}}) \tag{4.3}$$

For system (4.3) the following holds as an immediate consequence of Property (4.1.0.1):

Proposition 4.1.1 *The lateral velocity dynamics are locally input-to-state stable with respect to the disturbance input d_v. In particular, there exists a constant μ_v and a class-$\mathcal{K}$ function $\varphi_v(\cdot)$ such that, for all disturbance inputs satisfying $\|d_v\|_\infty \leq \mu_v$, the state $v(\cdot)$ of* (4.3) *satisfies the asymptotic bound*

$$\|v\|_a \leq \varphi_v\left(\|d_v\|_a\right) \tag{4.4}$$

Using repeatedly the relation [15, 16]

$$\varrho(a + b) \leq \varrho(2a) + \varrho(2b) \leq 2 \max\left\{\varrho(2a), \varrho(2b)\right\}$$

which holds for any class-$\mathcal{K}$ function $\varrho(\cdot)$ and any $a, b \geq 0$, one obtains

$$\begin{aligned}\|v\|_a &\leq 4 \max\left\{\varphi_v\left(4\|d_{v,1}\|_a\right), \varphi_v\left(4\|d_{v,2}\|_a\right), \varphi_v\left(4\|d_{v,3}\|_a\right), \varphi_v\left(4\|d_{v,\text{ref}}\|_a\right)\right\} \\ &= 4 \max\left\{\varphi_v\left(4\|d_{v,1}\|_a\right), \varphi_v\left(4\|d_{v,2}\|_a\right), \varphi_v\left(4\|d_{v,3}\|_a\right)\right\}\end{aligned} \tag{4.5}$$

It is noted that the asymptotic norm of $d_{v,\text{ref}}(\cdot)$ has been removed from the bound (4.4) due to the fact that $\|d_{v,\text{ref}}\|_a = 0$ by definition of the class of reference trajectories under consideration.

4.2 Control of the Vertical Velocity

The equation of the dynamics of the vertical velocity, introduced in (1.8), reads as

$$\dot{w} = qu - pv + mg\cos\phi\cos\theta + \tfrac{1}{m}Z_A(\alpha) + \tfrac{1}{m}Z_{A,\delta} \tag{4.6}$$

The goal is to use the available aerodynamic control force, $Z_{A,\delta}$ to let $w(t)$ track the reference trajectory $w_{\text{ref}}(t) := V_{T,\text{ref}}(t)\sin\alpha_{\text{ref}}(t)$, being $\alpha(t)$, $t \geq t$ the reference for the angle of attack, which is a degree of freedom in the design. For the time being, we assume that $\alpha_{\text{ref}}(t)$, $t \geq t$ is an signal that is available together with its derivatives.

The expression of the vertical aerodynamic force due to the fuselage, $Z_{A,\text{base}}$ has the following expression, as given in Table (1.3) and equations (1.6)

$$\begin{aligned} Z_{A,\text{base}} &= -D\sin\alpha - L\cos\alpha \\ &= -\bar{q}S\left(C_D^0 + C_D^{\alpha}\alpha + C_D^{\alpha^2}\alpha^2 + C_D^{M_\infty}M_\infty\right)\sin\alpha + \\ &\quad -\bar{q}S\left(C_L^0 + C_L^{\alpha}\alpha + C_L^{M_\infty\alpha}M_\infty\alpha + C_L^{\alpha^2}\alpha^2 + C_L^{M_\infty}M_\infty\right)\cos\alpha \end{aligned}$$

We approximate this expression with the following second-order expansion[1]

$$Z_{A,\text{base}} \approx -\bar{q}SC_z^{\alpha}\alpha - \bar{q}SC_z^{\alpha^2}\alpha^2 - \bar{q}SC_z^0 \tag{4.7}$$

where it is noted that $C_z^{\alpha} > 0$. It is also noted that C_z^{α} is the dominant term in the above expansion. With this in mind, we let

$$\tfrac{1}{m}Z_{A,\text{base}} = \boldsymbol{\phi}_w(\alpha)^T\boldsymbol{\vartheta}_w$$

where the vector of uncertain aerodynamic parameters reads as

$$\boldsymbol{\vartheta}_w = [C_z^0 \;\; C_z^{\alpha} \;\; C_z^{\alpha^2}]^T$$

and the regressor $\boldsymbol{\phi}_w(\alpha)$ has an obvious expression. The vertical dynamics is therefore written as follows

$$\dot{w} = qu - pv + g\cos\phi\cos\theta + \boldsymbol{\phi}_w(\alpha)^T\boldsymbol{\vartheta}_w + \tfrac{1}{m}Z_{A,\delta} \tag{4.8}$$

where the aerodynamic control force $Z_{A,\delta}$ reads as

$$Z_{A,\delta} = \boldsymbol{e}_3^T\boldsymbol{F}_{A,\delta} = \bar{q}\,\boldsymbol{e}_3^T\boldsymbol{B}_1\,\mathbf{diag}(\boldsymbol{\lambda})\boldsymbol{\delta} = \bar{q}\,\boldsymbol{e}_3^T\boldsymbol{B}_1\,\mathbf{diag}(\boldsymbol{\delta})\boldsymbol{\lambda}$$

Consider an adaptive observer for (4.8) of the form

$$\dot{\hat{w}} = qu - pv + g\cos\phi\cos\theta + \hat{\boldsymbol{\vartheta}}_w^T\boldsymbol{\phi}_w(\alpha) + \tfrac{\bar{q}}{m}\,\boldsymbol{e}_3^T\boldsymbol{B}_1\,\mathbf{diag}(\boldsymbol{\delta})\hat{\boldsymbol{\lambda}} + k_w\,(w - \hat{w}) \tag{4.9}$$

yielding the observer error dynamics

$$\dot{\tilde{w}}_{\text{obsv}} = -k_w\tilde{w}_{\text{obsv}} + \boldsymbol{\phi}_w(\alpha)^T\tilde{\boldsymbol{\vartheta}}_w + \tfrac{\bar{q}}{m}\,\boldsymbol{e}_3^T\boldsymbol{B}_1\,\mathbf{diag}(\boldsymbol{\delta})\tilde{\boldsymbol{\lambda}} \tag{4.10}$$

[1] Alternatively, one can obtain a curve-fitted model of $Z_{A,\text{base}}$ of the form (4.7) directly from wind-tunnel or CFD data.

where $\tilde{\boldsymbol{\vartheta}}_w := \boldsymbol{\vartheta}_w - \hat{\boldsymbol{\vartheta}}_w$, and $\tilde{\boldsymbol{\lambda}} := \boldsymbol{\lambda} - \hat{\boldsymbol{\lambda}}$ has been defined in (3.24). Using the Lyapunov function candidate

$$W_w(\tilde{w}_{\text{obs}}, \tilde{\boldsymbol{\vartheta}}_w, \tilde{\boldsymbol{\lambda}}) := \tfrac{1}{2}\tilde{w}_{\text{obs}}^2 + \tfrac{1}{2}\tilde{\boldsymbol{\vartheta}}_w^T \boldsymbol{\Gamma}_w^{-1}\tilde{\boldsymbol{\vartheta}}_w + \tfrac{1}{2}\tilde{\boldsymbol{\lambda}}^T \boldsymbol{\Gamma}_\lambda^{-1}\tilde{\boldsymbol{\lambda}} \tag{4.11}$$

the update law for the estimates $\hat{\boldsymbol{\vartheta}}_w$ is readily found to be

$$\dot{\hat{\boldsymbol{\vartheta}}}_w = \underset{\hat{\boldsymbol{\vartheta}}_w \in \mathcal{P}}{\operatorname{Proj}} \left\{ \boldsymbol{\Gamma}_w \boldsymbol{\phi}_w(\alpha)\tilde{w}_{\text{obs}} \right\}, \quad \hat{\boldsymbol{\vartheta}}_w(0) \in \operatorname{int}\mathcal{P} \tag{4.12}$$

where $\boldsymbol{\Gamma}_w = \boldsymbol{\Gamma}_w^T > \mathbf{0}$ is a gain matrix. The update law for $\hat{\boldsymbol{\lambda}}$ is obtained by augmenting the last equation of (3.26) with a term that serves the purpose of canceling the cross term that depends on $\tilde{\boldsymbol{\vartheta}}_w$ in the derivative of the Lyapunov function candidate (4.11). The expression of the redefined update law for $\hat{\boldsymbol{\lambda}}$ is given by

$$\dot{\hat{\boldsymbol{\lambda}}} = \underset{\hat{\boldsymbol{\lambda}} \in \Lambda}{\operatorname{Proj}} \left\{ \tfrac{\bar{q}}{m} \boldsymbol{\Gamma}_\lambda \mathbf{diag}(\boldsymbol{\delta})^T \left[\boldsymbol{B}_1^T \boldsymbol{e}_3 \tilde{w}_{\text{obsv}} + m \boldsymbol{B}_2^T \tilde{\boldsymbol{\omega}}_{\text{obsv}} \right] \right\}, \quad \hat{\boldsymbol{\lambda}}(0) = \mathbf{1} \tag{4.13}$$

where $\boldsymbol{\Gamma}_\lambda = \boldsymbol{\Gamma}_\lambda^T > \mathbf{0}$ is the adaptation gain matrix, and $\mathbf{1} = [1\ 1\ \cdots\ 1]^T$.

In the sequel, we concentrate on the analysis of the observer of the vertical dynamics only. With a substantial abuse, we will omit the coupling of the update law (4.13) with the angular velocity observer dynamics, with the underlying notion that the actual stability analysis should be carried out jointly with the observer error dynamics (3.27). As a result, we henceforth consider the update law

$$\dot{\hat{\boldsymbol{\lambda}}} = \underset{\hat{\boldsymbol{\lambda}} \in \Lambda}{\operatorname{Proj}} \left\{ \tfrac{\bar{q}}{m} \boldsymbol{\Gamma}_\lambda \mathbf{diag}(\boldsymbol{\delta})^T \boldsymbol{B}_1^T \boldsymbol{e}_3 \tilde{w}_{\text{obsv}} \right\}, \quad \hat{\boldsymbol{\lambda}}(0) = \mathbf{1} \tag{4.14}$$

in place of (4.13) only for the purpose of analyzing the stability properties of the vertical velocity observer. The overall stability result for the vertical velocity and angular velocity observers under the full update law (4.13) is just the combination of Proposition 3.3.2 and Proposition 4.2.1 below.

For the dynamics of the adaptive observer (4.10)–(4.12)–(4.14) in error coordinates, namely

$$\begin{aligned}
\dot{\tilde{w}}_{\text{obsv}} &= -k_w \tilde{w}_{\text{obsv}} + \boldsymbol{\phi}_w(\alpha)^T \tilde{\boldsymbol{\vartheta}}_w + \bar{q}\, \boldsymbol{e}_3^T \boldsymbol{B}_1\, \mathbf{diag}(\boldsymbol{\delta})\tilde{\boldsymbol{\lambda}} \\
\dot{\tilde{\boldsymbol{\vartheta}}}_w &= -\underset{\hat{\boldsymbol{\vartheta}}_w \in \mathcal{P}}{\operatorname{Proj}} \left\{ \boldsymbol{\Gamma}_w \boldsymbol{\phi}_w(\alpha)\tilde{w}_{\text{obs}} \right\} \\
\dot{\tilde{\boldsymbol{\lambda}}} &= -\underset{\hat{\boldsymbol{\lambda}} \in \Lambda}{\operatorname{Proj}} \left\{ \tfrac{\bar{q}}{m} \boldsymbol{\Gamma}_\lambda \mathbf{diag}(\boldsymbol{\delta})^T \boldsymbol{B}_1^T \boldsymbol{e}_3 \tilde{w}_{\text{obsv}} \right\}
\end{aligned} \tag{4.15}$$

the following result holds:

Proposition 4.2.1 *Assume that there exist a positive constant $\beta_{\max} < \pi/2$ such that, for all $t \geq 0$,*

$$|\beta(t)| \leq \beta_{\max}$$

and that the forward trajectory $V_T(t)$, $t \geq 0$, is bounded. Then, all forward trajectories of system (4.15) *originating from initial conditions as in* (4.9), (4.12) *and* (4.14) *are bounded and satisfy*

$$\|\tilde{w}_{\text{obs}}\|_a = 0$$

Proof. Since

$$\alpha = \arctan\left(\frac{w}{u}\right) = \arctan\left(\frac{w}{V_T\cos\alpha\cos\beta}\right)$$

one obtains

$$\sin\alpha = \frac{w}{V_T\cos\beta} \implies \alpha = \arcsin\left(\frac{w}{V_T\cos\beta}\right)$$

hence α is bounded if the assumptions hold. The proof then follows directly from application of La Salle/Yoshizawa Theorem using the Lyapunov function candidate (4.11). $\square$

Next, we devote our attention to the problem of controlling the observer (4.9)

$$\dot{\hat{w}} = qu - pv + g\cos\phi\cos\theta + \hat{\boldsymbol{\vartheta}}_w^T\boldsymbol{\phi}_w(\alpha) + \tfrac{1}{m}\hat{Z}_{A,\delta} + k_w\tilde{w}_{\text{obsv}}$$

with the purpose of regulating the tracking error $\tilde{w} = w - w_{\text{ref}}$. Define the estimated tracking error $\hat{e}_w := \hat{w} - w_{\text{ref}}$ and note that $\hat{e}_w = \tilde{w} + \tilde{w}_{\text{obsv}}$. Due to the limited control authority provided by the (estimated) aerodynamic force $\hat{Z}_{A,\delta}$, we avoid a control strategy that aims at canceling the known dynamics of the observer, but instead rely on the natural dissipation provided by $Z_{A,\text{base}}$. In. this sense, the control input will only be used to decouple the observed vertical velocity dynamics from the lateral velocity dynamics, to compensate for the gravity force and for imposing the required equilibrium at trim.

Specifically, note that

$$\begin{aligned} qu &= q_{\text{ref}}V_T\sin\alpha\cos\beta + \tilde{q}\,V_T\sin\alpha\cos\beta \\ &= q_{\text{ref}}V_{T,\text{ref}}\sin\alpha\cos\beta + q_{\text{ref}}\tilde{V}_T\sin\alpha\cos\beta + \tilde{q}\,V_T\sin\alpha\cos\beta \end{aligned}$$

and define the control

$$\begin{aligned} \hat{Z}_{A,\delta} = &-m\,\varepsilon_w\hat{e} + m\,pv - m\,q_{\text{ref}}V_{T,\text{ref}}\sin\alpha\cos\beta + m\bar{q}S\hat{C}_z^{\alpha}\alpha_{\text{ref}} - mg\cos\phi\cos\theta + m\,\bar{q}S\hat{C}_z^{\alpha^2}\alpha^2 \\ &+ m\,\bar{q}S\hat{C}_z^{0} - m\dot{w}_{\text{ref}} \end{aligned}$$

where $\varepsilon_w > 0$ is a *small* gain parameter[2], to obtain the observed tracking dynamics

$$\dot{\hat{e}}_w = -\varepsilon_w\hat{e} - \bar{q}S\hat{C}_z^{\alpha}\,(\alpha - \alpha_{\text{ref}}) + q_{\text{ref}}\tilde{V}_T\sin\alpha\cos\beta + \tilde{q}\,V_T\sin\alpha\cos\beta + k_w\tilde{w}_{\text{obsv}} \tag{4.16}$$

Recalling that

$$\alpha = \arctan\left(\frac{w}{u}\right), \qquad \alpha_{\text{ref}} = \arctan\left(\frac{w_{\text{ref}}}{u_{\text{ref}}}\right)$$

[2] It is emphasized that the small dissipation term $-\varepsilon_w\hat{e}$ provided by the controller may not be needed at all, as the term $-\bar{q}S\hat{C}_z^{\alpha}(\alpha - \alpha_{\text{ref}})$ usually provides ample stability margin for the estimated tracking error dynamics (4.16).

one obtains

$$
\begin{aligned}
\alpha - \alpha_{\mathrm{ref}} &= \arctan\left(\frac{w}{u}\right) - \arctan\left(\frac{w_{\mathrm{ref}}}{u_{\mathrm{ref}}}\right) \\
&= \arctan\left(\frac{w}{u_{\mathrm{ref}}}\right) - \arctan\left(\frac{w_{\mathrm{ref}}}{u_{\mathrm{ref}}}\right) + \underbrace{\arctan\left(\frac{w}{u}\right) - \arctan\left(\frac{w}{u_{\mathrm{ref}}}\right)}_{\varrho_1(\boldsymbol{\nu})} \\
&= \arctan\left(\frac{\hat{e}}{u_{\mathrm{ref}}}\right) + \underbrace{\arctan\left(\frac{w}{u_{\mathrm{ref}}}\right) - \arctan\left(\frac{\hat{e}}{u_{\mathrm{ref}}}\right) - \arctan\left(\frac{w_{\mathrm{ref}}}{u_{\mathrm{ref}}}\right)}_{\varrho_2(\boldsymbol{\nu})} + \varrho_1(\boldsymbol{\nu}) \\
&= \arctan\left(\frac{\hat{e}}{u_{\mathrm{ref}}}\right) + \varrho_1(\boldsymbol{\nu}) + \varrho_2(\boldsymbol{\nu})
\end{aligned}
$$

The term $\varrho_1(\boldsymbol{\nu})$ is expanded as follows:

$$
\begin{aligned}
\varrho_1(\boldsymbol{\nu}) &= \arctan\left(\frac{w}{u}\right) - \arctan\left(\frac{w}{u_{\mathrm{ref}}}\right) = \arctan\left(\frac{w}{\tilde{u} + u_{\mathrm{ref}}}\right) - \arctan\left(\frac{w}{u_{\mathrm{ref}}}\right) \\
&= \arctan\left(\frac{w}{\tilde{V}_T \cos\alpha\cos\beta + u_{\mathrm{ref}}}\right) - \arctan\left(\frac{w}{u_{\mathrm{ref}}}\right)
\end{aligned}
$$

as a result, $\varrho_1(\boldsymbol{\nu}) = \varrho_1(\tilde{V}_T)$ is a bounded perturbation vanishing at $\tilde{V}_T = 0$. Similarly, using the fact that

$$
w = w + w_{\mathrm{ref}} - w_{\mathrm{ref}} + \hat{w} - \hat{w} = \hat{e}_w + \tilde{w}_{\mathrm{obsv}} + w_{\mathrm{ref}}
$$

the term $\varrho_2(\boldsymbol{\nu})$ is expanded as follows:

$$
\varrho_2(\boldsymbol{\nu}) = \arctan\left(\frac{\tilde{w}_{\mathrm{obsv}} + \hat{e}_w + w_{\mathrm{ref}}}{u_{\mathrm{ref}}}\right) - \arctan\left(\frac{\hat{e}}{u_{\mathrm{ref}}}\right) - \arctan\left(\frac{w_{\mathrm{ref}}}{u_{\mathrm{ref}}}\right)
$$

which shows that $\varrho_2(\boldsymbol{\nu}) = \varrho_2(\tilde{w}_{\mathrm{obsv}}$ is a bounded perturbation vanishing at $\tilde{w}_{\mathrm{obsv}} = 0$. Finally, defining

$$
\begin{aligned}
d_{w,1}(\tilde{\boldsymbol{\omega}}, \tilde{V}_T) &:= -\bar{q}S\hat{C}_z^{\alpha}\varrho_1(\tilde{V}_T) + q_{\mathrm{ref}}\tilde{V}_T \sin\alpha\cos\beta + \tilde{q}\, V_T \sin\alpha\cos\beta \\
d_{w,2}(\tilde{w}_{\mathrm{obsv}}) &= -\bar{q}S\hat{C}_z^{\alpha}\varrho_2(\tilde{w}_{\mathrm{obsv}}) + k_w\tilde{w}_{\mathrm{obsv}}
\end{aligned}
$$

to obtain from (4.16)

$$
\dot{\hat{e}}_w = -\varepsilon_w\hat{e}_w - \kappa_w(\hat{e}) + d_{w,1}(\tilde{\boldsymbol{\omega}}, \tilde{V}_T) + d_{w,2}(\tilde{w}_{\mathrm{obsv}}) \tag{4.17}
$$

where

$$
\kappa_w(\hat{e}) := \bar{q}S\hat{C}_z^{\alpha} \arctan\left(\frac{\hat{e}}{u_{\mathrm{ref}}}\right)
$$

As the estimate $\hat{C}_z^{\alpha}$ is constrained by parameter projection to range over the same compact interval where C_z^{α} belongs, there exist constants $\underline{c}_z > 0$ and $\bar{c}_z > 0$ such that

$$
0 < \underline{c}_z \leq \hat{C}_z^{\alpha}(t) \leq \bar{c}_z
$$

As a result, since u_{ref} ranges over a compact set as well, there is a class-$\mathcal{K}$ function $\bar{\kappa}_w(\cdot)$ such that

$$
\hat{e}\kappa_w(\hat{e}) \geq |\hat{e}_w|\, \bar{\kappa}_w(|\hat{e}_w|)
$$

As a result, it is readily seen that system (4.17) is ISS with respect to the inputs $d_{w,1}(\tilde{\boldsymbol{\omega}}, \tilde{V}_T)$ and $d_{w,2}(\tilde{w}_{\mathrm{obsv}})$. More specifically, the following result holds:

Proposition 4.2.2 *The observed vertical velocity system* (4.16) *is input-to-state stable with respect to the disturbance inputs* $d_{w,1}(\tilde{\boldsymbol{\omega}}, \tilde{V}_T)$ *and* $d_{w,2}(\tilde{w}_{\text{obsv}})$. *In particular, the state* $\hat{e}_w(\cdot)$ *of* (4.16) *satisfies the asymptotic bound*

$$\|\hat{e}\|_a \leq 2 \max \left\{ \varphi_w^{-1}(2\|d_{w,1}\|_a), \varphi_w^{-1}(2\|d_{w,2}\|_a) \right\} \tag{4.18}$$

where $\varphi_w(\cdot)$ *is the class-*$\mathcal{K}_\infty$ *function defined as*

$$\varphi_w(s) := \frac{\varepsilon_w}{2} s + \bar{\kappa}_w(s)\,, \qquad s \in [0, \infty)$$

Chapter 5

Control Reconfiguration

In the previous sections, we have defined the expressions for the commanded control aerodynamic forces and moments, $\hat{\boldsymbol{F}}_{A,\text{cmd}}$ and $\hat{\boldsymbol{M}}_{A,\text{cmd}}$. The goal of this chapter is to present a method to solve for $\boldsymbol{\delta}_{\text{cmd}}$ the overdetermined equations

$$\hat{\boldsymbol{F}}_{A,\text{cmd}} = \bar{q}\boldsymbol{B}_1\,\mathbf{diag}(\hat{\boldsymbol{\lambda}})\boldsymbol{\delta}_{\text{cmd}}, \quad \hat{\boldsymbol{M}}_{A,\text{cmd}} = \bar{q}\boldsymbol{B}_2\,\mathbf{diag}(\hat{\boldsymbol{\lambda}})\boldsymbol{\delta}_{\text{cmd}}$$

while suitably exploiting the non-uniqueness of solutions. Control reconfiguration capabilities are provided by adaptation on the entries of the estimated actuator effectiveness $\hat{\boldsymbol{\lambda}}$ (developed in Chapter 3 and Chapter 4), and by dynamic control allocation (DCA), which is the subject of this chapter. The material in this chapter is adapted from the recent works [17–19].

5.1 Preliminary Transformations

Recall that the expression of the control effectiveness matrix

$$\boldsymbol{B} = \begin{pmatrix} \boldsymbol{B}_1 \\ \boldsymbol{B}_2 \end{pmatrix}$$

where

$$\boldsymbol{B}_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ -C_y^{\text{c}} & C_y^{\text{c}} & -C_y^{\text{f}} & C_y^{\text{f}} & C_y^{\text{t}} & -C_y^{\text{t}} \\ -C_z^{\text{c}} & -C_z^{\text{c}} & -C_z^{\text{f}} & -C_z^{\text{f}} & C_z^{\text{t}} & C_z^{\text{t}} \end{pmatrix}$$

$$\boldsymbol{B}_2 = \begin{pmatrix} -C_l^{\text{c}} & C_l^{\text{c}} & -C_l^{\text{f}} & C_l^{\text{f}} & C_l^{\text{t}} & -C_l^{\text{t}} \\ -C_m^{\text{c}} & -C_m^{\text{c}} & -C_m^{\text{f}} & -C_m^{\text{f}} & C_m^{\text{t}} & C_m^{\text{t}} \\ C_n^{\text{c}} & -C_n^{\text{c}} & C_n^{\text{f}} & -C_n^{\text{f}} & -C_n^{\text{t}} & C_n^{\text{t}} \end{pmatrix}$$

and that

$$\operatorname{rank} \boldsymbol{B} = 4$$

As a result, the system is input redundant, with $\dim\ker \boldsymbol{B} = 2$. To simplify the expression of the control effectiveness matrix, consider the isomorphism $\boldsymbol{T} : \mathbb{R}^6 \to \mathbb{R}^6$ defined by the non-singular matrix

$$
\boldsymbol{T} = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{pmatrix}
$$

and define the change of coordinates in the input space[1]

$$
\boldsymbol{\delta}_{\text{eff}} = \boldsymbol{T}\,\mathbf{diag}(\hat{\boldsymbol{\lambda}})\boldsymbol{\delta}_{\text{cmd}}, \quad \boldsymbol{\delta}_{\text{cmd}} = \mathbf{diag}(\hat{\boldsymbol{\lambda}})^{-1}\boldsymbol{T}^{-1}\boldsymbol{\delta}_{\text{eff}} \tag{5.1}
$$

The new coordinates $\boldsymbol{\delta}_{\text{eff}}$ represent the symmetric and anti-symmetric deflections of pairs of each pair of left and right actuators, weighted by their effectiveness. In the new coordinates, the expression of the control effectiveness matrices is greatly simplified, as

$$
\boldsymbol{B}_1\boldsymbol{T}^{-1} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -C_y^{\text{c}} & -C_y^{\text{f}} & C_y^{\text{t}} \\ -C_z^{\text{c}} & -C_z^{\text{f}} & C_z^{\text{t}} & 0 & 0 & 0 \end{pmatrix}
$$

$$
\boldsymbol{B}_2\boldsymbol{T}^{-1} = \begin{pmatrix} 0 & 0 & 0 & -C_l^{\text{c}} & -C_l^{\text{f}} & C_l^{\text{t}} \\ -C_m^{\text{c}} & -C_m^{\text{f}} & C_m^{\text{t}} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_n^{\text{c}} & C_n^{\text{f}} & -C_n^{\text{t}} \end{pmatrix}
$$

5.2 Baseline solution

We start be resolving the allocation of the commanded control moments. Assign

$$
\boldsymbol{\delta}_{\text{eff}} = \frac{1}{\bar{q}}\left(\boldsymbol{B}_2\boldsymbol{T}^{-1}\right)^{\dagger}\hat{\boldsymbol{M}}_{A,\text{cmd}} + \left(\boldsymbol{B}_2\boldsymbol{T}^{-1}\right)^{\perp}\boldsymbol{\tau}_1 \tag{5.2}
$$

where $\boldsymbol{\tau}_1 \in \mathbb{R}^3$ is an additional control,

$$
\left(\boldsymbol{B}_2\boldsymbol{T}^{-1}\right)^{\dagger} = \left(\boldsymbol{B}_2\boldsymbol{T}^{-1}\right)^{T}\left[\left(\boldsymbol{B}_2\boldsymbol{T}^{-1}\right)\left(\boldsymbol{B}_2\boldsymbol{T}^{-1}\right)^{T}\right]^{-1}
$$

is the Moore-Penrose pseudo-inverse of $\boldsymbol{B}_2\boldsymbol{T}^{-1}$ (recall that $\operatorname{rank}\boldsymbol{B}_2 = 3$), and $\left(\boldsymbol{B}_2\boldsymbol{T}^{-1}\right)^{\perp} \in \mathbb{R}^{6\times 3}$ is such that

$$
\operatorname{im}\left(\boldsymbol{B}_2\boldsymbol{T}^{-1}\right)^{\perp} = \ker\left(\boldsymbol{B}_2\boldsymbol{T}^{-1}\right)
$$

Note that $\boldsymbol{\tau}_1$ represents coordinates in the kernel of $\boldsymbol{B}_2$, hence the subspace of the effective deflections $\boldsymbol{\delta}_{\text{eff}}$ that do not produce control moments. Reverting back to the commanded deflections $\boldsymbol{\delta}_{\text{cmd}}$ via the second identity in (5.1), one obtains

$$
\bar{q}\boldsymbol{B}_2\,\mathbf{diag}(\hat{\boldsymbol{\lambda}})\boldsymbol{\delta}_{\text{cmd}} = \hat{\boldsymbol{M}}_{A,\text{cmd}}
$$

[1] Recall that $\mathbf{diag}(\hat{\boldsymbol{\lambda}}) \in \mathbb{R}^{6\times 6}$ is nonsingular for all $\hat{\boldsymbol{\lambda}} \in \Lambda$, as $\mathbf{diag}(\hat{\boldsymbol{\lambda}}) \geq \lambda_0^6 > 0$.

as desired. Next, we evaluate the effect of the selection (5.2) on the translational dynamics. Clearly,

$$\begin{aligned}\bar{q}\boldsymbol{B}_1\boldsymbol{\delta}_{\text{cmd}} &= \boldsymbol{B}_1\left(\boldsymbol{B}_2\boldsymbol{T}^{-1}\right)^{\dagger}\hat{\boldsymbol{M}}_{A,\text{cmd}} + \bar{q}\boldsymbol{B}_1\left(\boldsymbol{B}_2\boldsymbol{T}^{-1}\right)^{\perp}\boldsymbol{\tau}_1\\ &:= \bar{\boldsymbol{B}}_{11}\hat{\boldsymbol{M}}_{A,\text{cmd}} + \bar{q}\bar{\boldsymbol{B}}_{12}\boldsymbol{\tau}_1\end{aligned}$$

It can be shown via direct computation that $\bar{\boldsymbol{B}}_{11} \in \mathbb{R}^{3\times 3}$ and $\bar{\boldsymbol{B}}_{12} \in \mathbb{R}^{3\times 3}$ have the following structure

$$\bar{\boldsymbol{B}}_{11} = \begin{pmatrix} 0 & 0 & 0\\ * & 0 & *\\ 0 & * & 0\end{pmatrix}, \quad \bar{\boldsymbol{B}}_{12} = \begin{pmatrix} 0 & 0 & 0\\ 0 & 0 & 0\\ * & * & *\end{pmatrix}$$

where '$*$' represents a generic non-zero entry. It should be noted that the $(2,1)$-entry and $(2,3)$-entry of the matrix $\bar{\boldsymbol{B}}_{11}$ are much smaller in magnitude than the other entries, which is consistent with the assumption taken earlier to neglect the small-body force $Y_{A,\delta}$. Consequently, we set those entries to zero, and obtain

$$\bar{\boldsymbol{B}}_{11} = \begin{pmatrix} 0 & 0 & 0\\ 0 & 0 & 0\\ 0 & \dfrac{C_m^{\text{c}}C_z^{\text{c}} + C_m^{\text{f}}C_z^{\text{f}} + C_m^{\text{t}}C_z^{\text{t}}}{{C_m^{\text{c}}}^2 + {C_m^{\text{f}}}^2 + {C_m^{\text{t}}}^2} & 0\end{pmatrix}$$

$$\bar{\boldsymbol{B}}_{12} = \begin{pmatrix} 0 & 0 & 0\\ 0 & 0 & 0\\ \dfrac{C_m^{\text{f}}C_z^{\text{c}} - C_m^{\text{c}}C_z^{\text{f}}}{C_m^{\text{c}}} & \dfrac{C_m^{\text{c}}C_z^{\text{t}} - C_m^{\text{t}}C_z^{\text{c}}}{C_m^{\text{c}}} & \dfrac{C_m^{\text{f}}C_z^{\text{c}} - C_m^{\text{c}}C_z^{\text{f}}}{C_m^{\text{c}}}\end{pmatrix}$$

with $\operatorname{rank}\bar{\boldsymbol{B}}_{11} = \operatorname{rank}\bar{\boldsymbol{B}}_{12} = 1$. Recall also that

$$\hat{\boldsymbol{F}}_{A,\text{cmd}} = \begin{pmatrix} 0\\ 0\\ \hat{Z}_{A,\text{cmd}}\end{pmatrix}$$

hence $\hat{\boldsymbol{F}}_{A,\text{cmd}} \in \operatorname{im}\bar{\boldsymbol{B}}_{12}$. Furthermore,

$$\operatorname{im}\bar{\boldsymbol{B}}_{11} \subseteq \operatorname{im}\bar{\boldsymbol{B}}_{12}$$

hence the equation

$$\bar{\boldsymbol{B}}_{11}\hat{\boldsymbol{M}}_{A,\text{cmd}} + \bar{q}\bar{\boldsymbol{B}}_{12}\boldsymbol{\tau}_1 = \hat{\boldsymbol{F}}_{A,\text{cmd}}$$

can be solved (non uniquely) for $\boldsymbol{\tau}_1$. Let $\boldsymbol{\tau}_1^{\star}(\hat{\boldsymbol{F}}_{A,\text{cmd}})$ denote one such solution, for example

$$\boldsymbol{\tau}_1^{\star}(\hat{\boldsymbol{F}}_{A,\text{cmd}}) = \begin{pmatrix} 0\\ -\dfrac{C_m^{\text{c}}}{C_m^{\text{c}}C_z^{\text{t}} - C_m^{\text{t}}C_z^{\text{c}}}\dfrac{C_m^{\text{c}}C_z^{\text{c}} + C_m^{\text{f}}C_z^{\text{f}} + C_m^{\text{t}}C_z^{\text{t}}}{{C_m^{\text{c}}}^2 + {C_m^{\text{f}}}^2 + {C_m^{\text{t}}}^2}\hat{M}_{A,\text{cmd}} + \dfrac{C_m^{\text{c}}}{C_m^{\text{c}}C_z^{\text{t}} - C_m^{\text{t}}C_z^{\text{c}}}\hat{Z}_{A,\text{cmd}}\\ 0\end{pmatrix}$$

and define

$$\boldsymbol{\tau}_1 = \tfrac{1}{\bar{q}}\boldsymbol{\tau}_1^{\star} + \tfrac{1}{\bar{q}}\bar{\boldsymbol{B}}_{12}^{\perp}\boldsymbol{\tau}_2$$

where

$$\boldsymbol{B}_{12}^{\perp} = \begin{pmatrix} -1 & \dfrac{C_m^{\mathrm{c}} C_z^{\mathrm{t}} - C_m^{\mathrm{t}} C_z^{\mathrm{c}}}{C_m^{\mathrm{c}} C_z^{\mathrm{f}} - C_m^{\mathrm{f}} C_z^{\mathrm{c}}} \\ 0 & 1 \\ 1 & 0 \end{pmatrix}$$

is such that

$$\operatorname{im} \bar{\boldsymbol{B}}_{12}^{\perp} = \ker \bar{\boldsymbol{B}}_{12}$$

and $\boldsymbol{\tau}_2 \in \mathbb{R}^2$ represents coordinates on the kernel of $\bar{\boldsymbol{B}}_{12}$. Reverting back to the commanded deflections $\boldsymbol{\delta}_{\mathrm{cmd}}$ via the second identity in (5.1), one obtains

$$\bar{q}\boldsymbol{B}_1 \,\mathbf{diag}(\hat{\boldsymbol{\lambda}})\boldsymbol{\delta}_{\mathrm{cmd}} = \hat{\boldsymbol{F}}_{A,\mathrm{cmd}}$$

as desired. As a result, the assignment

$$\begin{aligned} \boldsymbol{\delta}_{\mathrm{cmd}} &= \mathbf{diag}(\hat{\boldsymbol{\lambda}})^{-1}\boldsymbol{T}^{-1}\boldsymbol{\delta}_{\mathrm{eff}} \\ \boldsymbol{\delta}_{\mathrm{eff}} &= \tfrac{1}{\bar{q}}\left(\boldsymbol{B}_2\boldsymbol{T}^{-1}\right)^{\dagger}\hat{\boldsymbol{M}}_{A,\mathrm{cmd}} + \tfrac{1}{\bar{q}}\left(\boldsymbol{B}_2\boldsymbol{T}^{-1}\right)^{\perp}\boldsymbol{\tau}_1^{\star}(\hat{\boldsymbol{F}}_{A,\mathrm{cmd}}) + \tfrac{1}{\bar{q}}\left(\boldsymbol{B}_2\boldsymbol{T}^{-1}\right)^{\perp}\bar{\boldsymbol{B}}_{12}^{\perp}\boldsymbol{\tau}_2 \end{aligned} \tag{5.3}$$

results in the generation of the desired forces and moments

$$\bar{q}\boldsymbol{B}\,\mathbf{diag}(\hat{\boldsymbol{\lambda}})\boldsymbol{\delta}_{\mathrm{cmd}} = \begin{pmatrix} \hat{\boldsymbol{F}}_{A,\mathrm{cmd}} \\ \hat{\boldsymbol{M}}_{A,\mathrm{cmd}} \end{pmatrix}$$

while providing *strong input redundancy* [2,3] in the form of the free vector $\boldsymbol{\tau}_2 \in \mathbb{R}^2$ expressing coordinates in $\ker \boldsymbol{B}$. The assignment (5.3) with $\boldsymbol{\tau}_2 = 0$ is referred to as the *baseline allocation.*

5.3 Dynamic Control Allocation

5.3.1 Strong Input Redundancy

The first DCA action presented in this section exploits the so-called *strong input redundancy* [2, 3] of the system due to the rank deficiency of $\boldsymbol{B}$. In this case, the DCA action aims at distributing the control effort among the available control surfaces in such a way that the actual aerodynamic surface deflections, $\boldsymbol{\delta}$ are closest to the commanded ones, $\boldsymbol{\delta}_{\mathrm{cmd}}$ in the least-square sense. To this end, write (5.3) in the concise form

$$\boldsymbol{\delta}_{\mathrm{cmd}} = \boldsymbol{L}_1\hat{\boldsymbol{M}}_{A,\mathrm{cmd}} + \boldsymbol{L}_2\boldsymbol{\tau}_1^{\star}(\hat{\boldsymbol{F}}_{A,\mathrm{cmd}}) + \boldsymbol{L}_3\boldsymbol{\tau}_2$$

with obvious definition of the matrices $\boldsymbol{L}_1 \in \mathbb{R}^{6\times 3}$, $\boldsymbol{L}_2 \in \mathbb{R}^{6\times 3}$ and $\boldsymbol{L}_3 \in \mathbb{R}^{6\times 2}$. Recall that $\boldsymbol{\tau}_2 \neq 0$ does not generate forces or moments, as

$$\boldsymbol{B}\boldsymbol{L}_3 = \mathbf{0}$$

However, the actual value of $\boldsymbol{\delta}_{\mathrm{cmd}}$ is modified from the baseline control

$$\boldsymbol{\delta}_{\mathrm{cmd}} = \boldsymbol{L}_1\hat{\boldsymbol{M}}_{A,\mathrm{cmd}} + \boldsymbol{L}_2\boldsymbol{\tau}_1^{\star}(\hat{\boldsymbol{F}}_{A,\mathrm{cmd}})$$

that assigns the required commanded aerodynamic forces and moments. Consequently, one may exploit this non-uniqueness in the control allocation (referred to as a strong input redundancy) to solve an additional optimization problem. Specifically, consider the cost function

$$\mathcal{J}_{\text{strong}} := \frac{1}{2}\left(\boldsymbol{\delta} - \boldsymbol{\delta}_{\text{cmd}}\right)^T \boldsymbol{Q}_1 \left(\boldsymbol{\delta} - \boldsymbol{\delta}_{\text{cmd}}\right) + \frac{1}{2}\varepsilon_1 \boldsymbol{\tau}_2^T \boldsymbol{\tau}_2$$

where $\boldsymbol{Q}_1 \in \mathbb{R}^{6\times 6}$, $\boldsymbol{Q}_1 = \boldsymbol{Q}_1^T > \mathbf{0}$ and $\varepsilon_1 > 0$ are weights. The allocation policy for the strong redundancy input $\boldsymbol{\tau}_2$ is simply obtain by the dynamic minimizer obtained via the gradient flow

$$\dot{\boldsymbol{\tau}}_2 = -\varrho_1 \nabla_{\boldsymbol{\tau}_2} \mathcal{J}_{\text{strong}}\,, \quad \boldsymbol{\tau}_2(0) = 0 \tag{5.4}$$

where $\varrho_1 > 0$ is the adaptation gain. The policy (5.4) yields the exponentially stable dynamics

$$\dot{\boldsymbol{\tau}}_2 = -\varrho_1 \left(\boldsymbol{L}_3^T \boldsymbol{Q}_1 \boldsymbol{L}_3 + \varepsilon_1 \boldsymbol{I}_6\right) \boldsymbol{\tau}_2 + \varrho_1 \boldsymbol{L}_3^T \boldsymbol{Q}_1 \left(\boldsymbol{\delta} - \boldsymbol{L}_1 \hat{\boldsymbol{M}}_{A,\text{cmd}} - \boldsymbol{L}_2 \boldsymbol{\tau}_1^{\star}(\hat{\boldsymbol{F}}_{A,\text{cmd}})\right)$$

Note that, since $\boldsymbol{B}\boldsymbol{L}_3 = \mathbf{0}$, the dynamics of the strong redundancy allocator (5.4) is completely decoupled from the rest of the dynamics, hence stability of the interconnection between the allocator and the closed-loop system is guaranteed as long as this latter remains stable.

5.3.2 Weak Input Redundancy

The second level of allocation regards the use of the assignable reference for the angle of attack, $\alpha_{\text{ref}}(t)$, to minimize the difference between the commanded forces and moments and the actual ones produced by the aerodynamic actuators via a suitable reconfiguration of the vehicle attitude that does not affect the tracking performance. This possibility is made possible by the fact that the system under investigation is also *weakly input redundant*, in the terminology of [2, 3].

Appendix A

Attitude Parameterization

The goal of this brief section is to fix the notation and introduce basic concepts regarding the attitude parameterization of a rigid body in a given coordinate frame.

A.1 Rotation Matrices

Let two coordinate systems be given, an inertial one, $\mathscr{F}_e = \{\vec{\boldsymbol{e}}_1, \vec{\boldsymbol{e}}_2, \vec{\boldsymbol{e}}_3\}$, and a body-fixed one, $\mathscr{F}_b = \{\vec{\boldsymbol{b}}_1, \vec{\boldsymbol{b}}_2, \vec{\boldsymbol{b}}_3\}$. All coordinate frames are assumed to have common origin of the axes defined by the vectors $\vec{\boldsymbol{e}}_i, \vec{\boldsymbol{b}}_i$, $i = 1, 2, 3$. To represent the relative orientation between $\mathscr{F}_e$ and $\mathscr{F}_b$, note that

$$\vec{\boldsymbol{b}}_i = \sum_{j=1}^{3} (\vec{\boldsymbol{b}}_i \circ \vec{\boldsymbol{e}}_j)\vec{\boldsymbol{e}}_j$$

from which one obtains the relation between the basis vectors

$$\begin{bmatrix} \vec{\boldsymbol{b}}_1 & \vec{\boldsymbol{b}}_2 & \vec{\boldsymbol{b}}_3 \end{bmatrix} = \begin{bmatrix} \vec{\boldsymbol{e}}_1 & \vec{\boldsymbol{e}}_2 & \vec{\boldsymbol{e}}_3 \end{bmatrix} \begin{pmatrix} \vec{\boldsymbol{b}}_1 \circ \vec{\boldsymbol{e}}_1 & \vec{\boldsymbol{b}}_2 \circ \vec{\boldsymbol{e}}_1 & \vec{\boldsymbol{b}}_3 \circ \vec{\boldsymbol{e}}_1 \\ \vec{\boldsymbol{b}}_1 \circ \vec{\boldsymbol{e}}_2 & \vec{\boldsymbol{b}}_2 \circ \vec{\boldsymbol{e}}_2 & \vec{\boldsymbol{b}}_3 \circ \vec{\boldsymbol{e}}_2 \\ \vec{\boldsymbol{b}}_1 \circ \vec{\boldsymbol{e}}_3 & \vec{\boldsymbol{b}}_2 \circ \vec{\boldsymbol{e}}_3 & \vec{\boldsymbol{b}}_3 \circ \vec{\boldsymbol{e}}_3 \end{pmatrix} =: \begin{bmatrix} \vec{\boldsymbol{e}}_1 & \vec{\boldsymbol{e}}_2 & \vec{\boldsymbol{e}}_3 \end{bmatrix} \boldsymbol{R}_{eb} \tag{A.1}$$

where $\boldsymbol{R}_{eb}$ is the *rotation matrix* describing the orientation of $\mathscr{F}_b$ in $\mathscr{F}_e$. Given a vector $\vec{\boldsymbol{\nu}}$, its expression in each of the two coordinate frames is given by

$$\vec{\boldsymbol{\nu}} = {}^e\nu_1\vec{\boldsymbol{e}}_1 + {}^e\nu_2\vec{\boldsymbol{e}}_2 + {}^e\nu_3\vec{\boldsymbol{e}}_3 = {}^b\nu_1\vec{\boldsymbol{b}}_1 + {}^b\nu_2\vec{\boldsymbol{b}}_2 + {}^b\nu_3\vec{\boldsymbol{b}}_3$$

yielding

$$\begin{bmatrix} \vec{\boldsymbol{e}}_1 & \vec{\boldsymbol{e}}_2 & \vec{\boldsymbol{e}}_3 \end{bmatrix} \begin{pmatrix} {}^e\nu_1 \\ {}^e\nu_2 \\ {}^e\nu_3 \end{pmatrix} = \begin{bmatrix} \vec{\boldsymbol{b}}_1 & \vec{\boldsymbol{b}}_2 & \vec{\boldsymbol{b}}_3 \end{bmatrix} \begin{pmatrix} {}^b\nu_1 \\ {}^b\nu_2 \\ {}^b\nu_3 \end{pmatrix} \tag{A.2}$$

Using (A.1) in the right-hand side of (A.2), one obtains

$$\vec{\boldsymbol{\nu}} = \begin{bmatrix} \vec{\boldsymbol{e}}_1 & \vec{\boldsymbol{e}}_2 & \vec{\boldsymbol{e}}_3 \end{bmatrix} \begin{pmatrix} {}^e\nu_1 \\ {}^e\nu_2 \\ {}^e\nu_3 \end{pmatrix} = \begin{bmatrix} \vec{\boldsymbol{e}}_1 & \vec{\boldsymbol{e}}_2 & \vec{\boldsymbol{e}}_3 \end{bmatrix} \boldsymbol{R}_{eb} \begin{pmatrix} {}^b\nu_1 \\ {}^b\nu_2 \\ {}^b\nu_3 \end{pmatrix}$$

hence

$$\begin{pmatrix} {}^e\nu_1 \\ {}^e\nu_2 \\ {}^e\nu_3 \end{pmatrix} = \boldsymbol{R}_{eb} \begin{pmatrix} {}^b\nu_1 \\ {}^b\nu_2 \\ {}^b\nu_3 \end{pmatrix}$$

More concisely, letting ${}^b\boldsymbol{\nu} \in \mathbb{R}^3$ and ${}^e\boldsymbol{\nu} \in \mathbb{R}^3$ be the coordinate vectors of $\vec{\boldsymbol{\nu}}$ in $\mathscr{F}_b$ and $\mathscr{F}_e$, respectively, one obtains

$${}^e\boldsymbol{\nu} = \boldsymbol{R}_{eb}\,{}^b\boldsymbol{\nu}$$

which expresses the fact that $\boldsymbol{R}_{eb} : \mathbb{R}^3 \to \mathbb{R}^3$ maps the coordinates of a vector expressed in the frame $\mathscr{F}_b$ into the coordinates of the same vector expressed in the frame $\mathscr{F}_e$. It is well known (but shall not be proven here) that $\boldsymbol{R}_{eb}$ belongs to the *special orthogonal group*

$$SO(3) := \left\{ \boldsymbol{R} \in \mathbb{R}^{3\times 3} : \boldsymbol{R}^T\boldsymbol{R} = \boldsymbol{I}_3 \text{ and } \det \boldsymbol{R} = 1 \right\}$$

It is known that $\dim SO(3) = 3$.

The rotational kinematic equations of a rigid body with body-fixed frame $\mathscr{F}_b$ and Earth-centered inertial frame $\mathscr{F}_e$ read as

$$\dot{\boldsymbol{R}}_{eb} = \boldsymbol{R}_{eb}\boldsymbol{S}({}^b\boldsymbol{\omega})$$

or as

$$\dot{\boldsymbol{R}}_{eb} = \boldsymbol{S}({}^e\boldsymbol{\omega})\boldsymbol{R}_{eb}$$

where $\boldsymbol{S}(\cdot)$ is the skew-symmetric operator defined by $\boldsymbol{S}(\boldsymbol{\omega})\boldsymbol{\nu} = \boldsymbol{\omega} \times \boldsymbol{\nu}$, and ${}^b\boldsymbol{\omega} \in \mathbb{R}^3$ and ${}^e\boldsymbol{\omega} \in \mathbb{R}^3$ are the coordinate vectors of the angular velocity of the body expressed in $\mathscr{F}_b$ and $\mathscr{F}_e$, respectively.

A.2 Euler Angle Parameterization

One of the most popular minimal parameterization (hence requiring three parameters) of the rotation matrix is obtained via a series of elementary rotations. This parameterization is often termed *Euler angles parameterization*: as a matter of fact, the Euler angles parameterization is but one out of twelve possible choices for the sequences of three elementary rotations. The method that we will use to bring the frame $\mathscr{F}_b$ in a desired orientation with respect to the frame $\mathscr{F}_e$ consists in aligning the two systems and perform a rotation of $\mathscr{F}_b$ around the axes $\vec{\boldsymbol{e}}_1$, $\vec{\boldsymbol{e}}_2$ and $\vec{\boldsymbol{e}}_3$ of $\mathscr{F}_e$ by angles equal to ϕ (roll), θ (pitch) and ψ (yaw). We shall obtain the corresponding final rotation matrix $\boldsymbol{R}_{eb}$ using the sequence of = intermediate rotations $\boldsymbol{R}_{x,\phi}$, $\boldsymbol{R}_{y,\theta}$, $\boldsymbol{R}_{z,\psi}$, yielding

$$\boldsymbol{R}_{eb} = \boldsymbol{R}^T_{z,\psi}\boldsymbol{R}^T_{y,\theta}\boldsymbol{R}^T_{x,\phi}$$

or, using *exponential notation*

$$\boldsymbol{R}_{eb} = \mathrm{e}^{\hat{\boldsymbol{z}}\psi}\mathrm{e}^{\hat{\boldsymbol{y}}\theta}\mathrm{e}^{\hat{\boldsymbol{x}}\phi}\,,$$

dove

$$\hat{\boldsymbol{x}} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \ \hat{\boldsymbol{y}} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \ \hat{\boldsymbol{z}} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}.$$

Letting $\boldsymbol{\eta} = \mathrm{col}(\phi, \theta, \psi)$ the vector of generalized coordinates, the relation between the time derivative of $\boldsymbol{\eta}(t)$ and the angular velocity ${}^b\boldsymbol{\omega}$ of $\mathscr{F}_b$ in $\mathscr{F}_e$, expressed in $\mathscr{F}_b$ is the following:

$$
{}^b\boldsymbol{\omega} = \begin{pmatrix} \dot{\phi} \\ 0 \\ 0 \end{pmatrix} + \boldsymbol{R}_{x,\phi} \begin{pmatrix} 0 \\ \dot{\theta} \\ 0 \end{pmatrix} + \boldsymbol{R}_{x,\phi}\boldsymbol{R}_{y,\theta} \begin{pmatrix} 0 \\ 0 \\ \dot{\psi} \end{pmatrix} .
$$

This relation yields the kinematic equation

$$
\dot{\boldsymbol{\eta}} = \boldsymbol{H}(\boldsymbol{\eta})\, {}^b\boldsymbol{\omega}
$$

where

$$
\boldsymbol{H}(\boldsymbol{\eta}) = \begin{pmatrix} 1 & \sin\phi\tan\theta & \cos\phi\tan\theta \\ 0 & \cos\phi & -\sin\phi \\ 0 & \frac{\sin\phi}{\cos\theta} & \frac{\cos\phi}{\cos\theta} \end{pmatrix}, \quad \boldsymbol{H}(\boldsymbol{\eta})^{-1} = \begin{pmatrix} 1 & 0 & -\sin\theta \\ 0 & \cos\phi & \cos\theta\sin\phi \\ 0 & -\sin\phi & \cos\theta\cos\phi \end{pmatrix}
$$

A.3 Euler Parameters and Unit Quaternions

An increasingly popular way of parameterizing rotations, which is globally nonsingular and is quite attractive from a computational viewpoint is offered by *Euler parameters*. The Euler parameters corresponding to a rotation of an angle θ about an axis $\vec{\boldsymbol{\lambda}}$ are given by

$$
\boldsymbol{\mathcal{Q}} = \left\{ \begin{array}{c} \boldsymbol{\varepsilon} \\ \eta \end{array} \right\} = \left\{ \begin{array}{c} \sin\frac{\theta}{2}\boldsymbol{\lambda} \\ \cos\frac{\theta}{2} \end{array} \right\},
$$

where $\boldsymbol{\lambda}$ is any representation of the vector $\vec{\boldsymbol{\lambda}}$ in coordinates. The Euler parameters form a *unit quaternion*, hence they can be manipulated using the *quaternion algebra*. We refer to $\boldsymbol{\varepsilon} \in \mathbb{R}^3$ as the *vector part* of the quaternion and to η as its *scalar part*. The unit quaternions satisfy the norm constraint

$$
\|\boldsymbol{\mathcal{Q}}\|^2 = \eta^2 + \boldsymbol{\varepsilon}^T\boldsymbol{\varepsilon} = 1\,.
$$

The Euler theorem states that the relative orientation between two coordinate frames can be expressed as a single rotation about a fixed axis. The associated rotation matrix R is given by the *exponential formula*

$$
\boldsymbol{R} = \exp\left(\theta\boldsymbol{S}(\boldsymbol{\lambda})\right)\,.
$$

It is not difficult to compute the expression of $\boldsymbol{R}$ in terms of its Euler parameters

$$
\boldsymbol{R}(\boldsymbol{\mathcal{Q}}) = \boldsymbol{I}_3 + 2\boldsymbol{S}^2(\boldsymbol{\varepsilon}) + 2\eta\boldsymbol{S}(\boldsymbol{\varepsilon})\,, \tag{A.3}
$$

$$
R = \begin{pmatrix} 1 - 2\,\epsilon_3^2 - 2\,\epsilon_2^2 & 2\,\epsilon_2\,\epsilon_1 - 2\,\eta\,\epsilon_3 & 2\,\epsilon_3\,\epsilon_1 + 2\,\eta\,\epsilon_2 \\ 2\,\epsilon_2\,\epsilon_1 + 2\,\eta\,\epsilon_3 & 1 - 2\,\epsilon_3^2 - 2\,\epsilon_1^2 & 2\,\epsilon_3\,\epsilon_2 - 2\,\eta\,\epsilon_1 \\ 2\,\epsilon_3\,\epsilon_1 - 2\,\eta\,\epsilon_2 & 2\,\varepsilon 3\,\epsilon_2 + 2\,\eta\,\epsilon_1 & 1 - 2\,\epsilon_2^2 - 2\,\epsilon_1^2 \end{pmatrix},
$$

from which it is readily seen that $\boldsymbol{R}(\boldsymbol{\mathcal{Q}}) = \boldsymbol{I}_3$ if and only if $\boldsymbol{\varepsilon} = 0$ and $\eta = \pm 1$. Note that $\{\boldsymbol{\varepsilon}, \eta\}$ and $\{\boldsymbol{\varepsilon}, -\eta\}$ represent the same rotation. The two fundamental operations of the algebra of quaternions are the quaternion addition $\boldsymbol{\mathcal{Q}} = \boldsymbol{\mathcal{Q}}_1 + \boldsymbol{\mathcal{Q}}_2$ and the quaternion multiplication $\boldsymbol{\mathcal{Q}} = \boldsymbol{\mathcal{Q}}_1 \star \boldsymbol{\mathcal{Q}}_2$. In the first case, $\boldsymbol{\mathcal{Q}}$ is simply defined as the quaternion whose vector part and scalar part are the sum of the vector parts and the scalar parts of $\boldsymbol{\mathcal{Q}}_1$ and $\boldsymbol{\mathcal{Q}}_2$ respectively, while the second operation is defined according to the following rule:

$$\boldsymbol{\mathcal{Q}}_1 \star \boldsymbol{\mathcal{Q}}_2 = \begin{bmatrix} \eta_1 \boldsymbol{I}_3 + \boldsymbol{S}(\boldsymbol{\varepsilon}_1) & \boldsymbol{\varepsilon}_1 \\ -\boldsymbol{\varepsilon}_1^T & \eta_1 \end{bmatrix} \begin{Bmatrix} \boldsymbol{\varepsilon}_2 \\ \eta_2 \end{Bmatrix} .$$

The quaternion product is used to derive the Euler parameters corresponding to composite rotations. Consider, for instance, a sequence of rotations that aligns the body-fixed frame $\mathscr{F}_b$ with an auxiliary frame $\mathscr{F}_c$ and then with an inertial frame $\mathscr{F}_i$. The resulting rotation matrix is given by

$$\boldsymbol{R}(\boldsymbol{\mathcal{Q}}_{\text{ib}}) = \boldsymbol{R}(\boldsymbol{\mathcal{Q}}_{\text{ic}})\boldsymbol{R}(\boldsymbol{\mathcal{Q}}_{\text{cb}}) ,$$

with the corresponding quaternions $\boldsymbol{\mathcal{Q}}_{\text{ib}}$, $\boldsymbol{\mathcal{Q}}_{\text{ic}}$, $\boldsymbol{\mathcal{Q}}_{\text{cb}}$ satisfying

$$\boldsymbol{\mathcal{Q}}_{\text{ib}} = \boldsymbol{\mathcal{Q}}_{\text{ic}} \star \boldsymbol{\mathcal{Q}}_{\text{cb}} .$$

Finally, we denote with $\boldsymbol{\mathcal{Q}}^\dagger$ the conjugate quaternion, defined as

$$\boldsymbol{\mathcal{Q}}^\dagger = \begin{Bmatrix} \boldsymbol{\varepsilon} \\ \eta \end{Bmatrix}^\dagger = \begin{Bmatrix} -\boldsymbol{\varepsilon} \\ \eta \end{Bmatrix} .$$

The conjugate quaternion satisfies

$$\boldsymbol{\mathcal{Q}} \star \boldsymbol{\mathcal{Q}}^\dagger = \boldsymbol{\mathcal{Q}}^\dagger \star \boldsymbol{\mathcal{Q}} = \begin{Bmatrix} 0 \\ 1 \end{Bmatrix} ,$$

and thus, if $\boldsymbol{\mathcal{Q}}$ parameterizes $\boldsymbol{R}$, the conjugate quaternion $\boldsymbol{\mathcal{Q}}^\dagger$ is the one associated with $\boldsymbol{R}^T$. The *quaternion propagation rule* relates the time derivative of the quaternion with the angular velocity vector of the rotation

$$\dot{\boldsymbol{\mathcal{Q}}} = \frac{1}{2}\boldsymbol{\mathcal{Q}} \star \begin{Bmatrix} \boldsymbol{\omega} \\ 0 \end{Bmatrix} ,$$

where the last term on the right-hand side is the quaternion whose vector part is the angular velocity vector and scalar part equal to zero. The inverse relationship is given by

$$\begin{Bmatrix} \boldsymbol{\omega} \\ 0 \end{Bmatrix} = 2\boldsymbol{\mathcal{Q}}^\dagger \star \dot{\boldsymbol{\mathcal{Q}}} .$$

A more convenient notation for the quaternion propagation rule is given as follows

$$\begin{aligned} \dot{\boldsymbol{\mathcal{Q}}} &= \tfrac{1}{2}\boldsymbol{E}(\boldsymbol{\mathcal{Q}})\omega \\ \dot{\boldsymbol{\mathcal{Q}}} &= \tfrac{1}{2}\boldsymbol{A}(\boldsymbol{\omega})\boldsymbol{\mathcal{Q}} \end{aligned}$$

where

$$\boldsymbol{E}(\boldsymbol{\mathcal{Q}}) = \begin{pmatrix} \eta \boldsymbol{I}_3 + \boldsymbol{S}(\boldsymbol{\varepsilon}) \\ -\boldsymbol{\varepsilon}^T \end{pmatrix} = \begin{pmatrix} \eta & -\epsilon_3 & \epsilon_2 \\ \epsilon_3 & \eta & -\epsilon_1 \\ -\epsilon_2 & \epsilon_1 & \eta \\ -\epsilon_1 & -\epsilon_2 & -\epsilon_3 \end{pmatrix} ,$$

$$
\boldsymbol{A}(\boldsymbol{\omega}) = \begin{pmatrix} -\boldsymbol{S}(\boldsymbol{\omega}) & \boldsymbol{\omega} \\ -\boldsymbol{\omega}^T & 0 \end{pmatrix} .
$$

Note that $\boldsymbol{E}^T(\boldsymbol{\mathcal{Q}})\boldsymbol{E}(\boldsymbol{\mathcal{Q}}) = \boldsymbol{I}_3$ and that $\|\boldsymbol{E}(\boldsymbol{\mathcal{Q}})| = 1$ for all $\boldsymbol{\mathcal{Q}}$. Therefore,

$$
\boldsymbol{\omega} = 2\boldsymbol{E}^T(\boldsymbol{\mathcal{Q}})\dot{\boldsymbol{\mathcal{Q}}} ,
$$

$$
\boldsymbol{E}^T(\boldsymbol{\mathcal{Q}}) = \begin{pmatrix} \eta & \epsilon_3 & -\epsilon_2 & -\epsilon_1 \\ -\epsilon_3 & \eta & \epsilon_1 & -\epsilon_2 \\ \epsilon_2 & -\epsilon_1 & \eta & -\epsilon_3 \end{pmatrix}
$$

A.4 Orientation Error

Consider the goal to let the body-fixed coordinate frame track a time-varying, desired orientation, represented by a reference coordinate frame $\mathscr{F}_d$. This latter is specified by means of a *reference rotation* $\boldsymbol{R}_d$ with respect to the inertial coordinate frame, with associated angular velocity vector $\vec{\boldsymbol{\omega}}_d(t)$. The reference rotation matrix satisfies the differential equation

$$
\dot{\boldsymbol{R}}_d = \mathbb{R}_d \boldsymbol{S}(\boldsymbol{\omega}_d^d) ,
$$

where $\boldsymbol{\omega}_d^d \in \mathbb{R}^3$ denotes the expression of $\vec{\boldsymbol{\omega}}_d$ in $\mathscr{F}_d$. The orientation error between the desired frame and the body-fixed frame is defined in terms of the orientation of $\mathscr{F}_b$ relative to $\mathscr{F}_d$

$$
\tilde{\boldsymbol{R}} = \boldsymbol{R}_d^T \boldsymbol{R} ,
$$

or the orientation of $\mathscr{F}_d$ relative to $\mathscr{F}_b$

$$
\bar{\boldsymbol{R}} = \boldsymbol{R}^T \boldsymbol{R}_d .
$$

According to the first choice ($\tilde{\boldsymbol{R}}$), the angular velocity error is the angular velocity of the body-fixed frame relative to the desired frame, expressed in body coordinates as

$$
\begin{aligned}
\tilde{\boldsymbol{\omega}}^b &= \boldsymbol{\omega}^b - \boldsymbol{\omega}_d^b \\
&= \boldsymbol{\omega}^b - \tilde{\boldsymbol{R}}^T \boldsymbol{\omega}_d^d ,
\end{aligned}
$$

and with respect to the desired frame as

$$
\begin{aligned}
\tilde{\boldsymbol{\omega}}^d &= \boldsymbol{\omega}^d - \boldsymbol{\omega}_d^d \\
&= \tilde{\boldsymbol{R}} \boldsymbol{\omega}^b - \boldsymbol{\omega}_d^d .
\end{aligned}
$$

In the second case ($\bar{\boldsymbol{R}}$), the angular velocity error is the angular velocity of the desired frame relative to the body-fixed frame, expressed in body coordinates simply as

$$
\bar{\boldsymbol{\omega}} = -\tilde{\boldsymbol{\omega}} .
$$

Let's adopt the $\tilde{\boldsymbol{R}}$ notation; consequently, letting $\boldsymbol{\mathcal{Q}}$ and $\boldsymbol{\mathcal{Q}}_d$ be the quaternion associated with $\boldsymbol{R}$ and $\boldsymbol{R}_d$ respectively, we obtain the quaternion associated with $\tilde{\boldsymbol{R}}$ as

$$
\tilde{\boldsymbol{\mathcal{Q}}} = \boldsymbol{\mathcal{Q}}_d^\dagger \star \boldsymbol{\mathcal{Q}} .
$$

A.5 Modified Rodrigues Parameters

Consider the rotation matrix $\boldsymbol{R} \in SO(3)$ mapping vectors from their representation in the body-fixed frame $\mathscr{F}_b$ into their representation in the inertial frame $\mathscr{F}_e$, i.e.,

$$\boldsymbol{R} = \boldsymbol{R}_{eb}\,, \quad \text{and} \quad \boldsymbol{\nu}^e = \boldsymbol{R}\boldsymbol{\nu}^b\,.$$

Let $\boldsymbol{\lambda}$ be the instantaneous axis of rotation of $\mathscr{F}_b$ with respect to $\mathscr{F}_e$, and let θ the angle of which $\mathscr{F}_b$ is rotated about $\boldsymbol{\lambda}$ (with a slight abuse of notation, we denote the axis of rotation by its coordinates, and not as a vector in the Euclidian space, as its representation is the same in both $\mathscr{F}_e$ and $\mathscr{F}_b$). The angle/axis parameterization of $\boldsymbol{R}$ gives

$$\boldsymbol{R} = \exp(\boldsymbol{S}(\boldsymbol{\lambda})\theta)$$

which yields, according to Rodrigues formula,

$$\boldsymbol{R} = \boldsymbol{I}_3 + \sin(\theta)\boldsymbol{S}(\boldsymbol{\lambda}) + (1-\cos(\theta))\boldsymbol{S}(\boldsymbol{\lambda})^2\,,$$

where $\boldsymbol{S}(\boldsymbol{\lambda})\boldsymbol{\nu} = \boldsymbol{\lambda} \times \boldsymbol{\nu}$. The Euler parameters, defined as

$$\eta = \cos\tfrac{\theta}{2}\,, \qquad \boldsymbol{\varepsilon} = \boldsymbol{\lambda}\sin\tfrac{\theta}{2}$$

form a unit quaternion

$$\eta^2 + \boldsymbol{\varepsilon}^T\boldsymbol{\varepsilon} = 1$$

and gives the following expression for the rotation matrix

$$\boldsymbol{R} = \boldsymbol{I}_3 + 2\eta\boldsymbol{S}(\boldsymbol{\varepsilon}) + 2\boldsymbol{S}(\boldsymbol{\varepsilon})^2\,.$$

Note that, since

$$\boldsymbol{S}(\boldsymbol{\nu})^2 = \boldsymbol{\nu}\boldsymbol{\nu}^T - \boldsymbol{\nu}^T\boldsymbol{\nu}\boldsymbol{I}_3\,,$$

the rotation matrix as a function of the Euler parameters is sometimes given as

$$\boldsymbol{R} = (\eta^2 - \boldsymbol{\varepsilon}^T\boldsymbol{\varepsilon})\boldsymbol{I}_3 + 2\boldsymbol{\varepsilon}\boldsymbol{\varepsilon}^T + 2\boldsymbol{S}(\boldsymbol{\varepsilon})\,.$$

The *modified Rodrigues parameters* of $\boldsymbol{R}$ are defined as

$$\boldsymbol{\sigma} = \frac{\boldsymbol{\varepsilon}}{1+\eta} = \tan\tfrac{\theta}{4}\boldsymbol{\lambda}\,.$$

The advantage of the MRP versus the standard Gibbs/Rodrigues parameters

$$\rho = \frac{\boldsymbol{\varepsilon}}{\eta} = \tan\tfrac{\theta}{2}\boldsymbol{\lambda}$$

lies in the fact that the latter become singular at $\eta = 0$, while the former have a singularity at $\eta = -1$. This implies that the MRP can describe rotations in the range $\theta = [0, 2\pi)$, while the standard Rodrigues parameters cannot be employed for rotations exceeding the range $\theta \in [0, \pi)$. In this respect, the modified Rodrigues parameters offer the best minimal parameterization of $SO(3)$.

Rotation matrix

To derive the rotation matrix as a function of $\boldsymbol{\sigma}$, note that

$$\begin{aligned}\boldsymbol{R} &= \boldsymbol{I}_3 + \sin(\theta)\boldsymbol{S}(\boldsymbol{\lambda}) + (1-\cos(\theta))\boldsymbol{S}(\boldsymbol{\lambda})^2 \\ &= \boldsymbol{I}_3 + 2\sin\tfrac{\theta}{2}\cos\tfrac{\theta}{2}\boldsymbol{S}(\boldsymbol{\lambda}) + 2\sin^2\tfrac{\theta}{2}\boldsymbol{S}(\boldsymbol{\lambda})^2\,.\end{aligned}$$

Since

$$\sin\tfrac{\theta}{2} = \frac{2\tan\frac{\theta}{4}}{1+\tan^2\frac{\theta}{4}}\,, \qquad \cos\tfrac{\theta}{2} = \frac{1-\tan^2\frac{\theta}{4}}{1+\tan^2\frac{\theta}{4}}\,,$$

and

$$1+\tan^2\tfrac{\theta}{4} = 1+\boldsymbol{\sigma}^T\boldsymbol{\sigma}\,,$$

we obtain

$$2\sin\tfrac{\theta}{2}\cos\tfrac{\theta}{2} = 4\frac{1-\boldsymbol{\sigma}^T\boldsymbol{\sigma}}{(1+\boldsymbol{\sigma}^T\boldsymbol{\sigma})^2}\tan\tfrac{\theta}{4}\,, \qquad 2\sin^2\tfrac{\theta}{2} = \frac{8}{(1+\boldsymbol{\sigma}^T\boldsymbol{\sigma})^2}\tan^2\tfrac{\theta}{4}\,.$$

Substitution of the latter formulas into the expression for $\boldsymbol{R}$ yields

$$\boldsymbol{R} = \boldsymbol{I}_3 + \frac{8}{(1+\boldsymbol{\sigma}^T\boldsymbol{\sigma})^2}\boldsymbol{S}(\boldsymbol{\sigma})\left[\boldsymbol{S}(\boldsymbol{\sigma}) + \tfrac{1}{2}(1-\boldsymbol{\sigma}^T\boldsymbol{\sigma})\boldsymbol{I}_3\right]\,.$$

Kinematic equation

To derive the kinematic equations in terms of the MRP, we proceed as follows: let $\boldsymbol{\omega}$ be the angular velocity on $\mathscr{F}_b$ with respect to $\mathscr{F}_e$ resolved in $\mathscr{F}_b$, and consider the kinematic equations in terms of the Euler parameters

$$\begin{aligned}\dot{\eta} &= -\tfrac{1}{2}\boldsymbol{\varepsilon}^T\boldsymbol{\omega} \\ \dot{\boldsymbol{\varepsilon}} &= \tfrac{1}{2}[\eta\boldsymbol{I}_3 + \boldsymbol{S}(\boldsymbol{\varepsilon})]\boldsymbol{\omega}\,.\end{aligned}$$

Since $\boldsymbol{\sigma} = \dfrac{\boldsymbol{\varepsilon}}{1+\eta}$,

$$\begin{aligned}\dot{\boldsymbol{\sigma}} &= \frac{1}{2(1+\eta)}[\eta\boldsymbol{I}_3 + \boldsymbol{S}(\boldsymbol{\varepsilon})]\boldsymbol{\omega} + \frac{1}{2(1+\eta)^2}\boldsymbol{\varepsilon}\boldsymbol{\varepsilon}^T\boldsymbol{\omega} \\ &= \tfrac{1}{2}\left[\frac{\eta}{1+\eta}\boldsymbol{I}_3 + \boldsymbol{S}(\boldsymbol{\sigma}) + \boldsymbol{\sigma}\boldsymbol{\sigma}^T\right]\boldsymbol{\omega} \\ &= \tfrac{1}{2}\left[\frac{1-\boldsymbol{\sigma}^T\boldsymbol{\sigma}}{2}\boldsymbol{I}_3 + \boldsymbol{S}(\boldsymbol{\sigma}) + \boldsymbol{\sigma}\boldsymbol{\sigma}^T\right]\boldsymbol{\omega} \\ &= \tfrac{1}{2}\boldsymbol{G}(\boldsymbol{\sigma})\boldsymbol{\omega}\end{aligned}$$

where we have used the fact that

$$\frac{\eta}{1+\eta} = \frac{1-\boldsymbol{\sigma}^T\boldsymbol{\sigma}}{2}\,.$$

Since $\boldsymbol{S}(\boldsymbol{\sigma}) = \boldsymbol{\sigma}\boldsymbol{\sigma}^T - \boldsymbol{\sigma}^T\boldsymbol{\sigma}\boldsymbol{I}_3$, we can also write

$$\boldsymbol{G}(\boldsymbol{\sigma}) = \left[\frac{1+\boldsymbol{\sigma}^T\boldsymbol{\sigma}}{2}\boldsymbol{I}_3 + \boldsymbol{S}(\boldsymbol{\sigma}) + \boldsymbol{S}(\boldsymbol{\sigma})^2\right].$$

Also, it can be verified that

$$\boldsymbol{\sigma}^T\boldsymbol{G}(\boldsymbol{\sigma}) = \left(\frac{1+\boldsymbol{\sigma}^T\boldsymbol{\sigma}}{2}\right)\boldsymbol{\sigma}^T$$

and that

$$\boldsymbol{G}^T(\boldsymbol{\sigma})\boldsymbol{G}(\boldsymbol{\sigma}) = \left(\frac{1+\boldsymbol{\sigma}^T\boldsymbol{\sigma}}{2}\right)^2\boldsymbol{I}_3 \implies \boldsymbol{G}^{-1}(\boldsymbol{\sigma}) = \left(\frac{2}{1+\boldsymbol{\sigma}^T\boldsymbol{\sigma}}\right)^2\boldsymbol{G}^T(\boldsymbol{\sigma}).$$

Additional results

The kinematic equation

$$\dot{\boldsymbol{\sigma}} = \tfrac{1}{2}\boldsymbol{G}(\boldsymbol{\sigma})\boldsymbol{\omega}$$

is *lossless* with respect to the input $u = \boldsymbol{\omega}$ and the output $y = \boldsymbol{\sigma}$, with storage function $V(\boldsymbol{\sigma}) = 2\ln(1+\boldsymbol{\sigma}^T\boldsymbol{\sigma})$. As a matter of fact,

$$\frac{\partial\ln(1+\boldsymbol{\sigma}^T\boldsymbol{\sigma})}{\partial\boldsymbol{\sigma}}\boldsymbol{G}(\boldsymbol{\sigma})\boldsymbol{\omega} = \boldsymbol{\sigma}^T\boldsymbol{\omega}\,.$$

This implies that all the nice results about stabilization using passivity that hold for the quaternion parameterization apply *mutatis mutandis* to the MRP. Also, since

$$2\ln(1+\boldsymbol{\sigma}^T\boldsymbol{\sigma}) \le 2\|\boldsymbol{\sigma}\|^2\,, \qquad \text{for all } \boldsymbol{\sigma}$$

and

$$\|\boldsymbol{\sigma}\|^2 \le 2\ln(1+\boldsymbol{\sigma}^T\boldsymbol{\sigma})\,, \qquad \text{for all } \boldsymbol{\sigma}:\ \|\boldsymbol{\sigma}\| \le 1\,,$$

it is easy to prove local exponential stability as well.

Appendix B

Stability Tools

B.1 Notation and Math Preliminaries

For vectors $x \in \mathbb{R}^n$, $z \in \mathbb{R}^m$, we let $\mathrm{col}(x, z) := (x^T\ z^T)^T$.

Norms and Function Spaces

Norms on finite-dimensional vector spaces $\mathcal{X} \cong \mathbb{R}^n$ are denoted by a single bar, $|\cdot| : \mathcal{X} \to \mathbb{R}_{\geq 0}$. This notation encompasses vector norms, $|x|$ for $x \in \mathbb{R}^n$, and matrix norms, $|A|$ for $A \in \mathbb{R}^{m\times n}$. Due to the fact that norms on finite-dimensional vector spaces are all equivalent, norm types are left unspecified, unless explicitly noted. As a result, there is little loss of generality in assuming that $|\cdot| : \mathcal{X} \to \mathbb{R}_{\geq 0}$ is the euclidean norm on $\mathcal{X}$, that is:

$$|x| = |x|_2 := \sqrt{\sum_{i=1}^{n} x_i^2}\,,\ x \in \mathbb{R}^n\,, \qquad |A| = |A|_2 := \sup_{\substack{x\in\mathbb{R}^n \\ x\neq 0}} \frac{|Ax|_2}{|x|_2} = \sqrt{\lambda_{\max}\left(A^TA\right)}\,,\ A \in \mathbb{R}^{m\times n}$$

where $\lambda_{\max}(M)$ denotes the largest eigenvalue of the matrix $M \in \mathbb{R}^{n\times n}$. Given a set $\mathcal{A} \subset \mathbb{R}^n$, the *norm on* $\mathbb{R}^n$ *with respect to* $\mathcal{A}$ is defined as the *point-to-set distance*

$$|x|_{\mathcal{A}} := \mathrm{dist}(x, \mathcal{A}) := \inf_{\xi\in\mathcal{A}} |x - \xi|$$

Conversely, for function spaces we distinguish several types of functional norms, denoted by a double bar, $\|\cdot\| : \mathcal{U} \to \mathbb{R}_{\geq 0}$, where $\mathcal{U}$ is a suitable space of functions. Functions spaces that will be considered in this class are the spaces of k-times differentiable functions, with $k \in [0, 1, \ldots, \infty]$. Specifically,

$$\mathcal{C}^k_{\mathcal{I}}(\mathcal{X})\,, \quad k \in [0, \infty]\,,\ \mathcal{I} \subset (-\infty, \infty)\,,\ \mathcal{X} \cong \mathbb{R}^n$$

denotes the space of k-times differentiable functions of a scalar variable defined on the interval $\mathcal{I}$ of the real line, with codomain $\mathcal{X}$. For example, $\mathcal{C}^0_{[0,\infty)}(\mathbb{R}^m)$ denotes the space of continuous functions $u(\cdot) : t \mapsto u(t) \in \mathbb{R}^m$, $t \in [0, \infty)$, whereas $\mathcal{C}^1_{(0,\infty)}(\mathbb{R}^{m\times n})$ denotes the space of continuously differentiable matrix-valued functions $u(\cdot) : t \mapsto u(t) \in \mathbb{R}^{m\times n}$, $t \in (0, \infty)$. Note that $\mathcal{C}^{k+1}_{\mathcal{I}}(\mathcal{X}) \subset \mathcal{C}^k_{\mathcal{I}}(\mathcal{X})$, for $k = 0, 1, \ldots, \infty$. For economy of notation, unless confusion may arise, we shall omit to specify the codomain. The following norms will be used on $\mathcal{C}^k_{\mathcal{I}}(\mathcal{X})$:

- ∞-norm: $\|u(\cdot)\|_\infty := \sup_{t\in\mathcal{I}} |u(t)|$
- 2-norm: $\|u(\cdot)\|_2 := \sqrt{\int_{\mathcal{I}} |u(\tau)|^2 \mathrm{d}\tau}$

If, for a given $u(\cdot) \in \mathcal{C}^k_{\mathcal{I}}(\mathcal{X})$, $\|u(\cdot)\|_\infty < \infty$, then the function $u(\cdot)$ is said to be *bounded* on its domain. If $\|u(\cdot)\|_2 < \infty$, $u(\cdot)$ is said to have *finite energy* on its domain. Note that if $\mathcal{I}$ is a compact interval (that is, $\mathcal{I} = [a, b]$ for some $-\infty < a < b < \infty$), then both $\|u(\cdot)\|_\infty < \infty$ and $\|u(\cdot)\|_2 < \infty$ hold by continuity. Consequently, it makes sense to define

$$\mathcal{L}^\infty_{(-\infty,\infty)}(\mathcal{X}) := \left\{u(\cdot) \in \mathcal{C}^0_{(-\infty,\infty)}(\mathcal{X}) : \|u(\cdot)\|_\infty < \infty\right\}$$
$$\mathcal{L}^\infty_{[0,\infty)}(\mathcal{X}) := \left\{u(\cdot) \in \mathcal{C}^0_{[0,\infty)}(\mathcal{X}) : \|u(\cdot)\|_\infty < \infty\right\}$$

as the spaces of bounded functions defined over $(-\infty, \infty)$ or $[0, \infty)$, and

$$\mathcal{L}^2_{(-\infty,\infty)}(\mathcal{X}) := \left\{u(\cdot) \in \mathcal{C}^0_{(-\infty,\infty)}(\mathcal{X}) : \|u(\cdot)\|_2 < \infty\right\}$$
$$\mathcal{L}^2_{[0,\infty)}(\mathcal{X}) := \left\{u(\cdot) \in \mathcal{C}^0_{[0,\infty)}(\mathcal{X}) : \|u(\cdot)\|_2 < \infty\right\}$$

as the spaces of square-integrable functions defined over $(-\infty, \infty)$ or $[0, \infty)$. Again, the domain of definition and the codomain will be dropped from the notation whenever convenient and appropriate, and use the simpler notation $\mathcal{L}^\infty$ and $\mathcal{L}^2$. For a given function $u(\cdot) \in \mathcal{C}^k_{\mathcal{I}}(\mathcal{X})$, where either $\mathcal{I} = (-\infty, \infty)$ or $\mathcal{I} = [0, \infty)$ and $\tau \in \mathcal{I}$, we define the *truncation of* $u(\cdot)$ *over* $(-\infty, \tau]$ (or over $[0, \tau]$) as the function $u_\tau(\cdot) : \mathcal{I} \to \mathcal{X}$ defined as

$$u_\tau(t) = \begin{cases} u(t) & t \in \mathcal{I} \text{ and } t \leq \tau \\ 0 & t \geq \tau \end{cases}$$

On the basis of this definition, one defines the *extended* $\mathcal{L}^\infty$ *and* $\mathcal{L}^2$ *spaces* respectively as follows:

$$\mathcal{L}^{\infty,e}_{(-\infty,\infty)}(\mathcal{X}) := \left\{u_\tau(\cdot) \in \mathcal{C}^0_{(-\infty,\infty)}(\mathcal{X}) : \|u_\tau(\cdot)\|_\infty < \infty \text{ for all } \tau \in \mathbb{R}\right\}$$
$$\mathcal{L}^{\infty,e}_{[0,\infty)}(\mathcal{X}) := \left\{u_\tau(\cdot) \in \mathcal{C}^0_{[0,\infty)}(\mathcal{X}) : \|u_\tau(\cdot)\|_\infty < \infty \text{ for all } \tau \geq 0\right\}$$

and

$$\mathcal{L}^{2,e}_{(-\infty,\infty)}(\mathcal{X}) := \left\{u_\tau(\cdot) \in \mathcal{C}^0_{(-\infty,\infty)}(\mathcal{X}) : \|u_\tau(\cdot)\|_2 < \infty \text{ for all } \tau \in \mathbb{R}\right\}$$
$$\mathcal{L}^{2,e}_{[0,\infty)}(\mathcal{X}) := \left\{u_\tau(\cdot) \in \mathcal{C}^0_{[0,\infty)}(\mathcal{X}) : \|u_\tau(\cdot)\|_2 < \infty \text{ for all } \tau \geq 0\right\}$$

Clearly, $\mathcal{L}^\infty \subset \mathcal{L}^{\infty,e}$ and $\mathcal{L}^2 \subset \mathcal{L}^{2,e}$, but not vice versa.
GIven a signal $u(\cdot) \in \mathcal{C}^k_{(-\infty,\infty)}(\mathcal{X})$, its *asymptotic norm*, $\|u(\cdot)\|_a$, is defined as

$$\|u(\cdot)\|_a := \limsup_{t\to\infty} |u(t)|$$

Comparison Functions

Definition B.1.1 (Class-$\mathcal{K}$ Functions) *A function $\alpha(\cdot) : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is said to be* of class-$\mathcal{K}$ *if it is continuous, strictly increasing and satisfies $\alpha(0) = 0$. A class-$\mathcal{K}$ function is said to be* of class-$\mathcal{K}_\infty$ *if, in addition, it satisfies* $\lim_{s\to\infty} \alpha(s) = +\infty$.

Definition B.1.2 (Class-$\mathcal{N}$ Functions) *A function $\eta(\cdot) : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is said to be* of class-$\mathcal{N}$ *if it is continuous and non-decreasing. Note that a class-$\mathcal{N}$ function $\eta(\cdot)$ does not necessarily satisfy $\eta(0) = 0$.*

Definition B.1.3 (Class-$\mathcal{KL}$ Functions) *A function $\beta(\cdot) : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is said to be* of class-$\mathcal{KL}$ *if $\beta(\cdot, r)$ is a class-$\mathcal{K}$ function for all $r \in \mathbb{R}_{\geq 0}$ and, for all $s \in \mathbb{R}_{\geq 0}$, $\beta(s, \cdot)$ is a continuous strictly decreasing function satisfying* $\lim_{r\to\infty} \beta(s, r) = 0$.

B.2 Stability Definitions

In what follows, we consider nonlinear nonautonomous systems of the form

$$\begin{aligned} \dot{x} &= f(t, x) \\ x(t_0) &= x_0 \end{aligned} \tag{B.1}$$

with state $x \in \mathbb{R}^n$. The vector field $f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ is assumed to be at least continuous in $t \in \mathbb{R}$, and locally Lipschitz in $x \in \mathbb{R}^n$, uniformly in t (note that this implies that $\sup_{t\geq 0} |f(t, x)| < \infty$, for all x belonging to arbitrary compact sets $\mathcal{M} \subset \mathbb{R}^n$.) Often, we will add the further assumption that f is continuously differentiable, or even smooth. The assumption on local Lipschitz continuity (uniformly in t) of the vector field f guarantees that there exists a unique absolutely continuous solution $x(t; t_0, x_0)$ of (B.20), which can be extended over a maximal open interval $\mathcal{I}_{t_0,x_0} = (t_0 - \delta_{\min}, t_0 + \delta_{\max})$. If $\delta_{\max} = +\infty$, (respectively, $\delta_{\min} = +\infty$) for all initial conditions x_0 and all initial times t_0, we say that (B.20) is *forward complete* (respectively, *backward complete*). A system that is both backward and forward complete is said to be *complete*. On the other hand, if $\delta_{\max}$ (respectively, $\delta_{\min}$) is finite, then the trajectory $x(t; t_0, x_0)$ leaves any compact set $\mathcal{M}$ containing x_0 as $t \to t_0 + \delta_{\max}$ (respectively, $t \to t_0 - \delta_{\min}$.) It is assumed that the origin $x = 0$ is an equilibrium for (B.20), that is $f(t, 0) = 0$, for all $t \in \mathbb{R}$.

Definition B.2.1 (Uniform Stability) *The origin of (B.20) is said to be* uniformly stable *if for any $\varepsilon > 0$ there exists $\delta_\varepsilon > 0$ such that for any $t_0 \in \mathbb{R}_{\geq 0}$ and any $|x_0| \leq \delta_\varepsilon$ the solution $x(t; t_0, x_0)$ satisfies $|x(t; t_0, x_0)| \leq \varepsilon$ for all $t \geq t_0$.*

Definition B.2.2 (Uniform Global Stability) *The origin of (B.20) is said to be* uniformly globally stable *if there exists a class-$\mathcal{K}_\infty$ function $\gamma(\cdot)$ such that for each $(t_0, x_0) \in \mathbb{R}_{\geq 0} \times \mathbb{R}^n$ the solution $x(t; t_0, x_0)$ satisfies*

$$|x(t; t_0, x_0)| \leq \gamma(|x_0|)\,, \qquad \forall\, t \geq t_0\,.$$

Note that the definition of uniform stability is different from that of uniform global stability, as the latter embeds the notion of forward completeness, while the former does not.

Definition B.2.3 (Uniform Global Attractivity) *The origin of (B.20) is said to be* uniformly globally attractive *if for any numbers $R > 0$ and $\varepsilon > 0$ there exists $T > 0$ (which depends only on R and ε) such that for any $t_0 \in \mathbb{R}_{\geq 0}$ and any $|x_0| \leq R$*

$$|x(t; t_0, x_0)| \leq \varepsilon , \qquad \forall\, t \geq t_0 + T .$$

Definition B.2.4 (Uniform Global Asymptotic Stability) *The origin of (B.20) is said to be* uniformly globally asymptotically stable *if it is uniformly stable and uniformly globally attractive.*

A well known result is the following:

Proposition B.2.5 *The origin of the system (B.20) is uniformly globally asymptotically stable if and only if there exists a class-$\mathcal{KL}$ function $\beta(\cdot,\cdot)$ such that all solutions of (B.20) satisfy*

$$|x(t; t_0, x)| \leq \beta(|x_0|, t - t_0) , \qquad \forall\, t \geq t_0$$

for all $(t_0, x_0) \in \mathbb{R}_{\geq 0} \times \mathbb{R}^n$.

Definition B.2.6 (Exponential Convergence) *The trajectories of the system (B.20) are said to be* (locally) exponentially convergent *if there exists an open neighborhood of the origin $\mathcal{D}$ such that* for each *pair of initial conditions $(t_0, x_0) \in \mathbb{R}_{\geq 0} \times \mathcal{D}$ there exist constants $\mu_0 > 0$, $\lambda_0 > 0$ such that the solution $x(t; t_0, x_0)$ satisfies*

$$|x(t; t_0, x)| \leq \mu_0 |x_0| \mathrm{e}^{-\lambda_0 (t - t_0)} , \qquad \forall\, t \geq t_0 . \tag{B.2}$$

The system (B.20) is said to be globally exponentially convergent *if* for each *pair of initial conditions $(t_0, x_0) \in \mathbb{R}_{\geq 0} \times \mathbb{R}^n$ there exist constants $\mu_0 > 0$, $\lambda_0 > 0$ such that (B.2) is satisfied.*

Definition B.2.7 (Exponential Stability) *The origin of (B.20) said to be* (locally) exponentially stable *if there exist constants $\mu > 0$, $\lambda > 0$ and a neighborhood $\mathcal{D}$ of the origin such that for any initial condition $(t_0, x_0) \in \mathbb{R}_{\geq 0} \times \mathcal{D}$ the corresponding solutions satisfy*

$$|x(t; t_0, x)| \leq \mu |x_0| \mathrm{e}^{-\lambda (t - t_0)} , \qquad \forall\, t \geq t_0 . \tag{B.3}$$

The system (B.20) is said to be globally exponentially stable *if there exist constants $\mu > 0$, $\lambda > 0$ such that (B.3) is satisfied for any $(t_0, x_0) \in \mathbb{R}_{\geq 0} \times \mathbb{R}^n$.*

B.3 Stability Theorems

In this section, we recall a few results on stability theory of the equilibrium of non-autonomous nonlinear systems of the form (B.20). Almost all the results presented in this section will be given without proof. The reader may refer to [20] for further details. The reader should be familiar with Lyapunov stability theory for autonomous nonlinear systems.

B.3.1 Lyapunov Theorems

Definition B.3.2 (Lyapunov Function Candidates) *A $\mathcal{C}^1$ function $V : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ is said to be a* Lyapunov Function Candidate *for (B.20) if there exist class-$\mathcal{K}_\infty$ functions $\underline{\alpha}(\cdot)$, $\overline{\alpha}(\cdot)$, such that*

$$\underline{\alpha}(|x|) \leq V(t, x) \leq \overline{\alpha}(|x|) \tag{B.4}$$

for all $t \in \mathbb{R}$ and all $x \in \mathbb{R}^n$.

In particular, the lower bound in (B.4) establishes the fact that $V(t, x)$ is *positive definite* and *radially unbounded*, that is, $V(t, x) > 0$ for all $t \in \mathbb{R}$ and all $x \in \mathbb{R}^n - \{0\}$, and

$$\lim_{|x| \to \infty} V(t, x) = +\infty$$

Conversely, the upper bound establishes the property that $V(t, x)$ is *decrescent*. The classic Lyapunov Theorems for non-autonomous systems can be summarized as follows:

Theorem B.3.3 *Assume that the $\mathcal{C}^1$ function $V : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ a Lyapunov Function Candidate for (B.20). Then, the equilibrium $x = 0$ of (B.20) is:*

- **Uniformly globally stable** *if*

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t, x) \leq 0$$

  for all $t \geq 0$ and all $x \in \mathbb{R}^n$;

- **Uniformly globally asymptotically stable** *if there exists a class-$\mathcal{K}$ function $\alpha(\cdot)$ such that*

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t, x) \leq -\alpha(|x|) \tag{B.5}$$

  for all $t \geq 0$ and all $x \in \mathbb{R}^n$;

- **Globally exponentially stable** *if (B.4) and (B.5) hold with quadratic functions, that is,*

$$\underline{\alpha}(s) = \underline{a}\, s^2 , \qquad \overline{\alpha}(s) = \overline{a}\, s^2 , \qquad \alpha(s) = a\, s^2$$

  for some constants $0 < \underline{a} \leq \overline{a}$, $a > 0$;

- **Uniformly globally asymptotically and locally exponentially stable** *if (B.4) and (B.5) hold with locally quadratic functions, that is, there exist positive numbers δ, $\underline{a}$, $\overline{a}$, a such that*

$$\underline{\alpha}(s) \geq \underline{a}\, s^2 , \qquad \overline{\alpha}(s) \leq \overline{a}\, s^2 , \qquad \alpha(s) \geq a\, s^2$$

  for all $s \in [0, \delta]$.

B.3.4 Converse Theorems and Related Results

We recall first some useful results on the inversion of the theorems of Lyapunov. Converse Lyapunov theorems play a crucial role in modern nonlinear control theory, as the existence of smooth Lyapunov functions is instrumental in assessing various forms of robustness with respect to vanishing and persistent perturbations. In regard to this, a fundamental example is given by the theorem of *total stability*, given later in the section. An introduction to the classic contributions by Massera and Kurzweil, can be found in the textbooks [20, 21]. For recent important results, the reader should consult [22] and [23], which also contain a very nice discussion of early work on the subject as well as detailed and precise bibliographic references. The first converse theorem is extremely important, and concerns the existence of a smooth Lyapunov function for locally Lipschitz systems possessing a UGAS equilibrium. For a proof, see [24] or [22].

Theorem B.3.5 (Massera) *Assume that in (B.20) the vector field f is locally Lipschitz in $x \in \mathbb{R}^n$, uniformly in t. Assume that the equilibrium $x = 0$ is uniformly globally asymptotically stable. Then, there exists a* smooth *function $V : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \to \mathbb{R}_{\geq 0}$, class-$\mathcal{K}_\infty$ functions $\underline{\alpha}(\cdot)$, $\overline{\alpha}(\cdot)$, and a class-$\mathcal{K}$ function $\alpha(\cdot)$ such that*

$$\underline{\alpha}(|x|) \leq V(t,x) \leq \overline{\alpha}(|x|)$$

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t,x) \leq -\alpha(|x|)$$

for all $t \geq 0$ and all $x \in \mathbb{R}^n$.

Appropriate versions of the above converse theorem for uniform local asymptotic stability and local exponential stability read as follows:

Theorem B.3.6 *Assume that in (B.20) the vector field f is continuously differentiable in $\mathbb{R}_{\geq 0} \times \mathcal{B}_r$, where $\mathcal{B}_r = \{x \in \mathbb{R}^n : \ |x| < r\}$, and that the Jacobian matrix $\partial f/\partial x$ is bounded on $\mathcal{B}_r$ uniformly in t. Assume that there exist a class-$\mathcal{KL}$ function $\beta(\cdot,\cdot)$ and a positive number r_0 satisfying $\beta(r_0, 0) < r$ such that the trajectories of* (B.20) *satisfy*

$$|x(t,t_0,x_0)| \leq \beta(|x_0|, t - t_0)\,, \qquad \forall\, x_0 \in \mathcal{B}(r_0)\,,\ \forall\, t \geq t_0 \geq 0\,.$$

Then, there exists a continuously differentiable function $V : \mathbb{R}_{\geq 0} \times \mathcal{B}_{r_0} \to \mathbb{R}^n$ satisfying

$$\alpha_1(|x|) \leq V(t,x) \leq \alpha_2(|x|)$$

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t,x) \leq -\alpha_3(|x|)$$

$$\left|\frac{\partial V}{\partial x}\right| \leq \alpha_4(|x|)$$

for some class-$\mathcal{K}$ functions $\alpha_i(\cdot)$, $i = 1, \ldots, 4$, defined on $[0, r_0]$. *If, in addition, the system* (B.20) *is autonomous, the function V can be chosen independent of t.*

Theorem B.3.7 *Assume that in (B.20) the vector field f is continuously differentiable in $\mathbb{R}_{\geq 0} \times \mathcal{B}_r$, where $\mathcal{B}_r = \{x \in \mathbb{R}^n : \ |x| < r\}$, and that the Jacobian matrix $\partial f/\partial x$ is bounded on $\mathcal{B}_r$ uniformly in t. Assume that there exist positive constants κ, λ, r_0, with $r_0 < r/\kappa$*

such that for any initial condition $(t_0, x_0) \in \mathbb{R}_{\geq 0} \times \mathcal{B}_{r_0}$ *the corresponding solution of (B.20) satisfies*

$$|x(t; t_0, x)| \leq \kappa |x_0| \mathrm{e}^{-\lambda(t-t_0)}, \qquad \forall\, t \geq t_0 \,.$$

Then, there exists a continuously differentiable function $V : \mathbb{R}_{\geq 0} \times \mathcal{B}_{r_0} \to \mathbb{R}^n$ *satisfying*

$$\begin{aligned} c_1|x|^2 \leq V(t,x) \leq c_2|x|^2 \\ \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t,x) \leq -c_3|x|^2 \\ \left|\frac{\partial V}{\partial x}\right| \leq c_4|x| \end{aligned}$$

for some positive constants c_i, $i = 1, \ldots, 4$, *for all* $t \geq 0$, *and all* $x \in \mathcal{B}_r$. *If, in addition, the equilibrium* $x = 0$ *is globally uniformly exponentially stable, the above inequalities hold on* $\mathbb{R}^n$. *Moreover, if the system is autonomous, the function* V *can be chosen independent of* t.

A proof of Theorem B.3.6 and Theorem B.3.7 can be found in [20] and [21]. It is worth noting that Theorem B.3.5 and Theorem B.3.7 can be combined, retaining the more restrictive assumptions on the regularity of the vector field f stated in Theorem B.3.7, to obtain a converse Lyapunov theorem for UGAS and LES equilibria yielding a continuously differentiable Lyapunov function which is locally quadratic at the origin.

Theorem B.3.8 *Assume that in (B.20) the vector field* f *is continuously differentiable in* $\mathbb{R}_{\geq 0} \times \mathbb{R}^n$, *and the Jacobian matrix* $\partial f / \partial x$ *is bounded on any compact set, uniformly in* t. *Then, the equilibrium* $x = 0$ *is uniformly globally asymptotically stable (UGAS) and locally exponentially stable (LES)* if and only if *there exist a continuously differentiable function* $V : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \to \mathbb{R}_{\geq 0}$, *class-*$\mathcal{K}_\infty$ *functions* $\underline{\alpha}(\cdot)$, $\overline{\alpha}(\cdot)$, *a class-*$\mathcal{K}$ *function* $\alpha(\cdot)$, *and positive numbers* δ, $\underline{a}$, $\overline{a}$, a *such that*

$$\begin{aligned} \underline{\alpha}(|x|) \leq V(t,x) \leq \overline{\alpha}(|x|) \\ \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t,x) \leq -\alpha(|x|) \end{aligned}$$

for all $t \geq 0$ *and all* $x \in \mathbb{R}^n$, *and*

$$\underline{\alpha}(s) \geq \underline{a}\, s^2 \,, \qquad \overline{\alpha}(s) \leq \overline{a}\, s^2 \,, \qquad \alpha(s) \geq a\, s^2$$

for all $s \in [0, \delta]$.

A proof of Theorem B.3.8 can be obtained following the same lines, *mutatis mutandis*, of [25, Lemma 10.1.5]. For autonomous systems possessing a locally asymptotically stable equilibrium, the following theorem due to Kurzweil [26] establishes the existence of a smooth Lyapunov function which is *proper* on the domain of attraction, generalizing in a significant way the classic theorem by Zubov [27]. A nice self-contained proof can be found in [20, Theorem 4.17].

Theorem B.3.9 (Kurzweil) *Assume that the system* (B.20) *is autonomous, and let* $f : \mathcal{D} \to \mathbb{R}^n$ *be locally Lipschitz on the domain* $\mathcal{D} \subset \mathbb{R}^n$ *containing the origin. Assume that the origin is a (locally) asymptotically stable equilibrium, and denote with* $\mathcal{A} \subset \mathcal{D}$ *its domain of attraction. Then, there exists a smooth, positive definite function* $V : \mathcal{A} \to \mathbb{R}_{\geq 0}$ *and a continuous, positive definite function* $W : \mathcal{A} \to \mathbb{R}_{\geq 0}$ *satisfying*

$$\lim_{x \to \partial \mathcal{A}} V(x) = +\infty$$

$$\frac{\partial V}{\partial x} f(x) \leq -W(x)\,, \qquad \forall\, x \in \mathcal{A}\,.$$

In particular, for any $c > 0$*, the level set* $\Omega_c = \{x \in \mathbb{R}^n : V(x) \leq c\}$ *is a positively invariant compact subset of* $\mathcal{A}$*.*

B.3.10 Stability of Perturbed Systems

The following theorem, known as the *Theorem of Total Stability*, establishes the fact that uniform asymptotic stability of an equilibrium of a nonlinear systems provides robustness against small non-vanishing perturbations. In particular, the theorem establishes boundedness of all trajectories of a perturbed system that originate sufficiently close to the equilibrium, if the perturbation is "sufficiently small" in a meaningful sense.

Theorem B.3.11 (Total Stability)

Consider system (B.20), and assume that the vector field f *is continuously differentiable in* $\mathbb{R}_{\geq 0} \times \mathcal{B}_r$*, where* $\mathcal{B}_r = \{x \in \mathbb{R}^n : \ |x| < r\}$*, that the Jacobian matrix* $\partial f/\partial x$ *is bounded on* $\mathcal{B}_r$ *uniformly in* t*, and that* $f(t, 0) = 0$ *for all* $t \geq 0$*. Let* $g : \mathbb{R}_{\geq 0} \times \mathcal{B}_r \to \mathbb{R}^n$ *be such that* $g(t, x)$ *is piecewise continuous in* t *and locally Lipschitz in* x*, uniformly in* t*. Assume, in addition, that the equilibrium at the origin of (B.20) is (locally) uniformly asymptotically stable. Then, given any* $\varepsilon > 0$*, there exists* $\delta_1 > 0$ *and* $\delta_2 > 0$ *such that if*

$$\begin{aligned} &|x_0| \leq \delta_1 \\ &|g(t, x)| \leq \delta_2 \quad \forall\, t \geq 0 \quad \forall\, x \in \mathcal{B}_\varepsilon \end{aligned}$$

then the trajectory $x(t) = x(t; t_0, x_0)$ *of the perturbed system*

$$\begin{aligned} \dot{x} &= f(t, x) + g(t, x) \\ x(t_0) &= x_0 \end{aligned}$$

satisfies $|x(t)| \leq \varepsilon$ *for all* $t \geq t_0 \geq 0$*.*

Next, we restrict our attention to systems affected by bounded external disturbances, namely systems of the form

$$\begin{aligned} \dot{x} &= f(t, x, d) \\ x(t_0) &= x_0 \end{aligned} \tag{B.6}$$

where $d(\cdot) \in \mathcal{L}^\infty(\mathbb{R}^m)$. For these systems, we introduce and introduce a few important notions related to bounded-input bounded-state stability. The first such notion is improperly referred to as a stability property, as it does not entail contractivity of the forward trajectory over the semi-infinite interval (in terms of its $\mathcal{L}^\infty$-norm) with respect to the $\mathcal{L}^\infty$-norm of the disturbance. It is, however, an important property related to unifirm boundedness of trajectories in the face of bounded disturbances:

Definition B.3.12 (Global Uniform Ultimate Boundedness) *System* (B.6) *is said to possess the* global uniform ultimate boundedness property (GUUB) with respect to d *if there exists a class-$\mathcal{N}$ function $\eta(\cdot)$ such that for any initial condition $(t_0, x_0) \in \mathbb{R} \times \mathbb{R}^n$ and any $d(\cdot) \in \mathcal{L}^\infty(\mathbb{R}^m)$, the forward solution $x(t) := x(t; t_0, x_0, d_{[t_0,t)})$, $t \geq t_0$, of* (B.6) *satisfies*

$$\|x(\cdot)\|_a \leq \eta(\|d(\cdot)\|_\infty) , \qquad \|d(\cdot)\|_\infty := \sup_{t \geq t_0} |d(t)| \tag{B.7}$$

The GUUB property admits a (partial, as the converse statement does not hold) Lyapunov-like characterization, as follows:

Theorem B.3.13 *Let $V : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ be a continuously differentiable function satisfying*

$$\underline{\alpha}(|x|) \leq V(t, x) \leq \overline{\alpha}(|x|)$$

for all $(t, x) \in \mathbb{R} \times \mathbb{R}^n$, where $\underline{\alpha}(\cdot)$ and $\overline{\alpha}(\cdot)$ are class-$\mathcal{K}_\infty$ functions. Assume that there exists a class-$\mathcal{N}$-function $\chi(\cdot)$ such that for all $t \in \mathbb{R}$ and all $d \in \mathbb{R}^m$

$$|x| > \chi(|d|) \implies \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t, x, d) < 0 \tag{B.8}$$

Then, system (B.6) *has the GUUB property with respect to d. Morover, the bound* (B.7) *holds with $\eta(\cdot) = \underline{\alpha}^{-1} \circ \overline{\alpha} \circ \chi(\cdot)$.*

The second notion is a generalization of the concept of *internal stability* of an LTI system, as it provides a complete characterization of bounded-input bounded-state behavior together with global uniform asymtotic stability of the origin when the disturbance is inactive:

Definition B.3.14 (Input-to-State Stability) *System* (B.6) *is said to be* input-to-state stable (ISS) *if there exist class-$\mathcal{K}$ functions $\gamma_0(\cdot)$, $\gamma(\cdot)$ such that for any initial condition $(t_0, x_0) \in \mathbb{R} \times \mathbb{R}^n$ and any $d(\cdot) \in \mathcal{L}^\infty(\mathbb{R}^m)$, the forward solution $x(t) := x(t; t_0, x_0, d_{[t_0,t)})$, $t \geq t_0$, of* (B.6) *satisfies*

$$\begin{aligned} \|x(\cdot)\|_\infty &\leq \max\{\gamma_0(|x_0|), \gamma(\|d(\cdot)\|_\infty)\} \\ \|x(\cdot)\|_a &\leq \gamma(\|d(\cdot)\|_a) \end{aligned} \tag{B.9}$$

where $\|x(\cdot)\|_\infty := \sup_{t \geq t_0} |x(t)|$ and $\|d(\cdot)\|_\infty := \sup_{t \geq t_0} |d(t)|$.

The ISS property entails UGAS of the origin of the system when $d = 0$. The GUUB property admits a complete Lyapunov-like characterization, as follows:

Theorem B.3.15 *System* (B.6) *ISS (with respect to d as an input) if and only if there exist a continuously differentiable function $V : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}_{\geq 0}$, class-$\mathcal{K}_\infty$ functions $\underline{\alpha}(\cdot)$ and $\overline{\alpha}(\cdot)$, and a class-$\mathcal{K}$-function $\chi(\cdot)$ such that*

$$\underline{\alpha}(|x|) \leq V(t, x) \leq \overline{\alpha}(|x|) \tag{B.10}$$

for all $(t, x) \in \mathbb{R} \times \mathbb{R}^n$ and

$$|x| > \chi(|d|) \implies \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t, x, d) < 0 \tag{B.11}$$

for all $t \in \mathbb{R}$ and all $d \in \mathbb{R}^m$ Morover, given (B.10) *and* (B.11)*, the bounds* (B.9) *hold with $\gamma_0(\cdot) = \underline{\alpha}^{-1} \circ \overline{\alpha}(\cdot)$ and $\gamma(\cdot) = \underline{\alpha}^{-1} \circ \overline{\alpha} \circ \chi(\cdot)$.*

B.3.16 Invariance-like Theorems

A peculiar characteristic of direct adaptive control techniques is that the Lie derivative of certain candidate Lyapunov functions is rendered negative semi-definite by design. For autonomous systems, this situation of usually handled resorting to La Salle's invariance principle and the theorem of Krasovskii and Barbashin. However, for non-autonomous systems, the situation is far more complicated, and available results are in general much weaker. The reason lies in the fact that ω-limit sets of bounded trajectories of non-autonomous systems are not necessarily invariant, as it is indeed the case for autonomous or periodic systems. Invariance of ω-limit sets is the fundamental technical result that enables a "reduction principle" in determining the behavior of solutions when restricted to the zeroing manifold for the derivative of a Lyapunov function candidate, and unfortunately this method of analysis can not be carried over to the non-autonomous case. However, a weaker extension of La Salle's invariance principle can be used to infer certain properties of the asymptotic behavior of systems for which a Lyapunov-like function admitting a negative semi-definite derivative can be found. We begin with a classic result, ubiquitous in the literature of adaptive control, which is a key technical lemma in establishing convergence of integrable signals.

Lemma B.3.17 (Barbălat's lemma) *Let $\phi : \mathbb{R} \to \mathbb{R}$ be a uniformly continuous function over $[0, \infty)$. Assume also that $\lim_{t\to\infty} \int_0^t \phi(\tau)\mathrm{d}\tau$ exists and is finite. Then, $\lim_{t\to\infty} \phi(t) = 0$.*

The proof of Barbălat's lemma can be found in nearly every book on adaptive control, see for instance [28, Lemma 3.2.6], whereas a significant generalization has been recently suggested in [29].

The main result on "invariance-like" theorems for non-autonomous systems is due to Yoshizawa [30], and it is commonly referred to as the La Salle/Yoshizawa theorem.

Theorem B.3.18 (La Salle/Yoshizawa) *Consider the nonautonomous system (B.20) where the vector field $f(t,x)$ is piecewise continuous in $t \in \mathbb{R}$, and locally Lipschitz in $x \in \mathbb{R}^n$ uniformly in t. Assume that $x = 0$ is an equilibrium point for (B.20), that is $f(t,0) = 0$ for all t. Let $V : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ be a continuously differentiable function satisfying*

$$\underline{\alpha}(|x|) \leq V(t,x) \leq \overline{\alpha}(|x|) \tag{B.12}$$

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t,x) \leq -W(x) \tag{B.13}$$

for all $t \geq 0$, for all $x \in \mathbb{R}^n$, where $\underline{\alpha}(\cdot)$ and $\overline{\alpha}(\cdot)$ are class-$\mathcal{K}_\infty$ functions, and $W : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ is a continuous positive semi-definite function. Then, system (B.20) is uniformly globally stable, and satisfy

$$\lim_{t\to\infty} W(x(t)) = 0\,.$$

Proof. Since $f(t,x)$ is piecewise continuous in t and locally Lipschitz in x, the solution $x(t) := x(t : t_0, x_0)$ of (B.20) originating from any initial condition $(t_0, x_0) \in \mathbb{R}_{\geq 0} \times \mathbb{R}^n$ exists uniquely over a maximal interval $\mathcal{I}(t_0, x_0) = [t_0, t_0 + \delta_{\max})$. Next, we show that $x(t)$ is uniformly bounded over $\mathcal{I}(t_0, x_0)$, and thus $\mathcal{I}(t_0, x_0) = [t_0, \infty)$. Let

$$V(t) := V(t, x(t; t_0, x_0))$$

and note that since $\dot{V}(t) \leq 0$

$$V(t, x(t; t_0, x_0)) \leq V(t_0, x_0)\,, \qquad \forall\, t \in \mathcal{I}(t_0, x_0)\,.$$

Therefore

$$|x(t)| \leq (\underline{\alpha}^{-1} \circ \overline{\alpha})(|x_0|) =: B_{x_0}$$

for all $t \in \mathcal{I}(t_0, x_0)$, with $t_0 \in \mathbb{R}_{\geq 0}$ and $x_0 \in \mathbb{R}^n$ arbitrary. This shows that $\delta_{\max} = +\infty$, as otherwise $x(t)$ would leave the compact set $\{x : |x| \leq B_{x_0}\}$ as $t \to t_0 + \delta_{\max}$. Moreover, letting $\rho(\cdot) = (\underline{\alpha}^{-1} \circ \overline{\alpha})(\cdot)$, we obtain

$$|x(t; t_0, x_0)| \leq \rho(|x_0|)\,, \forall\, t_0 \geq 0\,,\ \forall\, t \geq t_0$$

hence global uniform stability of the origin is established. Since $V(t)$ is non increasing and bounded from below, $\lim_{t\to\infty} V(t) = V_\infty$ exists and is finite. Since $\dot{V}(t) \leq -W(x(t))$, it turns out that

$$\int_{t_0}^{t} W(x(\tau))\mathrm{d}\tau \leq V(t_0, x_0) - V(t)$$

and thus $\lim_{t\to\infty} \int_{t_0}^{t} W(x(\tau))\mathrm{d}\tau$ exists and is finite. Next, we show that $W(x(t))$ is a uniformly continuous function of t over $[t_0, \infty)$. Since, by definition,

$$x(t_2; t_0, x_0) = x(t_1; t_0, x_0) + \int_{t_1}^{t_2} f(\tau, x(\tau))\mathrm{d}\tau\,, \quad \forall\, t_2 \geq t_1 \geq t_0$$

and by virtue of the uniform local Lipschitz property there exist $L > 0$ such that

$$|f(t, x)| \leq L|x|\,, \qquad \forall\, t \geq t_0\,,\ \forall\, x : |x| \leq B_{x_0}$$

we obtain

$$|x(t_2) - x(t_1)| \leq \int_{t_1}^{t_2} L|x(\tau)|\mathrm{d}\tau \leq LB_{x_0}|t_1 - t_2|$$

for all $t_2 \geq t_1 \geq t_0$. Given any $\varepsilon > 0$, choose $\delta = \dfrac{\varepsilon}{LB_{x_0}}$ to obtain

$$|t_1 - t_2| < \delta \implies |x(t_1) - x(t_2)| < \varepsilon\,,$$

hence uniform continuity of $x(t)$ is established. Since $W(x)$ is a continuous function of x, it is uniformly continuous over the compact set $\{x : |x| \leq B_{x_0}\}$. Therefore, $W(x(t))$ is a uniformly continuous function of t, and the result of the theorem follows from Barbălat's lemma. □

It is worth noting that the La Salle/Yoshizawa theorem yields a much weaker result than its counterpart for autonomous systems (i.e., La Salle's invariance principle), as in the non-autonomous case the trajectory does not converge in general to an invariant set contained in $\mathcal{S} := \{x \in \mathbb{R}^n : W(x) = 0\}$. Furthermore, since convergence is established by means of Barbalăt's lemma, the set $\mathcal{S}$ is not guaranteed to be uniformly attractive.

To determine the behavior of the trajectory on the set $\mathcal{S}$, it may prove instrumental to use an additional *auxiliary function* $H : \mathbb{R} \times \mathbb{R}^n$, when appropriate conditions hold. The first result, due to Anderson and Moore [31], employs the auxiliary function

$$H(t, x) = \int_{t}^{t+\delta} \dot{V}(\tau, \chi(\tau, t, x))d\tau$$

where $\chi(\tau, t, x)$ is the solution of (B.20) originating from the initial condition x at time t, and $\delta > 0$ is a given constant. For a proof, see [31] or [20, Theorem 8.5].

Theorem B.3.19 (Anderson and Moore) *Let the assumptions of Theorem B.3.18 hold for the system* (B.20)*, with* (B.13) *replaced by the weaker condition*

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t,x) \leq 0 \qquad \forall\,(t,x) \in \mathbb{R} \times \mathbb{R}^n$$

and assume, in addition, that there exist $\delta > 0$ and $\lambda \in (0,1)$ such that

$$\int_t^{t+\delta} \dot{V}(t, \chi(\tau; t, x))\mathrm{d}\tau \leq -\lambda V(t,x)$$

for all $t \in \mathbb{R}_{\geq 0}$ and all $x \in \mathbb{R}^n$, where $\chi(\tau; t, x)$ is the solution of (B.20) at time τ originating from the initial condition x at the initial time t. Then, the equilibrium $x = 0$ is uniformly globally asymptotically stable. Furthermore, if for some positive numbers a_1, a_2, and ρ the comparison functions $\underline{\alpha}(\cdot)$ and $\overline{\alpha}(\cdot)$ satisfy

$$\underline{\alpha}(s) \geq a_1 s^2\,, \qquad \overline{\alpha}(s) \leq a_2 s^2$$

for all $s \in [0, \rho)$, then the equilibrium $x = 0$ is uniformly globally asymptotically and locally exponentially stable.

The most important application of Theorem B.3.19 regards the appropriate extension to the time-varying case of the familiar notion that an *observable* LTI system having a convergent output response under zero input is necessarily asymptotically stable.

Proposition B.3.20 *Consider the linear time-varying system*

$$\begin{aligned} \dot{x} &= A(t)x \\ y &= C(t)x \end{aligned} \tag{B.14}$$

where the mappings $A : \mathbb{R}_{\geq 0} \to \mathbb{R}^{n\times n}$ and $C : \mathbb{R} \to \mathbb{R}^{m\times n}$ are continuous and bounded. Assume that (B.14) *is* uniformly completely observable[1]*, that is, there exist constants $\delta > 0$ and $\kappa > 0$ such that the* observability gramian

$$W(t_1, t_2) = \int_{t_1}^{t_2} \Phi^T(\tau, t_1) C^T(\tau) C(\tau) \Phi(\tau, t_1) d\tau\,, \quad t_1 \leq t_2$$

satisfies

$$\kappa I \leq W(t, t+\delta)\,, \qquad \forall\, t \geq 0\,.$$

Furthermore, assume that there exists a continuously differentiable, symmetric mapping $P : \mathbb{R} \to \mathbb{R}^{n\times n}$ solution of the differential equation

$$\dot{P}(t) + A^T(t)P(t) + P(t)A(t) \leq -C^T(t)C(t)$$

satisfying

$$c_1 I \leq P(t) \leq c_2 I$$

for all $t \geq 0$ and some $c_1 > 0$, $c_2 > 0$. Then, the origin is a uniformly (globally) asymptotically stable equilibrium of (B.14)*.*

[1] The reader should be aware of the fact that the definition given here is only valid for *bounded* realizations. The reader should consult [32, 33] for the more general situation in which $A(\cdot)$ and $C(\cdot)$ are measurable and locally essentially bounded.

Proof. Consider the Lyapunov function candidate

$$V(t,x) = x^T P(t) x$$

yielding, along trajectories of (B.14),

$$\dot{V}(t,x) \le -x^T C^T(t) C(t) x \le 0\,.$$

It is easy to see that the assumptions of Theorem B.3.18 are satisfied, and thus trajectories of (B.14) are bounded, and satisfy $\lim_{t\to\infty} y(t) = 0$. Consider the auxiliary function

$$\begin{aligned} H(t,x) &= \int_t^{t+\delta} \dot{V}(\tau, \chi(\tau,t,x)) d\tau \\ &\le -\int_t^{t+\delta} \chi^T(\tau,t,x) C^T(\tau) C(\tau) \chi^T(\tau,t,x) d\tau \\ &= -x^T \int_t^{t+\delta} \Phi^T(\tau,t) C^T(\tau) C(\tau) \Phi(\tau,t) d\tau\, x\,, \end{aligned}$$

as $\chi(\tau,t,x) = \Phi(\tau,t)x$. Therefore,

$$H(t,x) \le -\kappa |x|^2 \le -\frac{\kappa}{c_2} V(t,x)\,,$$

and, since c_2 can always be chosen to be such that $c_2 > \kappa$, the result follows directly from Theorem B.3.19. □

The second result on uniform asymptotic stability of systems having a Lyapunov function with negative semi-definite derivative is due to Matrosov [34], and it is presented here in a simplified version. The interested reader is referred to [35] for the proof of a more general version, and to [36] for recent important generalizations.

Theorem B.3.21 (Matrosov) *Consider the nonlinear system* (B.20)*, where f is continuous in t, and locally Lipschitz in x, uniformly in t. Assume that there exists a continuously differentiable function $V : \mathbb{R}_{\ge 0} \times \mathbb{R}^n \to \mathbb{R}_{\ge 0}$ such that the assumptions of Theorem B.3.18 hold. Assume, in addition, that there exists a continuously differentiable function $H : \mathbb{R}_{\ge 0} \times \mathbb{R}^n \to \mathbb{R}$ with the following properties:*

i. For any fixed $x \in \mathbb{R}^n$, there exists a number $M > 0$ such that

$$|H(t,x)| \le M \qquad \forall\, t \ge 0\,.$$

i1. Let $\mathcal{E}$ be the set of all points $x \in \mathbb{R}^n$ such that $x \ne 0$ and $W(x) = 0$, that is, $\mathcal{E} = \{x : W(x) = 0\} \cap \{x \ne 0\}$. Assume that $\mathcal{E}$ is nonempty[2]*. Assume that the function $H(t,x)$ satisfies*

$$\dot{H}(t,x) := \frac{\partial H}{\partial t} + \frac{\partial H}{\partial x} f(t,x) > 0 \qquad \forall\, t \ge 0\,, \forall\, x \in \mathcal{E}\,.$$

Then, the equilibrium $x = 0$ is globally uniformly asymptotically stable

[2] Note that this rules out the possibility that $W(x)$ is positive definite.

B.4 Passivity

Passivity theory plays a fundamental role in the analysis and design of adaptive systems. Roughly speaking, the concept of passivity is a generalization of the notion of conservation of energy, in the sense that the rate of change of the energy stored in the system does not exceed the power supplied externally. Adaptive laws are usually designed to either exploit natural passivity properties of given plant models (as in the case of Euler-Lagrange or Hamiltonian systems) or to enforce passivity of the resulting closed-loop system. Passivity theory (or, more generally, the theory of dissipative systems) is usually formulated for autonomous systems, where, used in combination with La Salle's invariance principle and specific notions of observability, it yields a powerful tool to assess global asymptotic stability from Lyapunov functions admitting negative semi-definite derivatives. Furthermore, passivity theory and the related concept of finite $\mathcal{L}_2$-gain stability offer a natural extension to nonlinear systems of the concept of $\mathcal{H}_\infty$ norm of a stable transfer function, with all the advantages given by a Lyapunov-like characterization. The excellent monograph [37] provides a standard reference and a rewarding reading, while the reader interested in quickly grasping the fundamental concepts will find a lucid introduction in [25, Sections 10.7–10.9] and [20, Chapter 6]. Here, we will limit ourselves to giving only the most basic definitions and properties, extended to non-autonomous systems, that will be used in the sequel, adopting a simpler (albeit more restrictive) "differential" characterization of dissipativity.

Consider the following non-autonomous system in affine form

$$\begin{aligned} \dot{x} &= f(t,x) + g(t,x)u \\ y &= h(t,x) \end{aligned} \tag{B.15}$$

with state $x \in \mathbb{R}^n$, input $u \in \mathbb{R}^m$, and output $y \in \mathbb{R}^m$. It is assumed that $f(t,x)$, $g(t,x)$, and $h(t,x)$ are continuous in t and smooth in x. Also, assume that $f(t,0) = 0$ and $h(t,0) = 0$ for all t.

Definition B.4.1 (Passivity) *System (B.15) is said to be* passive *if there exists a smooth nonnegative function $V : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ (usually called a* storage function*) satisfying*

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t,x) \leq 0$$

$$\frac{\partial V}{\partial x} g(t,x) = h^T(t,x)$$

for all $t \in \mathbb{R}_{\geq 0}$, and all $x \in \mathbb{R}^n$.

Definition B.4.2 (Strict passivity) *System (B.15) is said to be* strictly passive *if there exists a smooth positive definite storage function $V : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}_{\geq 0}$, and a positive definite function $\alpha(\cdot)$ (called* dissipation rate*) satisfying*

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t,x) \leq -\alpha(x)$$

$$\frac{\partial V}{\partial x} g(t,x) = h^T(t,x)$$

for all $t \in \mathbb{R}_{\geq 0}$, and all $x \in \mathbb{R}^n$.

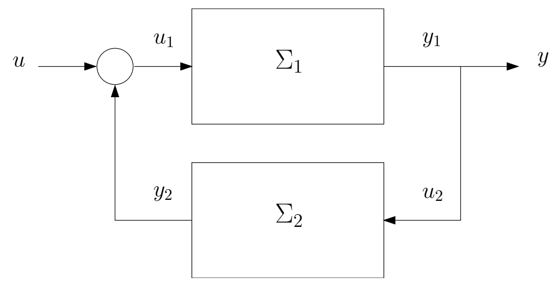


Figure B.1: Feedback interconnection of passive systems

Corollary B.4.3 *Assume that (B.15) is passive with respect to a positive definite and decrescent storage function $V(t,x)$, that is, such that*

$$W_1(x) \leq V(t,x) \leq W_2(x)$$

for all $t \in \mathbb{R}_{\geq 0}$ and all $x \in \mathbb{R}^n$, for some positive definite functions $W_1(\cdot)$, $W_2(\cdot)$. Then, the equilibrium $x = 0$ of the unforced system (that is, when $u = 0$) is uniformly stable.

Corollary B.4.4 *Assume that (B.15) is strictly passive with respect to a positive definite, decrescent, and radially unbounded storage function $V(t,x)$, that is, such that*

$$\gamma_1(|x|) \leq V(t,x) \leq \gamma_2(|x|)$$

for all $t \in \mathbb{R}_{\geq 0}$ and all $x \in \mathbb{R}^n$, for some class-$\mathcal{K}_\infty$ functions $\gamma_1(\cdot)$ and $\gamma_2(\cdot)$. Then, the equilibrium $x = 0$ of the unforced system (that is, when $u = 0$) is uniformly globally asymptotically stable.

Among the desirable properties of passive systems, one of the most useful is the fact that passivity is preserved under negative feedback interconnection. Specifically, let systems Σ_1 and Σ_2 be described respectively by

$$\Sigma_1 : \begin{cases} \dot{x}_1 &= f_1(t,x_1) + g_1(t,x_1)u_1 \\ y_1 &= h_1(t,x_1) \end{cases}$$

$$\Sigma_2 : \begin{cases} \dot{x}_2 &= f_2(t,x_2) + g_2(t,x_2)u_2 \\ y_2 &= h_2(t,x_2) \end{cases}$$

where $x_1 \in \mathbb{R}^{n_1}$, $x_2 \in \mathbb{R}^{n_2}$, $u_1 \in \mathbb{R}^m$, $u_2 \in \mathbb{R}^m$, $y_1 \in \mathbb{R}^m$, and $y_2 \in \mathbb{R}^m$. Consider the negative feedback interconnection of Σ_1 and Σ_2, defined by the relations

$$\begin{aligned} u_1 &= -y_2 + u \\ u_2 &= y_1 \\ y &= y_1 \end{aligned} \tag{B.16}$$

where u and y are the overall input and output of the feedback system (see Figure B.1).

Proposition B.4.5 *Assume that Σ_1 is passive with storage function $V_1(t, x_1)$, and that Σ_2 is passive with storage function $V_2(t, x_2)$. Then, the negative feedback interconnection defined by (B.16) is passive, with storage function $V(t, x_1, x_2) = V_1(t, x_1) + V_2(t, x_2)$. If both subsystems are strictly passive, with dissipation rates given by $\alpha_1(x_1)$ and $\alpha_2(x_2)$ respectively, then the feedback interconnection defined by (B.16) is strictly passive, with storage function $V(t, x_1, x_2) = V_1(t, x_1) + V_2(t, x_2)$ and dissipation rate $\alpha(x_1, x_2) = \alpha_1(x_1) + \alpha_2(x_2)$.*

Proposition B.4.6 *Assume that Σ_1 is strictly passive with positive definite, decrescent and radially unbounded storage function $V_1(t, x_1)$ and dissipation rate $\alpha_1(x_1)$. Let Σ_2 be passive with positive definite and decrescent storage function $V_2(t, x_2)$. Then, when $u = 0$, the negative feedback interconnection defined by (B.16) has a uniformly stable equilibrium at the origin $(x_1, x_2) = (0, 0)$. Moreover, if $V_2(t, x_2)$ is radially unbounded, then all trajectories are uniformly bounded, and satisfy $\lim_{t\to\infty} x_1(t) = 0$.*

For LTI systems of the form

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned} \tag{B.17}$$

with $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, and $y \in \mathbb{R}^m$, the following result applies.

Proposition B.4.7 *Consider system* (B.17). *Suppose there exist a symmetric positive definite matrix $P \in \mathbb{R}^{n\times n}$ and a symmetric positive semi-definite matrix $Q \in \mathbb{R}^{n\times n}$ such that*

$$A^T P + PA \leq -Q$$

$$PB = C^T .$$

Then, system (B.17) *is passive, and the pair (C, A) is detectable if and only if the pair (A, B) is stabilizable. If, in addition, $Q > 0$, then the system is strictly passive.*

Finally, we recall some useful results for LTI SISO systems with strictly proper transfer function. The reader is referred to [28, Section 3.5] and [20, Chapter 6] for further details. Consider again system (B.17), assume $u \in \mathbb{R}$, $y \in \mathbb{R}$, and let $G(s) = C(sI - A)^{-1}B$ denote its transfer function.

Definition B.4.8 *A rational proper transfer function $G(s)$ is called* positive real *(PR) if*

(i) $G(s)$ *is real for real s.*

(ii) $\mathrm{Re}[G(s)] \geq 0$ *for all* $\mathrm{Re}[s] > 0$.

Furthermore, assume that $G(s)$ is not identically zero. Then, $G(s)$ is called strictly positive real *(SPR) if $G(s - \epsilon)$ is positive real for some $\epsilon > 0$.*

Lemma B.4.9 *A rational proper transfer function $G(s)$ is PR if and only if*

(i) $G(s)$ *is real for real s.*

(ii) $G(s)$ *is analytic in* $\mathrm{Re}[s] > 0$, *and the poles on the $j\omega$-axis are simple and such that the associated residues are real and positive.*

(iii) For all real value ω for which $s = j\omega$ is not a pole of $G(s)$, one has $\mathrm{Re}[G(j\omega)] \geq 0$.

For a proof, see [28, Lemma 3.5.1]. The connection between (strict) positive realness of $G(s)$ and (strict) passivity of the realization (B.17) is given by the celebrated KYP lemma, and its subsequent variations:

Lemma B.4.10 (Kalman, Yakubovich, Popov) *Assume that* (B.17) *is a minimal realization of $G(s)$. Then, $G(s)$ is PR if and only if there exist a symmetric positive definite matrix $P \in \mathbb{R}^{n\times n}$ and a vector $q \in \mathbb{R}^n$ such that*

$$\begin{aligned} &A^T P + PA = -qq^T \\ &PB = C^T \,. \end{aligned}$$

Lemma B.4.11 (Meyer, Lefschetz, Kalman, Yakubovich) *A necessary condition for the transfer function $G(s) = C(sI-A)^{-1}B$ to be SPR is that for any positive definite matrix $L \in \mathbb{R}^{n\times n}$ there exist a symmetric positive definite matrix $P \in \mathbb{R}^{n\times n}$, a scalar $\nu > 0$ and a vector $q \in \mathbb{R}^n$ such that*

$$\begin{aligned} &A^T P + PA = -qq^T - \nu L \\ &PB = C^T \,. \end{aligned}$$

If (B.17) *is a minimal realization of $G(s)$, the above condition is also sufficient.*

The KYP and MLKY lemmas imply that for a minimal realization of a SISO system, positive realness of $G(s)$ (respectively, strictly positive realness) is equivalent to passivity (respectively, strict passivity). In case the realization is not minimal, but the matrix A is Hurwitz, strict positive realness implies strict passivity.

B.5 Input-Output Stability of Nonlinear Systems

B.5.1 Definitions

In this lecture, we consider systems of nonlinear ODEs that depend on external input functions and are equipped with output functions:

$$\begin{aligned} \dot{x} &= f(x,u)\,, \qquad x(0) = x_0 \\ y &= h(x) \end{aligned}$$

where

- $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is a $\mathcal{C}^k$ vector field, $k \geq 1$
- $h : \mathbb{R}^n \to \mathbb{R}^p$ is a $\mathcal{C}^k$ map, $k \geq 1$
- $u(\cdot) \in \mathcal{C}^0_{[0,\infty)}(\mathbb{R}^m)$ is the *input function*
- $y(\cdot) \in \mathcal{C}^1_{[0,T_{\max})}(\mathbb{R}^p)$ is the *output function*[3]

It is assumed that $x = 0$ is an equilibrium on the 0-input systems, i.e., $f(0,0) = 0$. Recall the following **norms for signals** $u(\cdot) \in \mathcal{C}^0_{[0,\infty)}(\mathbb{R}^m)$:

- For $p \in \{1, 2, \ldots\}$:
$$\|u(\cdot)\|_p := \left(\int_0^\infty |u(\tau)|^p \, \mathrm{d}\tau \right)^{\frac{1}{p}}$$
- For $p = \infty$:
$$\|u(\cdot)\|_\infty := \sup_{t \geq 0} |u(t)|$$

and, for $p \in \{1, 2, \ldots, \infty\}$, the **function spaces**[4]

$$\mathcal{L}_p := \left\{ u(\cdot) \in \mathcal{C}^0_{[0,\infty)}(\mathbb{R}^m) : \|u(\cdot)\|_p < \infty \right\}$$

Let $u : [0,\infty) \to \mathbb{R}^m$ be a signal, and let $T \geq 0$. The **truncation** or **truncated signal** $u_T : [0,\infty) \to \mathbb{R}^m$ is defined as follows:

$$u_T(t) = \begin{cases} u(t) & t \in [0,T) \\ 0 & t \geq T \end{cases}$$

$$\mathcal{L}_{p,e} := \{ u : [0,\infty) \to \mathbb{R}^m : u_T(\cdot) \in \mathcal{L}_p \quad \forall T \geq 0 \}\,, \qquad p \in \{1, 2, \ldots, \infty\}$$

Note that:

- $\mathcal{L}_p \subset \mathcal{L}_{p,e}\,, \quad p \in \{1, 2, \ldots, \infty\}$

[3] The function $y : \mathbb{R} \to \mathbb{R}^p$ is defined on the maximal interval $[0, T_{\max})$ where the forward trajectory $x(\cdot, x_0, u(\cdot))$ exists.

[4] The assumption $u(\cdot) \in \mathcal{C}^0_{[0,\infty)}(\mathbb{R}^m)$ is not necessary. For example, one can replace $\mathcal{C}^0_{[0,\infty)}(\mathbb{R}^m)$ with $\mathcal{PC}^0_{[0,\infty)}(\mathbb{R}^m)$, the space of piece-wise continuous functions.

- $\|u(\cdot)\|_p = \lim_{T\to\infty} \|u_T(\cdot)\|_p$

Example B.5.2 *The signal $u : t \mapsto \mathrm{e}^t$ satisfies*

$$u(\cdot) \in \mathcal{L}_{\infty,e} \text{ and } u(\cdot) \notin \mathcal{L}_\infty$$

Definition B.5.3 ($\mathcal{L}_p$-stability) *Let $p \in \{1, 2, \dots, \infty\}$. The system*

$$\begin{aligned} \dot{x} &= f(x,u)\,, \qquad x(0) = x_0 \\ y &= h(x) \end{aligned} \tag{B.18}$$

is said to be:

1. **$\mathcal{L}_p$-stable** *if for all $x_0 \in \mathbb{R}^n$*

$$u(\cdot) \in \mathcal{L}_p \implies y(\cdot) \in \mathcal{L}_p$$

2. **Finite $\mathcal{L}_p$-gain stable** *if there exist $\gamma > 0$ and, for each $x_0 \in \mathbb{R}^n$, a number $b(x_0) \geq 0$ such that for all $u(\cdot) \in \mathcal{L}_{p,e}$*

$$\|y_T(\cdot)\|_p \leq \gamma \|u_T(\cdot)\|_p + b(x_0) \qquad \forall\, T \geq 0$$

3. **Finite $\mathcal{L}_p$-gain stable with zero bias** *if it is finite $\mathcal{L}_p$-gain stable and $b(x_0) = 0$ for all $x_0 \in \mathbb{R}^n$.*

Note the following:

- If the system is finite $\mathcal{L}_p$-gain stable, then $u(\cdot) \in \mathcal{L}_{p,e} \implies y(\cdot) \in \mathcal{L}_{p,e}$
- If the system is finite $\mathcal{L}_p$-gain stable, it is $\mathcal{L}_p$-stable

Assume that the system is finite $\mathcal{L}_p$-gain stable. The **$\mathcal{L}_p$-gain of the system** is defined as

$$\begin{aligned} \gamma_p := \inf\{\gamma > 0 : \exists\, b \geq 0 \text{ such that } \quad & \|y_T(\cdot)\|_p \leq \gamma \|u_T(\cdot)\|_p + b(x_0) \\ & \forall\, u(\cdot) \in \mathcal{L}_{p,e}\ \forall\, T \geq 0\} \end{aligned}$$

Note that, in general, only an upper bound of γ_p can be computed. The question is to determine whether a system is finite $\mathcal{L}_p$-stable. For LTI systems, the answer is remarkably simple:

Proposition B.5.4 (LTI Systems) *Assume that the linear system*

$$\begin{aligned} \dot{x} &= Ax + Bu\,, \qquad x(0) = x_0 \\ y &= Cx \end{aligned}$$

is **internally stable***, that is,* $\operatorname{spec} A \subset \mathbb{C}^-$*. Then, the system is finite $\mathcal{L}_2$-gain and finite $\mathcal{L}_\infty$-gain stable.*

Proof.

Finite $\mathcal{L}_2$-gain stability: Let $P = P^T \succ 0$ be the unique solution of the Algebraic Lyapunov Equation

$$A^T P + PA = -I$$

and consider the Lyapunov function candidate $V(x) = x^T P x$. Then

$$\dot{V}(x) = -x^T x + 2x^T PBu$$

Select, arbitrarily, $x_0 \in \mathbb{R}^n$ and $u(\cdot) \in \mathcal{L}_{\infty,e}$, and let $V(t) := V(x(t))$, where

$$x(t) = \mathrm{e}^{At}x_0 + \int_0^t \mathrm{e}^{A(t-\tau)}Bu(\tau)\mathrm{d}\tau$$

Then

$$\begin{aligned} \frac{\mathrm{d}V(t)}{\mathrm{d}t} &\leq -\left|x(t)\right|_2^2 + 2\left|x(t)^T PBu(t)\right|_2 \\ &\leq -\left|x(t)\right|_2^2 + \underbrace{2\left|PB\right|_2}_{=:\mu}\left|x(t)\right|_2\left|u(t)\right|_2 \qquad t \geq 0 \end{aligned}$$

Using Young's inequality:

$$a, b \geq 0 \implies ab \leq \lambda a^2 + \frac{1}{4\lambda}b^2, \ \forall\, \lambda > 0$$

one obtains (setting $\lambda = \frac{1}{2}$)

$$\frac{\mathrm{d}V(t)}{\mathrm{d}t} \leq -\left|x(t)\right|_2^2 + \frac{1}{2}\left|x(t)\right|_2^2 + \frac{\mu^2}{2}\left|u(t)\right|_2^2 = -\frac{1}{2}\left|x(t)\right|_2^2 + \frac{\mu^2}{2}\left|u(t)\right|_2^2$$

Integrating both sides on $[0, T]$, with $T \geq 0$ arbitrary, yields

$$\begin{aligned} &\int_0^T \frac{\mathrm{d}V(t)}{\mathrm{d}t}\mathrm{d}t \leq -\frac{1}{2}\int_0^T \left|x(t)\right|_2^2 \mathrm{d}t + \frac{\mu^2}{2}\int_0^T \left|u(t)\right|_2^2 \mathrm{d}t \\ &\implies V(x(T)) - V(x_0) \leq -\frac{1}{2}\|x_T(\cdot)\|_2^2 + \frac{\mu^2}{2}\|u_T(\cdot)\|_2^2 \\ &\implies \|x_T(\cdot)\|_2^2 \leq 2V(x_0) - 2V(x(T)) + \mu^2\|u_T(\cdot)\|_2^2 \\ &\implies \|x_T(\cdot)\|_2 \leq \sqrt{2V(x_0)} + \mu\|u_T(\cdot)\|_2^2 \end{aligned}$$

Since

$$\|y_T(\cdot)\|_2 = \|Cx_T(\cdot)\|_2 \leq |C|\,\|x_T(\cdot)\|_2$$

one obtains

$$\|y_T(\cdot)\|_2 \leq |C|\sqrt{2V(x_0)} + \mu\,|C|\,\|u_T(\cdot)\|_2^2$$

Finite $\mathcal{L}_\infty$-gain stability: Since $\operatorname{spec} A \subset \mathbb{C}^-$, there exist $\kappa, \lambda > 0$ such that

$$\|\mathrm{e}^{At}\| \leq \kappa\,\mathrm{e}^{-\lambda t} \qquad \forall\, t \geq 0$$

Select, arbitrarily, $x_0 \in \mathbb{R}^n$, $u(\cdot) \in \mathcal{L}_{\infty,e}$ and $T \geq 0$. Then, for all $t \in [0, T]$

$$\begin{aligned} |x(t)| &\leq \left|\mathrm{e}^{At} x_0\right| + \left| \int_0^t \mathrm{e}^{A(t-\tau)} B u(\tau) \mathrm{d}\tau \right| \\ &\leq \left|\mathrm{e}^{At}\right| |x_0| + |B| \int_0^t \left|\mathrm{e}^{A(t-\tau)}\right| |u(\tau)| \, \mathrm{d}\tau \end{aligned}$$

Since for all $\tau \in [0, T]$

$$|u(\tau)| \leq \max_{\tau \in [0,T]} |u(\tau)| = \|u_T(\cdot)\|_\infty$$

one obtains

$$|x(t)| \leq \left|\mathrm{e}^{At}\right| |x_0| + |B| \int_0^t \left|\mathrm{e}^{A(t-\tau)}\right| \mathrm{d}\tau \, \|u_T(\cdot)\|_\infty \qquad \forall\, t \in [0, T]$$

Therefore

$$\begin{aligned} |x(t)| &\leq \left|\mathrm{e}^{At}\right| |x_0| + |B| \int_0^t \left|\mathrm{e}^{A(t-\tau)}\right| \mathrm{d}\tau \, \|u_T(\cdot)\|_\infty \\ &\leq \kappa \, \mathrm{e}^{-\lambda t} |x_0| + \kappa \, |B| \int_0^t \mathrm{e}^{-\lambda(t-\tau)} \mathrm{d}\tau \, \|u_T(\cdot)\|_\infty \\ &\leq |x_0| + \frac{\kappa \, |B|}{\lambda} \|u_T(\cdot)\|_\infty \end{aligned}$$

hence

$$\|y_T(\cdot)\|_\infty \leq |C| \max_{t \in [0,T]} |x(t)| \leq |C| \, |x_0| + \frac{\kappa \, |B| \, |C|}{\lambda} \|u_T(\cdot)\|_\infty$$

□

B.5.5 Dissipative Systems

Let us go back to the nonlinear system (B.18). Define:[5]

- A **supply rate** for system (B.18) as any continuous function $q : \mathbb{R}^m \times \mathbb{R}^p \to \mathbb{R}$ such that $q(0, 0) = 0$ and $q(0, y) \leq 0$ for all $y \in \mathbb{R}^p$.

- A **storage function** for system (B.18) as any continuously differentiable function $V : \mathbb{R}^n \to \mathbb{R}$ satisfying

  $$\underline{\alpha}(|x|) \leq V(x) \leq \overline{\alpha}(|x|) \qquad \forall\, x \in \mathbb{R}^n$$

  for class-$\mathcal{K}_\infty$ functions $\underline{\alpha}(\cdot)$, $\overline{\alpha}(\cdot)$.

Definition B.5.6 (Dissipativity) *System* (B.18) *is said to be* **dissipative** *with respect to the supply rate $q(\cdot, \cdot)$ if there exists a storage function $V(\cdot)$ such that*

$$\dot{V}(x) = \frac{\partial V}{\partial x} f(x, u) \leq q(u, h(x)) \qquad \forall (x, u) \in \mathbb{R}^n \times \mathbb{R}^m$$

[5] These definitions are not the most general, but are the most suitable for our analysis tools.

Definition B.5.7 (Strict Dissipativity) *System* (B.18) *is said to be* **strictly dissipative** *with respect to the supply rate* $q(\cdot,\cdot)$ *if there exist a storage function* $V(\cdot)$ *and a class-*$\mathcal{K}$ *function* $\alpha(\cdot)$ *such that*

$$\dot{V}(x) = \frac{\partial V}{\partial x} f(x,u) \leq -\alpha(|x|) + q(u, h(x)) \qquad \forall (x,u) \in \mathbb{R}^n \times \mathbb{R}^m$$

Note that, with respect to the 0-input system

$$\begin{aligned} \dot{x} &= f(x,0)\,, \qquad x(0) = x_0 \\ y &= h(x) \end{aligned}$$

1. Dissipativity $\implies$ stability of $x = 0$ for the 0-input system
2. Strict dissipativity $\implies$ global asymptotic stability of $x = 0$ for the 0-input system

Definition B.5.8 (Zero-state detectability) *System* (B.18) *is said to be* zero-state detectable (ZSD) *if for all* $x_0 \in \mathbb{R}^n$ *such that the solution* $x^\star(\cdot, x_0)$ *of the* 0*-input system*

$$\begin{aligned} \dot{x} &= f(x,0)\,, \qquad x(0) = x_0 \\ y &= h(x) \end{aligned} \tag{B.19}$$

is forward complete, the condition $h(x^\star(t,x_0)) = 0$ *for all* $t \geq 0$ *implies*

$$\lim_{t\to\infty} |x^\star(t,x_0)| = 0$$

Theorem B.5.9 (LaSalle's Theorem for dissipative systems) *Assume that a supply rate* $q(\cdot,\cdot)$ *satisfies* $q(0,y) < 0$ *for all* $y \in \mathbb{R}^p - \{0\}$*. Then, dissipativity with respect to* $q(\cdot,\cdot)$ *and zero-state detectability implies that the equilibrium* $x = 0$ *of the* 0*-input system* (B.19) *is GAS.*

Proof: Similar to the proof of LaSalle's Invariance Principle.

Theorem B.5.10 (Finite $\mathcal{L}_2$-gain stability of dissipative systems) *Assume that system* (B.18) *is dissipative with respect to the supply rate*

$$q(u,y) = \kappa\,|u|^2 - \lambda\,|y|^2\,, \qquad \kappa, \lambda > 0$$

Then, the system is finite $\mathcal{L}_2$*-gain stable.*

Proof: Let $V : \mathbb{R}^n \to \mathbb{R}$ be a storage function such that

$$\dot{V}(x,u) = \frac{\partial V}{\partial x} f(x,u) \leq \kappa\,|u|^2 - \lambda\,|h(x)|^2 \qquad \forall\,(x,u) \in \mathbb{R}^n \times \mathbb{R}^m$$

For simplicity, rescale $V(\cdot)$ as $W(x) := \lambda^{-1} V(x)$, and let $\gamma := \sqrt{\kappa/\lambda}$ to obtain

$$\dot{W}(x,u) \leq \gamma^2\,|u|^2 - |h(x)|^2 \qquad \forall\,(x,u) \in \mathbb{R}^n \times \mathbb{R}^m$$

Notice that dissipativity with respect to $q(u,y) = \kappa\,|u|^2 - \lambda\,|y|^2$ implies dissipativity with respect to $\bar{q}(u,v) = \gamma^2\,|u|^2 - |y|^2$, with $\gamma := \sqrt{\kappa/\lambda}$.

Fix, arbitrarily, $x_0 \in \mathbb{R}^n$ and $u(\cdot) \in \mathcal{L}_{2,e}$, and let $x(\cdot) := x(\cdot, x_0, u(\cdot))$ be the corresponding solution of the system.

Claim: The solution $x(\cdot)$ exists uniquely over $[0, \infty)$. Let $[0, T_{\max})$ denote the maximal interval of existence and uniqueness of the forward solution. Along this solution

$$\begin{aligned}\frac{\mathrm{d}W(x(t))}{\mathrm{d}t} = \dot{W}(x(t), u(t)) &\leq \gamma^2 |u(t)|^2 - |h(x(t))|^2 \qquad \forall\, t \in [0, T_{\max}) \\ &\leq \gamma^2 |u(t)|^2 \qquad \forall\, t \in [0, T_{\max})\end{aligned}$$

therefore

$$\begin{aligned}W(x(t)) &\leq W(x_0) + \gamma^2 \int_0^t |u(\tau)|^2 \, \mathrm{d}\tau \qquad \forall\, t \in [0, T_{\max}) \\ &\leq W(x_0) + \gamma^2 \int_0^{T_{\max}} |u(\tau)|^2 \, \mathrm{d}\tau =: M_0 \qquad \forall\, t \in [0, T_{\max})\end{aligned}$$

where $0 \leq M_0 < \infty$, since $u(\cdot) \in \mathcal{L}_{2,e}$. Using the fact that

$$\underline{\alpha}(|x|) \leq W(x) \qquad \forall\, x \in \mathbb{R}^n$$

for some class-$\mathcal{K}_\infty$ function $\underline{\alpha}(\cdot)$, one obtains

$$|x(t)| \leq \underline{\alpha}^{-1}(M_0) \qquad \forall\, t \in [0, T_{\max})$$

As a result

$$\limsup_{t \to T_{\max}} |x(t)| < \infty \implies T_{\max} = \infty$$

$\diamond$

Going back to the proof of the theorem, notice that we have established that

$$\begin{aligned}&\frac{\mathrm{d}W(x(t))}{\mathrm{d}t} \leq \gamma^2 |u(t)|^2 - |h(x(t))|^2 \qquad \forall\, t \geq 0 \\ \implies &\int_0^t |h(x(\tau))|^2 \, \mathrm{d}\tau \leq \gamma^2 \int_0^t |u(\tau)|^2 \, \mathrm{d}\tau + W(x_0) \qquad \forall\, t \geq 0\end{aligned}$$

Note that

$$\begin{aligned}\int_0^t |h(x(\tau))|^2 \, \mathrm{d}\tau &\leq \gamma^2 \int_0^t |u(\tau)|^2 \, \mathrm{d}\tau + W(x_0) \qquad \forall\, t \geq 0 \\ &\implies \|y_T\|_2^2 \leq \gamma^2 \|u_T\|_2^2 + W(x_0) \qquad \forall\, T \geq 0 \\ &\implies \|y_T\|_2 \leq \gamma \|u_T\|_2 + \sqrt{W(x_0)} \qquad \forall\, T \geq 0\end{aligned}$$

$\square$

To prove that

$$\sqrt{\gamma^2 \|u_T\|_2^2 + W(x_0)} \leq \gamma \|u_T\|_2 + \sqrt{W(x_0)}$$

let $v, w \in \mathbb{R}^2$ be defined as

$$v := (\gamma \|u_T\|_2 \;\; 0)^T, \quad w := (0 \;\; \sqrt{W(x_0)})^T$$

Then, use the triangle inequality $|v + w|_2 \leq |v|_2 + |w|_2$, being

$$|v + w|_2 = \sqrt{\gamma^2 \|u_T\|_2^2 + W(x_0)}, \quad |v|_2 = \gamma \|u_T\|_2, \quad |w|_2 = \sqrt{W(x_0)}$$

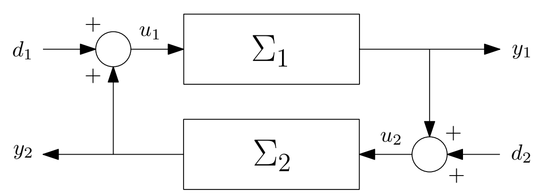
B.5.11 The Small-Gain Theorem for Finite $\mathcal{L}_2$-gain Systems

We present a version of one of the most important results in systems theory, the so-called *Small-Gain Theorem.* Small-gain theorems arise when studying **feedback interconnections** of systems, occurring naturally in feedback control.

Consider the $\mathcal{C}^1$ systems

$$\Sigma_1 : \begin{cases} \dot{x}_1 = f_1(x_1, u_1), & x_1(0) = x_{10} \\ y_1 = h_1(x_1) \end{cases} \qquad \Sigma_2 : \begin{cases} \dot{x}_2 = f_2(x_2, u_2), & x_2(0) = x_{20} \\ y_2 = h_2(x_2) \end{cases}$$

where $x_1 \in \mathbb{R}^{n_1}$, $x_2 \in \mathbb{R}^{n_2}$, $u_1, y_2 \in \mathbb{R}^{m_1}$, and $u_2, y_1 \in \mathbb{R}^{m_2}$, and their *feedback interconnection*



defined by

$$u_1 := d_1 + y_2, \quad d_1 \in \mathbb{R}^{m_1} \qquad u_2 := d_2 + y_1, \quad d_2 \in \mathbb{R}^{m_2}$$

The feedback interconnection defines an augmented system

$$\begin{aligned} \dot{x} &= f(x, d), \quad x(0) = x_0 \\ y &= h(x) \end{aligned}$$

with aggregate state $x \in \mathbb{R}^n$, overall input $d \in \mathbb{R}^m$ and overall output $y \in \mathbb{R}^m$

$$x := \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \qquad d := \begin{pmatrix} d_1 \\ d_2 \end{pmatrix}, \qquad y := \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

where $n := n_1 + n_2$, $m := m_1 + m_2$, and $x_0 := (x_{10} \;\; x_{20})^T$.

Theorem B.5.12 ($\mathcal{L}_2$ Small-Gain Theorem for Dissipative Systems) *Assume that:*

1. *System Σ_1 is dissipative with respect to $q_1(u_1, y_1) = \gamma_1^2 |u_1|^2 - |y_1|^2$*
2. *System Σ_2 is dissipative with respect to $q_2(u_2, y_2) = \gamma_2^2 |u_2|^2 - |y_2|^2$*

If the compositions of the gains γ_1 and γ_2 form a simple contraction, that is, if

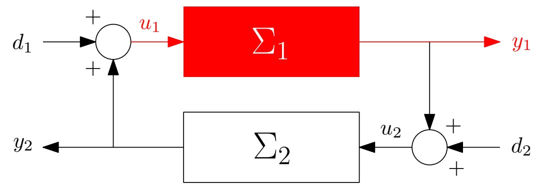
$$\gamma_1 \gamma_2 < 1 \qquad \textit{(small-gain condition)}$$

then the feedback-interconnected system

$$\begin{aligned} \dot{x} &= f(x, d), \quad x(0) = x_0 \\ y &= h(x) \end{aligned}$$

is finite $\mathcal{L}_2$-gain stable.

Proof: Fix, arbitrarily, x_0 and $d(\cdot) \in \mathcal{L}_{2,e}$, and let $[0, T_{\max})$ denote the maximal interval of existence and uniqueness of the corresponding forward solution of the **interconnected** system.



By assumption of dissipativity with respect to $q_1(u_1, y_1) = \gamma_1^2 |u_1|^2 - |y_1|^2$, the Σ_1-subsystem is finite $\mathcal{L}_2$-gain stable. Therefore, for all $T \in [0, T_{\max})$

$$
\begin{aligned}
\|y_{1T}(\cdot)\|_2 &\leq \gamma_1 \|u_{1T}(\cdot)\|_2 + b_1(x_{10}) \\
&\leq \gamma_1 \|d_{1T}(\cdot)\|_2 + \gamma_1 \|y_{2T}(\cdot)\|_2 + b_1(x_{10})
\end{aligned}
$$

Similarly, for the Σ_2-subsystem one obtains, for all $T \in [0, T_{\max})$,

$$
\|y_{2T}(\cdot)\|_2 \leq \gamma_2 \|d_{2T}(\cdot)\|_2 + \gamma_2 \|y_{1T}(\cdot)\|_2 + b_2(x_{20})
$$

Combining the two expressions, one obtains, for all $T \in [0, T_{\max})$,

$$
\begin{aligned}
\|y_{1T}(\cdot)\|_2 \leq \gamma_1 \|d_{1T}(\cdot)\|_2 + \gamma_1\gamma_2 \|y_{1T}(\cdot)\|_2 + \gamma_1\gamma_2 \|d_{2T}(\cdot)\|_2 \\
+ b_1(x_{10}) + \gamma_1 b_2(x_{20})
\end{aligned}
$$

$$
(1 - \gamma_1\gamma_2)\|y_{1T}(\cdot)\|_2 \leq \gamma_1 \|d_{1T}(\cdot)\|_2 + \gamma_1\gamma_2 \|d_{2T}(\cdot)\|_2 + b_1(x_{10}) + \gamma_1 b_2(x_{20})
$$

Since $\gamma_1\gamma_2 < 1$,

$$
\|y_{1T}(\cdot)\|_2 \leq \frac{\gamma_1}{1 - \gamma_1\gamma_2}\left[\|d_{1T}(\cdot)\|_2 + \gamma_2 \|d_{2T}(\cdot)\|_2\right] + \frac{1}{1 - \gamma_1\gamma_2}\left[b_1(x_{10}) + \gamma_1 b_2(x_{20})\right]
$$

Similarly,

$$
\|y_{2T}(\cdot)\|_2 \leq \frac{\gamma_1}{1 - \gamma_1\gamma_2}\left[\|d_{2T}(\cdot)\|_2 + \gamma_1 \|d_{1T}(\cdot)\|_2\right] + \frac{1}{1 - \gamma_1\gamma_2}\left[b_2(x_{20}) + \gamma_2 b_1(x_{10})\right]
$$

The above bounds would prove the result if $T_{\max} = \infty$. We are left to show that $T_{\max} = \infty$. By dissipativity of Σ_1, there exists a storage function $V_1 : \mathbb{R}^{n_1} \to \mathbb{R}$ such that

$$
\begin{aligned}
\dot{V}_1(x_1, u_1) &\leq \gamma_1^2 |u_1|^2 - |h_1(x_1)|^2 \\
&\leq \gamma_1^2 |d_1|^2 + \gamma_1^2 |h_2(x_2)|^2 - |h_1(x_1)|^2 \\
&\leq \gamma_1^2 |d_1|^2 + \gamma_1^2 |h_2(x_2)|^2
\end{aligned}
$$

Evaluating both sides of the above inequality along the solution $x(\cdot, x_0, d(\cdot))$, and integrating over $[0, t]$, $t \in [0, T_{\max})$, yields

$$
\begin{aligned}
V_1(x_1(t)) &\leq V_1(x_{10}) + \gamma_1^2 \int_0^t |d_1(\tau)|^2 \, \mathrm{d}\tau + \gamma_1^2 \int_0^t |y_2(\tau)|^2 \, \mathrm{d}\tau \\
&= V_1(x_{10}) + \gamma_1^2 \|d_{1t}(\cdot)\|_2^2 + \gamma_1^2 \|y_{2t}(\cdot)\|_2^2
\end{aligned}
$$

Note that $\|d_{1t}(\cdot)\|_2 < \infty$ for all $t \geq 0$, and

$$\|y_{2t}(\cdot)\|_2 \leq \frac{\gamma_2}{1-\gamma_1\gamma_2}\big[\|d_{2t}(\cdot)\|_2 + \gamma_1\|d_{1t}(\cdot)\|_2\big] + \frac{1}{1-\gamma_1\gamma_2}\big[b_2(x_{20}) + \gamma_2 b_1(x_{10})\big]$$

Consequently, for all $t \in [0, T_{\max})$

$$\begin{aligned}\|y_{2t}(\cdot)\|_2 \leq \frac{\gamma_2}{1-\gamma_1\gamma_2}&\big[\|d_{2T_{\max}}(\cdot)\|_2 + \gamma_1\|d_{1T_{\max}}(\cdot)\|_2\big]\\ &+ \frac{1}{1-\gamma_1\gamma_2}\big[b_2(x_{20}) + \gamma_2 b_1(x_{10})\big] < \infty\end{aligned}$$

As a result

$$\limsup_{t\to T_{\max}} V_1(x_1(t)) < \infty \implies x_1(\cdot) \text{ bounded over } [0, T_{\max})$$

Applying the same reasoning to the Σ_2-subsystem yields

$$\limsup_{t\to T_{\max}} V_2(x_2(t)) < \infty \implies x_2(\cdot) \text{ bounded over } [0, T_{\max})$$

hence $T_{\max} = \infty$. □

B.6 Input-to-State Stability

B.6.1 Preliminaries

We consider systems of nonlinear ODEs that depend on external input functions, but we drop the presence of an output function:

$$\dot{x} = f(x,u)\,, \qquad x(0) = x_0 \tag{B.20}$$

use where

- $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is a $\mathcal{C}^k$ vector field, $k \geq 1$
- $u(\cdot) \in \mathcal{C}^0_{[0,\infty)}(\mathbb{R}^m)$ is the *input function*

It is assumed that $x = 0$ is an equilibrium on the 0-input systems, i.e., $f(0,0) = 0$

The fundamental question to ask is the following: Assume that the equilibrium $x = 0$ of the 0-input system

$$\dot{x} = f(x,0)\,, \qquad x(0) = x_0$$

is GAS (that is, the system with input is 0-GAS). Does this imply any bounded-input bounded-state properties (as it is the case for LTI systems)? The answer is **no**, as seen from the following example:

Example B.6.2 *Consider the scalar system*

$$\dot{x} = -x + (x^2+1)u, \qquad x(0) = x_0$$

The system is clearly 0-GAS (actually, 0-GES). However, the system with input does not exhibit any bounded-input bounded state property. To see why this is the case, consider first the selection

$$x_0 = \frac{1}{2}, \qquad u(t) = 1, \quad t \geq 0$$

Note that $u(\cdot) \in \mathcal{L}_\infty$. The forward solution is given by

$$x(t) = \frac{1}{2} + \frac{\sqrt{3}}{2}\tan\left(\frac{\sqrt{3}}{2}t\right)$$

which is maximally defined over $[0, \pi/\sqrt{3})$, hence it explodes in finite time.

As a result, the 0-GAS property (even in its stronger 0-GES form) does not imply any type of $\mathcal{L}_\infty$-stability between the input and the state.

B.6.3 Definitions and Properties

Definition B.6.4 (Input-to-State Stability (E.D. Sontag, 1989)) *System* (B.20) *is said to be* input-to-state stable (ISS) *if it is dissipative*[6] *with respect to the supply rate*

$$q(u,x) = \theta(|u|) - \alpha(|x|)$$

for some class-$\mathcal{K}$ function $\theta(\cdot)$ and some class-$\mathcal{K}_\infty$ function $\alpha(\cdot)$.

[6] Note: Dissipativity may be used, as one defines the dummy output $y = x$.

Recall that this means that there exist a $\mathcal{C}^1$ function $V : \mathbb{R}^n \to \mathbb{R}$ and class-$\mathcal{K}_\infty$ functions $\underline{\alpha}(\cdot)$, $\overline{\alpha}(\cdot)$ such that

$$\underline{\alpha}(|x|) \leq V(x) \leq \overline{\alpha}(|x|) \qquad \forall\, x \in \mathbb{R}^n$$

and

$$\dot{V}(x,u) := \frac{\partial V}{\partial x} f(x,u) \leq \theta(|u|) - \alpha(|x|) \qquad \forall\, (x,u) \in \mathbb{R}^n \times \mathbb{R}^m$$

From the previous dissipation inequality and the fact that $V(\cdot)$ is a radially unbounded PDF, it follows that, setting $u = 0$,

$$\frac{\partial V}{\partial x} f(x,0) \leq -\alpha(|x|) \qquad \forall\, x \in \mathbb{R}^n$$

hence ISS implies 0-GAS

Is the converse statement also true? Not at all, as it is clear from the previous example and the following result:

Theorem B.6.5 (Equivalent definition of ISS) *System* (B.20) *is ISS if and only if there exist a class-$\mathcal{KL}$ function $\beta(\cdot,\cdot)$ and a class-$\mathcal{K}$ function $\sigma(\cdot)$ such that, for all $x_0 \in \mathbb{R}^n$ and all $u(\cdot) \in \mathcal{L}_{\infty,e}$, the forward solution $x(\cdot) = x(\cdot, x_0, u(\cdot))$ satisfies*

$$|x(t)| \leq \beta(|x_0|, t) + \sigma(\|u_t(\cdot)\|_\infty) \qquad \forall\, t \geq 0$$

What are the consequences of the class-$\mathcal{KL}$ / class-$\mathcal{K}$ estimate

$$|x(t)| \leq \beta(|x_0|, t) + \sigma(\|u_t(\cdot)\|_\infty) \qquad \forall\, t \geq 0$$

on the forward solution $x(\cdot) = x(\cdot, x_0, u(\cdot))$?

1. **If $u(\cdot) \in \mathcal{L}_{\infty,e}$, then $x(\cdot)$ is defined over $[0,\infty)$ for all $x_0 \in \mathbb{R}^n$.**

   Let $x(\cdot)$ be maximally defined on $[0, T_{\max})$. Then

   $$\begin{aligned} |x(t)| &\leq \beta(|x_0|, t) + \sigma(\|u_t(\cdot)\|_\infty) \qquad \forall\, t \in [0, T_{\max}) \\ &\leq \beta(|x_0|, 0) + \sigma(\|u_{T_{\max}}(\cdot)\|_\infty) \qquad \forall\, t \in [0, T_{\max}) \end{aligned}$$

   hence

   $$\limsup_{t \to T_{\max}} |x(t)| < \infty \implies T_{\max} = \infty$$

2. **ISS systems are $\mathcal{L}_\infty$-stable.**

   Fix, arbitarily, $x_0 \in \mathbb{R}^n$ and $u(\cdot) \in \mathcal{L}_\infty$. Then, for all $t \geq 0$

   $$\begin{aligned} |x(t)| &\leq \beta(|x_0|, t) + \sigma(\|u_t(\cdot)\|_\infty) \\ &\leq \beta(|x_0|, 0) + \sigma(\|u(\cdot)\|_\infty) \end{aligned}$$

   therefore

   $$\sup_{t \geq 0} |x(t)| \leq \beta(|x_0|, 0) + \sigma(\|u(\cdot)\|_\infty) < \infty$$

Note also that

$$\begin{aligned}\|x(\cdot)\|_\infty &\le \beta(|x_0|,0)+\sigma(\|u(\cdot)\|_\infty)\\ &\le 2\max\{\beta(|x_0|,0),\ \sigma(\|u(\cdot)\|_\infty)\}\end{aligned}$$

hence

$$\|x(\cdot)\|_\infty \le \max\{\gamma_0(|x_0|),\ \gamma(\|u(\cdot)\|_\infty)\} \tag{B.21}$$

where $\gamma_0(\cdot) := 2\beta(\cdot,0)$ and $\gamma(\cdot) := 2\sigma(\cdot)$ are class-$\mathcal{K}$ functions.

3. **The state of an ISS system satisfies an asymptotic bound**.

   Fix, arbitrarily, $x_0 \in \mathbb{R}^n$ and $u(\cdot) \in \mathcal{L}_\infty$. Then

$$|x(t)| \le \beta(|x_0|,t)+\sigma(\|u_t(\cdot)\|_\infty) \qquad \forall\, t \ge 0$$

$$\limsup_{t\to\infty}|x(t)| \le \limsup_{t\to\infty}\beta(|x_0|,t)+\limsup_{t\to\infty}\sigma(\|u_t(\cdot)\|_\infty)$$

   Note that

$$\limsup_{t\to\infty}\beta(|x_0|,t)=0$$

   and, since $\sigma(\cdot)$ is continuous and strictly increasing,

$$\begin{aligned}\limsup_{t\to\infty}\sigma\left(\|u_t(\cdot)\|_\infty\right) &= \sigma\left(\limsup_{t\to\infty}\|u_t(\cdot)\|_\infty\right)=\sigma\left(\limsup_{t\to\infty}\sup_{\tau\in[0,t)}|u(\tau)|\right)\\ &= \sigma\left(\limsup_{t\to\infty}|u(t)|\right)\end{aligned}$$

   Defining the **asymptotic norm** of a signal $v : [0,\infty) \to \mathbb{R}^p$ as

$$\|v(\cdot)\|_a := \limsup_{t\to\infty}|v(t)|$$

   one obtains, for all $u(\cdot) \in \mathcal{L}_{\infty,e}$

$$\|x(\cdot)\|_a \le \sigma\left(\|u(\cdot)\|_a\right) \tag{B.22}$$

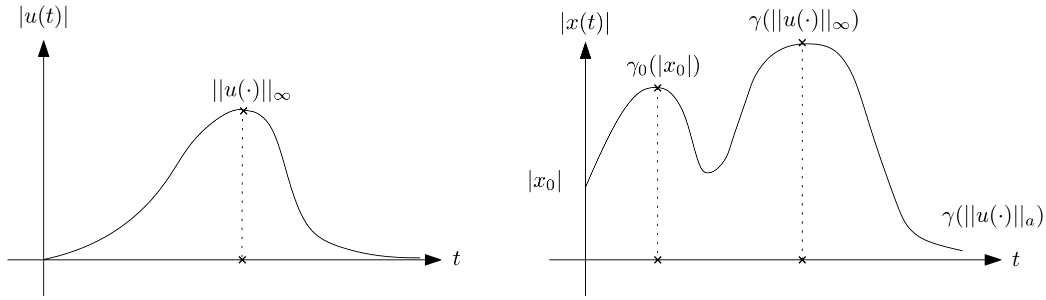
   regardless of the initial condition $x_0 \in \mathbb{R}^n$.

Since $\sigma\left(\|u(\cdot)\|_a\right) \le 2\sigma\left(\|u(\cdot)\|_a\right) = \gamma\left(\|u(\cdot)\|_a\right)$, combining (B.21) and (B.22) yields

$$\begin{cases}\|x(\cdot)\|_\infty \le \max\{\gamma_0(|x_0|),\ \gamma(\|u(\cdot)\|_\infty)\} & \text{worst-case estimate}\\ \|x(\cdot)\|_a \le \gamma\left(\|u(\cdot)\|_a\right) & \text{asymptotic estimate}\end{cases}$$

for all $x_0 \in \mathbb{R}^n$ and all $u(\cdot) \in \mathcal{L}_\infty$.

$$\begin{cases}\|x(\cdot)\|_\infty \le \max\{\gamma_0(|x_0|),\ \gamma(\|u(\cdot)\|_\infty)\} & \text{worst-case estimate}\\ \|x(\cdot)\|_a \le \gamma\left(\|u(\cdot)\|_a\right) & \text{asymptotic estimate}\end{cases}$$



For ISS systems, either the effect of the initial condition x_0 of that of the input may dominate the state response initially. The effect of x_0 is "forgotten" as $t \to \infty$, and the long-term behavior of the norm of the state becomes function of that of the input.

B.6.6 Alternative Definition of ISS

An alternative definition of ISS is due to A. R. Teel (1995). This characterization is more suitable to analyze interconnections of systems, and leads – in particular – to a simple version of the small-gain theorem for ISS systems.

Definition B.6.7 (a-$\mathcal{L}_\infty$ Bounds (A.R. Teel, 1995)) *The state $x(\cdot)$ of the system*

$$\dot{x} = f(x, u)\,, \qquad x(0) = x_0$$

is said to satisfy an a-$\mathcal{L}_\infty$ bound with respect to the input $u(\cdot)$ if there exist class-$\mathcal{K}$ functions $\gamma_0(\cdot)$, $\gamma(\cdot)$ (referred to as gain functions*) such that, for all $x_0 \in \mathbb{R}^n$ and all $u(\cdot) \in \mathcal{L}_\infty$*

$$\begin{aligned} \|x(\cdot)\|_\infty &\leq \max\{\gamma_0(|x_0|),\ \gamma(\|u(\cdot)\|_\infty)\} \\ \|x(\cdot)\|_a &\leq \gamma\left(\|u(\cdot)\|_a\right) \end{aligned}$$

Proposition B.6.8 (Equivalence between ISS and a-$\mathcal{L}_\infty$ bound) *A system is ISS if and only if its state satisfies an a-$\mathcal{L}_\infty$ bound wrt the input.*

B.6.9 ISS Lyapunov Functions

One of the advantages of Teel's characterization of ISS versus the original definition of Sontag is that the *nonlinear gains* $\gamma_0(\cdot)$, $\gamma(\cdot)$ can be computed from knowledge of a so-called *ISS-Lyapunov function.*

Definition B.6.10 (ISS-Lyapunov function) *A smooth function $V : \mathbb{R}^n \to \mathbb{R}$ is said to be an* ISS-Lyapunov function *if there exist class-$\mathcal{K}_\infty$ functions $\underline{\alpha}(\cdot)$, $\overline{\alpha}(\cdot)$, and a class-$\mathcal{K}$ function $\chi(\cdot)$ such that:*

$$\underline{\alpha}(|x|) \leq V(x) \leq \overline{\alpha}(|x|) \qquad \forall\, x \in \mathbb{R}^n$$

and

$$\forall\, u \in \mathbb{R}^m \qquad |x| > \chi(|u|) \implies \frac{\partial V}{\partial x} f(x, u) < 0$$

Theorem B.6.11 (Lyapunov characterization of ISS) *A system is input-to-state stable if and only if it admits an ISS-Lyapunov function.*

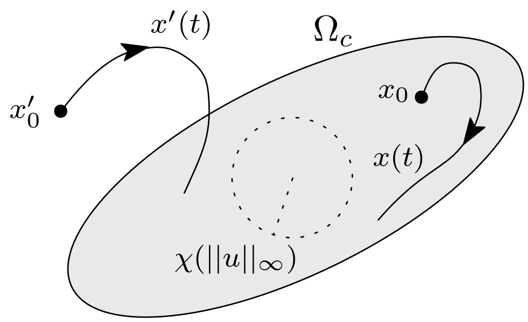
An intuitive explanation of the concept of ISS-Lyapunov function is the following: Fix $u(\cdot) \in \mathcal{L}_\infty$, and let $c > 0$ be such that the level set Ω_c of $V(\cdot)$ satisfies

$$\bar{\mathcal{B}}_{\chi(\|u(\cdot)\|_\infty)} \subset \Omega_c$$

so that

$$\frac{\partial V}{\partial x} f(x,u) < 0 \qquad \forall\, x \in \left(\mathbb{R}^n \setminus \Omega_c\right) \cup rtial\Omega_c, \quad \forall\, u : |u| \leq \|u(\cdot)\|_\infty$$

As a result, the level set Ω_c is forward invariant, and attractive (uniformly on compact sets of initial conditions.)



One important consequence of the availability of an ISS-Lyapunov functions is that the gain functions $\gamma_{10}(\cdot)$ and $\gamma(\cdot)$ of the a-$\mathcal{L}_\infty$ bound can be computed as

$$\gamma_{10}(\cdot) = \underline{\alpha}^{-1} \circ \overline{\alpha}(\cdot), \qquad \gamma_{10}(\cdot) = \underline{\alpha}^{-1} \circ \overline{\alpha} \circ \chi(\cdot)$$

Example B.6.12 *Consider the scalar system*

$$\dot{x} = -x^3 + (1 + \sin^2 x)u$$

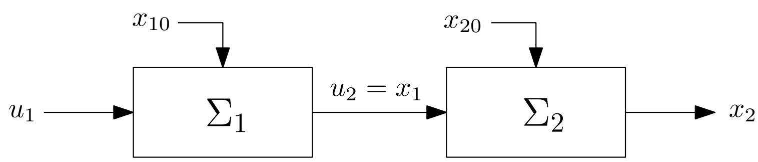
and the Lyapunov function candidate $V(x) = x^2$, *for which* $\underline{\alpha}(s) = \overline{\alpha}(s) = s^2$. *Since*

$$\begin{aligned} \dot{V}(x,u) &= -2x^4 + 2x(1 + \sin^2 x)u \\ &\leq -2\,|x|^4 + 2\,|x|\,\left|1 + \sin^2 x\right|\,|u| \\ &\leq -2\,|x|\,\big[-|x|^3 + 2\,|u|\,\big] \end{aligned}$$

one has $|x| > \sqrt[3]{2\,|u|} =: \chi(|u|) \implies \dot{V} < 0$, *hence the gain functions*

$$\gamma_0(s) = s, \quad \gamma(s) = \sqrt[3]{2s} \qquad s \geq 0$$

B.6.13 ISS of Interconnected Systems



Consider the series interconnection $u_2 = x_1$ of two nonlinear systems

$$\Sigma_1 : \begin{cases} \dot{x}_1 = f_1(x_1, u_1) \\ x_1(0) = x_{10} \end{cases} \qquad \Sigma_2 : \begin{cases} \dot{x}_2 = f_2(x_2, u_2) \\ x_2(0) = x_{20} \end{cases}$$

where $x_1 \in \mathbb{R}^{n_1}$, $x_2 \in \mathbb{R}^{n_2}$, $u_1 \in \mathbb{R}^{m_1}$ and $u_2 \in \mathbb{R}^{n_1}$.

Theorem B.6.14 (Series interconnection of ISS systems) *Assume that Σ_1 and Σ_2 are ISS. Then, the series interconnection*

$$\begin{aligned} \dot{x}_1 &= f_1(x_1, u_1) \\ \dot{x}_2 &= f_2(x_2, x_1) \end{aligned}$$

is ISS with respect to the input u_1.

Proof: Using Teel's characterization of ISS, the two systems satisfy a-$\mathcal{L}_\infty$ bounds of the form:

$$\Sigma_1 : \begin{cases} \|x_1(\cdot)\|_\infty \leq \max\{\gamma_{10}(|x_{10}|),\ \gamma_1(\|u_1(\cdot)\|_\infty)\} \\ \|x_1(\cdot)\|_a \leq \gamma_1\left(\|u_1(\cdot)\|_a\right) \end{cases}$$

$$\Sigma_2 : \begin{cases} \|x_2(\cdot)\|_\infty \leq \max\{\gamma_{20}(|x_{20}|),\ \gamma_2(\|u_2(\cdot)\|_\infty)\} \\ \|x_2(\cdot)\|_a \leq \gamma_2\left(\|u_2(\cdot)\|_a\right) \end{cases}$$

Letting $u_2 = x_1$ and combining the bounds, one obtains for the Σ_2-subsystem

$$\Sigma_2 : \begin{cases} \|x_2(\cdot)\|_\infty \leq \max\{\bar{\gamma}_{10}(|x_{10}|),\ \gamma_{20}(|x_{20}|),\ \bar{\gamma}_1(\|u_1(\cdot)\|_\infty),\ \gamma_2(\|u_2(\cdot)\|_\infty)\} \\ \|x_2(\cdot)\|_a \leq \max\{\bar{\gamma}_1\left(\|u_1(\cdot)\|_a\right),\ \gamma_2\left(\|u_2(\cdot)\|_a\right)\} \end{cases}$$

where

$$\bar{\gamma}_{10}(\cdot) := \gamma_2 \circ \gamma_{10}(\cdot), \qquad \bar{\gamma}_1(\cdot) := \gamma_2 \circ \gamma_1(\cdot)$$

Since for any vector $v = (v_1^T \;\; v_2^T)^T$ it holds that $|v_1| \leq |v|$, $|v_2| \leq |v|$, one obtains

$$\Sigma_1 : \begin{cases} \|x_1(\cdot)\|_\infty \leq \max\{\gamma_{10}(|x_0|),\ \gamma_1(\|u(\cdot)\|_\infty)\} \\ \|x_1(\cdot)\|_a \leq \gamma_1\left(\|u(\cdot)\|_a\right) \end{cases}$$

$$\Sigma_2 : \begin{cases} \|x_2(\cdot)\|_\infty \leq \max\{\bar{\gamma}_{10}(|x_0|),\ \gamma_{20}(|x_0|),\ \bar{\gamma}_1(\|u(\cdot)\|_\infty),\ \gamma_2(\|u(\cdot)\|_\infty)\} \\ \|x_2(\cdot)\|_a \leq \max\{\bar{\gamma}_1\left(\|u(\cdot)\|_a\right),\ \gamma_2\left(\|u(\cdot)\|_a\right)\} \end{cases}$$

Also, since $|v| \leq |v_1| + |v_2| \leq 2\max\{|v_1|, |v_2|\}$, it follows that the augmented state satisfies the a-$\mathcal{L}_\infty$ bound

$$normx(\cdot)_\infty \leq \max\{\tilde{\gamma}_0(|x_0|),\ \tilde{\gamma}(\|u(\cdot)\|_\infty)\}$$
$$\|x(\cdot)\|_a \leq \tilde{\gamma}\left(\|u(\cdot)\|_a\right)$$

where

$$\tilde{\gamma}_0(\cdot) := 2\max\{\gamma_{10}(\cdot), \bar{\gamma}_{10}(\cdot), \gamma_{20}(\cdot)\}, \quad \tilde{\gamma}(\cdot) := 2\max\{\gamma_1(\cdot), \bar{\gamma}_1(\cdot), \gamma_2(\cdot)\}$$

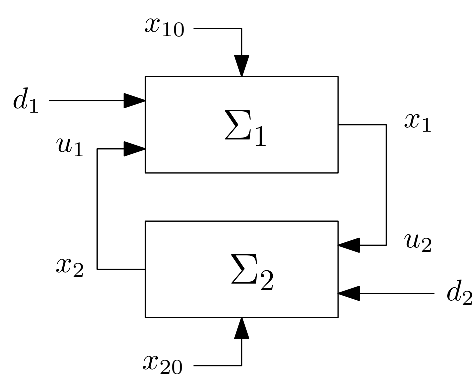
□

We conclude this section with an extremely important result, due to Teel (1995), which provides a small-gain theorem in the a-$\mathcal{L}_\infty$ framework.

Consider the feedback interconnection $u_1 = x_2$, $u_2 = x_1$ of the nonlinear systems

$$\Sigma_1 : \begin{cases} \dot{x}_1 = f_1(x_1, u_1, d_1) \\ x_1(0) = x_{10} \end{cases} \qquad \Sigma_2 : \begin{cases} \dot{x}_2 = f_2(x_2, u_2, d_2) \\ x_2(0) = x_{20} \end{cases}$$

where $x_1, u_2 \in \mathbb{R}^{n_1}$, $x_2, u_1 \in \mathbb{R}^{n_2}$, $d_1 \in \mathbb{R}^{m_1}$ and $d_2 \in \mathbb{R}^{m_d}$.



It is assumed that the two systems satisfy a-$\mathcal{L}_\infty$ bounds of the form

$$\|x_i(\cdot)\|_\infty \leq \max\{\gamma_{i0}(|x_{i0}|),\ \gamma_{i1}(\|u_i(\cdot)\|_\infty),\ \gamma_{i2}(\|d_i(\cdot)\|_\infty)\}$$
$$\|x_i(\cdot)\|_a \leq \max\{\gamma_{i1}\|u_i(\cdot)\|_a),\ \gamma_{i2}\|d_i(\cdot)\|_a)), \qquad i = 1, 2$$

Theorem B.6.15 (Small-gain Theorem for ISS Systems (Teel, 1995)) *Assume that the composition of the* channel-1 gains *of the two systems forms a simple contraction, that is, assume that*

$$\gamma_{11} \circ \gamma_{21}(s) < s \qquad \forall\, s > 0$$

Then, the augmented state $x = (x_1^T \;\; x_2^T)^T$ *of the feedback-interconnected system*

$$\dot{x} = f(x, d), \quad x(0) = x_0$$

satisfies an a-$\mathcal{L}_\infty$ *bound with respect to the overall input* $d := (d_1^T \;\; d_2^T)^T$.

Proof: Fix, arbitrarily, $x_0 = (x_{10}^T \;\; x_{20}^T)^T \in \mathbb{R}^n$ and $d(\cdot) \in \mathcal{L}_\infty$ and let $[0, T_{\max})$ be the maximal interval of existence and uniqueness of the corresponding forward trajectory. Then, for all $\tau \in [0, T_{\max})$ one obtains the bounds

$$\begin{aligned}
\|x_{1\tau}(\cdot)\|_\infty &\le \max\{\gamma_{10}(|x_{10}|),\ \gamma_{11}(\|x_{2\tau}(\cdot)\|_\infty),\ \gamma_{12}(\|d_{1\tau}(\cdot)\|_\infty)\} \\
\|x_{2\tau}(\cdot)\|_\infty &\le \max\{\gamma_{20}(|x_{20}|),\ \gamma_{21}(\|x_{1\tau}(\cdot)\|_\infty),\ \gamma_{22}(\|d_{2\tau}(\cdot)\|_\infty)\}
\end{aligned}$$

Substituting the second bound into the first one yields

$$\begin{aligned}
\|x_{1\tau}(\cdot)\|_\infty \le \max\{&\gamma_{10}(|x_{10}|),\ \bar{\gamma}_{20}(|x_{20}|),\ \bar{\gamma}_{21}(\|x_{1\tau}(\cdot)\|_\infty), \\
&\gamma_{12}(\|d_{1\tau}(\cdot)\|_\infty),\ \bar{\gamma}_{22}(\|d_{2\tau}(\cdot)\|_\infty)\}
\end{aligned}$$

where

$$\bar{\gamma}_{20}(\cdot) := \gamma_{11} \circ \gamma_{20}(\cdot), \quad \bar{\gamma}_{21}(\cdot) := \gamma_{11} \circ \gamma_{21}(\cdot), \quad \bar{\gamma}_{22}(\cdot) := \gamma_{11} \circ \gamma_{22}(\cdot)$$

Lemma B.6.16 *Let $a, b, c, d \in \mathbb{R}$ and consider the inequality*

$$a \le \max\{b, c, d\}$$

If $b < a$ then $\max\{b, c, d\} = \max\{c, d\}$, thus

$$a \le \max\{c, d\}$$

Since

$$\bar{\gamma}_{21}(\|x_{1\tau}(\cdot)\|_\infty) = \gamma_{11} \circ \gamma_{21}(\|x_{1\tau}(\cdot)\|_\infty) < \|x_{1\tau}(\cdot)\|_\infty$$

one can drop this term from the bound for $\|x_{1\tau}(\cdot)\|_\infty$, yielding

$$\begin{aligned}
\|x_{1\tau}(\cdot)\|_\infty &\le \max\{\gamma_{10}(|x_{10}|),\ \bar{\gamma}_{20}(|x_{20}|),\ \gamma_{12}(\|d_{1\tau}(\cdot)\|_\infty),\ \bar{\gamma}_{22}(\|d_{2\tau}(\cdot)\|_\infty)\} \\
&\le \max\{\gamma_{10}(|x_{10}|),\ \bar{\gamma}_{20}(|x_{20}|),\ \gamma_{12}(\|d_1(\cdot)\|_\infty),\ \bar{\gamma}_{22}(\|d_2(\cdot)\|_\infty)\}
\end{aligned}$$

hence

$$\limsup_{\tau \to T_{\max}} \|x_{1\tau}(\cdot)\|_\infty < \infty$$

Repeating the procedure for the bound involving $\|x_{2\tau}(\cdot)\|_\infty$, and noticing that the small-gain condition $\gamma_{11} \circ \gamma_{21}(s) < s$ for all $s > 0$ implies $\gamma_{21} \circ \gamma_{11}(s) < s$ for all $s > 0$, sone obtains

$$\|x_{2\tau}(\cdot)\|_\infty \le \max\{\gamma_{20}(|x_{20}|),\ \bar{\gamma}_{10}(|x_{10}|),\ \gamma_{22}(\|d_2(\cdot)\|_\infty),\ \bar{\gamma}_{12}(\|d_1(\cdot)\|_\infty)\}$$

hence

$$\limsup_{\tau \to T_{\max}} \|x_{2\tau}(\cdot)\|_\infty < \infty$$

Consequently, $T_{\max} = \infty$ and

$$\begin{aligned}
\|x_1(\cdot)\|_\infty &\le \max\{\gamma_{10}(|x_{10}|),\ \bar{\gamma}_{20}(|x_{20}|),\ \gamma_{12}(\|d_1(\cdot)\|_\infty),\ \bar{\gamma}_{22}(\|d_2(\cdot)\|_\infty)\} \\
\|x_2(\cdot)\|_\infty &\le \max\{\gamma_{20}(|x_{20}|),\ \bar{\gamma}_{10}(|x_{10}|),\ \gamma_{22}(\|d_2(\cdot)\|_\infty),\ \bar{\gamma}_{12}(\|d_1(\cdot)\|_\infty)\}
\end{aligned}$$

Furthermore, using the previous reasoning, one obtains

$$\begin{aligned}
\|x_1(\cdot)\|_a &\le \max\{\gamma_{12}(\|d_1(\cdot)\|_a),\ \bar{\gamma}_{22}(\|d_2(\cdot)\|_a)\} \\
\|x_2(\cdot)\|_a &\le \max\{\gamma_{22}(\|d_2(\cdot)\|_a),\ \bar{\gamma}_{12}(\|d_1(\cdot)\|_a)\}
\end{aligned}$$

□

Appendix C

A Primer on Adaptive Systems

C.1 Introduction

In this chapter, we introduce fundamental issues concerning stability of equilibria for classes of systems that arise in direct adaptive control systems design. We start from a few motivating examples, and introduce a typical system structure that we regard as a *standard adaptive control problem.* We then specialize the tools introduced in Chapter B to deal with the stability analysis for the standard problem.

C.1.1 Adaptive Stabilization of Nonlinear Systems in Normal Form

Suppose we are given a parameterized family of nonlinear time-invariant systems of the form

$$\dot{x} = f(x, \mu) + g(x, \mu)u \tag{C.1}$$

with state $x \in \mathbb{R}^n$, control input $u \in \mathbb{R}$ and unknown constant parameter vector $\mu \in \mathbb{R}^q$. We make the usual assumptions on smoothness of the vector fields, and assume that the origin $x = 0$ is an equilibrium of the unforced system, i.e., $f(0, \mu) = 0$ for all $\mu \in \mathbb{R}^q$.

The problem we want to address is the design of controllers of fixed structure that enforces certain properties for the trajectories of the closed-loop system, regardless of the actual value of the unknown parameter vector. The simplest (and most fundamental) problem that can be carved out from the above setup is the design of a (possibly dynamic) state-feedback controller, that is, a system of the form

$$\begin{aligned} \dot{\xi} &= \alpha(\xi, x) \\ u &= \beta(\xi, x) \end{aligned} \tag{C.2}$$

that renders the origin of the closed-loop system (C.1)-(C.2) a *globally uniformly asymptotically stable* equilibrium, *robustly* with respect to μ. Note that it is explicitly assumed that the entire state vector is available for feedback. Clearly, a general solution of the above problem is not available unless more structure is specified for the plant model. In particular, the problem can be considerably simplified if additional properties hold, namely the existence of a globally defined normal form in which the uncertain parameters enter linearly. Specifically, we make the following (quite restrictive) assumption:

Assumption C.1.2 *There exists a globally-defined diffeomorphism*[1] $\Phi : \mathbb{R}^n \to \mathbb{R}^n$, *which preserves the origin, such that the system in the new coordinates* $z = \Phi(x)$ *reads as*

$$\begin{aligned}\dot{z} &= \frac{\partial \Phi}{\partial x} f(\Phi^{-1}(z), \mu) + \frac{\partial \Phi}{\partial x} g(\Phi^{-1}(z), \mu) u \\ &= A_b z + B_b \left[\phi^T(z)\theta + u\right]\end{aligned} \tag{C.3}$$

where A_b, B_b *are in Brunovsky form, i.e.,*

$$A_b = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \quad B_b = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix},$$

the function $\phi(\cdot) : \mathbb{R}^n \to \mathbb{R}^p$ *is known, and* $\theta \in \mathbb{R}^p$, *with* $p \geq q$, *is a re-parametrization of the vector* μ, *that is, a continuous map* $\theta : \mu \mapsto \theta(\mu)$.

If this is the case, if the actual value of the parameter vector θ was available, the obvious memoryless control law that globally asymptotically (and exponentially) stabilizes the origin would be given by

$$u = Kz - \phi^T(z)\theta \tag{C.4}$$

with $K \in \mathbb{R}^{1\times n}$ chosen in such a way that $A_b + B_b K$ is Hurwitz. Since θ is unknown, one may resort to the principle of *certainty equivalence*, and substitute θ in (C.4) with an estimate $\hat{\theta}$, and apply the control

$$u = Kz - \phi^T(z)\hat{\theta}(t)$$

instead. The design must then be completed by a suitable *update law*

$$\dot{\hat{\theta}} = \varphi(\hat{\theta}, z) \tag{C.5}$$

that guarantees stability of the closed-loop system, and, hopefully, convergence of $z(t)$ to the origin, and of $\hat{\theta}(t)$ to θ. To find such an update law, let P be the symmetric, positive definite solution of the Lyapunov matrix equation

$$P(A_b + B_b K) + (A_b + B_b K)^T P = -I$$

and consider the Lyapunov function candidate

$$V(z, \tilde{\theta}) = z^T P z + \tfrac{1}{\gamma}\tilde{\theta}^T\tilde{\theta}$$

where $\gamma > 0$ is a positive constant that plays the role of an *adaptation gain*, and $\tilde{\theta} = \theta - \hat{\theta}$ is a change of coordinates that shifts the origin of the coordinate system for the state of (C.5)

[1] That is, a continuously differentiable map whose inverse exists and is continuously differentiable as well.

to the "true" value of the parameter vector. Evaluating the derivative of V along solutions of (C.3)-(C.4) yields

$$\dot{V}(z,\tilde{\theta}) = -|z|^2 + 2z^T P B_b \phi^T(z)\tilde{\theta} + \tfrac{2}{\gamma}\tilde{\theta}\dot{\tilde{\theta}}$$

from which, keeping in mind that $\dot{\tilde{\theta}} = -\dot{\hat{\theta}}$, one obtains

$$\dot{V}(z,\tilde{\theta}) = -|z|^2 + \tfrac{2}{\gamma}\left[\gamma\phi(z)B_b^T P z - \varphi(\hat{\theta},z)\right].$$

The obvious choice

$$\dot{\hat{\theta}} = \gamma\phi(z)B_b^T P z$$

yields

$$\dot{V}(z,\tilde{\theta}) = -|z|^2\,, \tag{C.6}$$

and this renders the equilibrium $(z,\tilde{\theta}) = (0,0)$ *uniformly globally stable*, as for any initial condition $(z_0,\tilde{\theta}_0) \in \mathbb{R}^n \times \mathbb{R}^p$ the corresponding trajectory of the closed-loop system

$$\begin{aligned} \dot{z} &= (A_b + B_b K)z + B_b\phi^T(z)\tilde{\theta} \\ \dot{\tilde{\theta}} &= -\gamma\phi(z)B_b^T P z \end{aligned} \tag{C.7}$$

satisfies

$$V(z(t),\tilde{\theta}(t)) \le V(z_0,\tilde{\theta}_0)\,, \qquad \forall\, t \ge 0$$

and thus

$$|(z(t),\tilde{\theta}(t))| \le a\,|(z_0,\tilde{\theta}_0)|\,, \qquad \forall\, t \ge 0$$

for some $a > 0$ which depends only on the given choice of the controller parameters K and γ. The asymptotic properties of the trajectories of (C.7), on the other hand, can be determined by a simple application of La Salle's invariance principle, as (C.7) is an *autonomous* system. In particular, trajectories converge to the largest invariant set $\mathcal{M}$ contained in the set $\mathcal{S} = \{(z,\tilde{\theta}) \in \mathbb{R}^n \times \mathbb{R}^p : \ \dot{V} = 0\}$. It is easy to see that any trajectory $(z^\star(t),\tilde{\theta}^\star(t))$ which originates in $\mathcal{M}$ remains in $\mathcal{M}$ for all $t \ge 0$ (recall that (C.7) is forward complete) and satisfies

$$z^\star(t) \equiv 0\,, \qquad \tilde{\theta}^\star(t) = \tilde{\theta}^\star = \text{const}\,.$$

As a result, the set $\mathcal{M}$ is given by

$$\mathcal{M} = \{(z,\tilde{\theta}) \in \mathbb{R}^n \times \mathbb{R}^p : \ z = 0\,,\ \phi^T(0)\tilde{\theta} = 0\}\,.$$

Note that $\mathcal{M}$ is a closed set, but in general not compact. As a matter of fact, the only case in which $\mathcal{M}$ is compact is when $p = 1$ and $\phi(0) \ne 0$, and thus $\mathcal{M} = \{(0,0)\}$. As a result, it is not possible to conclude that the origin is an asymptotically stable equilibrium of (C.7), apart from the rather trivial case discussed above. The only conclusions that can be drawn are the following:

a.) The origin is a (uniformly) globally stable equilibrium of (C.7).

b.) The closed set $\mathcal{M}$ is globally attractive[2].

[2] It is worth noting that convergence to $\mathcal{M}$ is not guaranteed to be uniform, since $\mathcal{M}$ is not compact.

Assuming that $p > 1$, the closed-loop system achieves boundedness of all trajectories and regulation of $z(t)$ in place of global asymptotic stability of the origin in $\mathbb{R}^{n+p}$, which may still seem a reasonable outcome. In the literature, this result is referred to as "partial stabilization," that is, regulation of a certain subset of the state variables to zero, while preserving boundedness of all trajectories. The problem is that this is not enough to guarantee that (C.7) possesses even the mildest form of robustness ensured by the theorem of total stability. In particular, trajectories of (C.7) may grow unbounded in presence of arbitrarily small non-vanishing perturbations, as will be shown later in this chapter.

The question is whether (C.6) can be used to assess uniform global asymptotic stability of the equilibrium set $\mathcal{M}$, as opposed to asymptotic stability of the equilibrium at the origin. As a matter of fact, it is true that the Lyapunov function candidate V admits a class-$\mathcal{K}_\infty$ estimate from below which is a function of the point-to-set distance from $\mathcal{M}$ alone, as

$$V(z,\tilde{\theta}) \geq z^T P z \geq \lambda_{\min}(P)|z|^2 = \lambda_{\min}(P)|(z,\tilde{\theta})|^2_{\mathcal{M}}\,,$$

and that obviously the estimate

$$\dot{V}(z,\tilde{\theta}) \leq -|(z,\tilde{\theta})|^2_{\mathcal{M}}$$

holds for the derivative of V along (C.7). However, the function V does not admit a class-$\mathcal{K}_\infty$ estimate *from above* which is a function of $|z|$ alone, thus missing a *crucial* ingredient in the Lyapunov characterization of global uniform asymptotic stability with respect to a set. The following counterexample shows that, indeed, equation (C.6) does not imply stability of $\mathcal{M}$ in the sense of Lyapunov, and thus, for the system (C.7), a Lyapunov function with respect to the set $\mathcal{M}$ does not exist.

Example C.1.3 Consider the simple problem of global asymptotic stabilization of the origin of the scalar system

$$\dot{x} = \mu x^2 + u$$

where $\mu > 0$ is an unknown parameter. The origin is semi-globally stabilizable by means of the simple high-gain feedback $u = -kx$, $k > 0$, meaning that the origin is rendered locally asymptotically stable, with domain of attraction given by the open interval $\mathcal{A} = (-\infty, k/\mu)$. However, it is clear that global asymptotic stabilization is not attainable by linear feedback alone. Applying the principle of certainty equivalence, a candidate controller is given by the control law

$$u = -kx - \hat{\theta}x^2\,, \qquad k > 0$$

with update law

$$\dot{\hat{\theta}} = \gamma x^3\,, \qquad \gamma > 0$$

obtained using the obvious Lyapunov function candidate $V(x,\tilde{\theta}) = x^2 + \gamma^{-1}\tilde{\theta}^2$, where $\tilde{\theta} = \hat{\theta} - \mu$. Clearly, in this case we have adopted the trivial re-parametrization $\theta(\mu) = \mu$ for the unknown plant parameter.

Application of La Salle's invariance principle shows that trajectories of the closed-loop system

$$\begin{aligned} \dot{x} &= -kx - x^2\tilde{\theta} \\ \dot{\tilde{\theta}} &= \gamma x^3 \end{aligned} \tag{C.8}$$

are bounded, and converge asymptotically to the equilibrium set $\mathcal{M} = \{0\} \times \mathbb{R}$. We will first show that, while the set $\mathcal{M}$ is obviously attractive, it is not stable in the sense of Lyapunov. Recall that, for the set $\mathcal{M}$ to be stable in the sense of Lyapunov, for any $\epsilon > 0$ there must exist $\delta > 0$ so that for any initial condition $(x_0, \tilde{\theta}_0)$ satisfying $|(x_0, \tilde{\theta}_0)|_{\mathcal{M}} \leq \delta$ the corresponding trajectory $(x(t), \tilde{\theta}(t))$ satisfies $|(x(t), \tilde{\theta}(t))|_{\mathcal{M}} \leq \epsilon$ for all $t \geq 0$.

Fix $k > 0$ and $\gamma > 0$, and consider the set $\mathcal{S}_1 = \{(x, \tilde{\theta}) : x \geq 0,\, x\,\tilde{\theta} \geq -k\}$. This set is forward invariant for the closed-loop system (C.8), as the lower boundary $\{x = 0\}$ is made of trajectories (the set $\mathcal{M}$, which is an equilibrium set), whereas on the boundary $\{x\,\tilde{\theta} = -k\}$ the vector fields point inward (note that $\dot{x} = 0$ and $\dot{\tilde{\theta}} > 0$ on $\{x\,\theta = -k\}$.) On the other hand, the set $\mathcal{S}_2 = \{(x, \tilde{\theta}) : x \geq 0,\, x\,\tilde{\theta} < -k\}$ is backward invariant. Choose, arbitrarily, $\epsilon > 0$ and an initial condition $p(0) = (x(0), \tilde{\theta}(0))$ such that $|p(0)|_{\mathcal{M}} > \epsilon$. Without loss of generality, assume that $p(0)$ lies on the first quadrant, so that $x\,\tilde{\theta} > 0$ (see Figure C.1). Since $\mathrm{d}x/\mathrm{d}\tilde{\theta} < 0$ on $\mathcal{S}_1$, the trajectory $p(t)$ originating from $p(0)$ remains in $\mathcal{S}_1$ and converge asymptotically to $\mathcal{M}$. Note also that the trajectory in question approaches $\mathcal{M}$ along the normal direction to the set, since $\mathrm{d}x/\mathrm{d}\tilde{\theta} \to -\infty$ as $x \to 0$. Integrating the system backward from the initial condition p moves the trajectory towards the boundary $\{x\,\tilde{\theta} = -k\}$, since in this case

$$\frac{\mathrm{d}x}{\mathrm{d}(-t)} = kx + x^2\tilde{\theta} \quad \text{and} \quad \frac{\mathrm{d}\tilde{\theta}}{\mathrm{d}(-t)} = -\gamma x^3$$

Since $x(t)$ is increasing in backward time, it is bounded away from zero, and so is $\mathrm{d}\tilde{\theta}/\mathrm{d}(-t)$. As a result, there exists a finite time $-\tau$ such that $x(-\tau)\tilde{\theta}(-\tau) = -k$. At the boundary, the vector field of the backward system points inward $\mathcal{S}_2$. Once the backward trajectory has entered the invariant set $\mathcal{S}_2$, the sign of $\mathrm{d}x/\mathrm{d}(-t)$ is reversed, and thus $\lim_{t\to-\infty} x(t) = 0$. This implies that for any $0 < \delta < \epsilon$ there exists $T > 0$ such that $|p(-T)|_{\mathcal{M}} < \delta$. Therefore, the *forward* trajectory originating from $p(-T)$ leaves the ball $\{p : |p|_{\mathcal{M}} \leq \epsilon\}$ in finite time. By virtue of the fact that δ is arbitrary, this implies that the set $\mathcal{M}$ is not stable in the sense of Lyapunov. □

The Role of Passivity

The structure of the closed-loop system (C.7) lends itself to an interpretation that is of fundamental importance in the analysis of adaptive systems: system (C.7) can be seen as the negative feedback interconnection, shown in Figure C.2, between the system

$$\Sigma_1 : \left\{ \begin{array}{rcl} \dot{z} & = & Az + B\phi^T(z)u_1 \\ y_1 & = & \phi(z)Cz\,, \end{array} \right.$$

where $A = (A_b + B_bK)$, $B = B_b$, and $C = B_b^T P$, and the system

$$\Sigma_2 : \left\{ \begin{array}{rcl} \dot{\tilde{\theta}} & = & \gamma\, u_2 \\ y_2 & = & \tilde{\theta}\,. \end{array} \right.$$

Note that, by construction, the triplet (A, B, C) is strictly positive real, since it possesses the KYP property,[3] and that the system Σ_1 is strictly passive, with positive definite and

[3] See Lemma 3.5.2, Lemma 3.5.3, and Lemma 3.5.4 in [28].

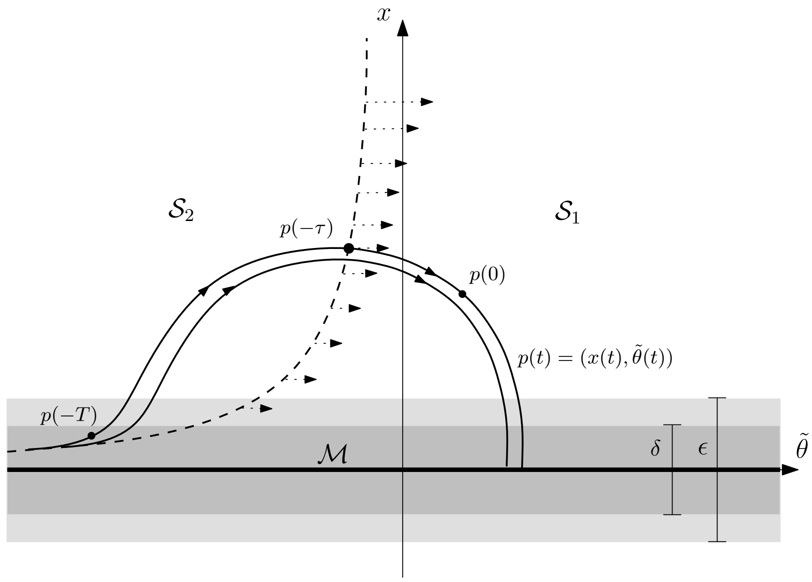


Figure C.1: Example C.8

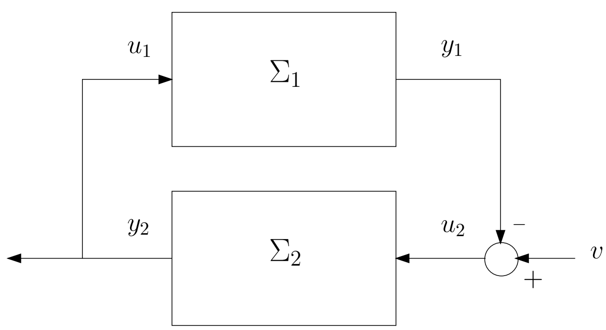


Figure C.2: Adaptive feedback loop

proper storage function given precisely by $V_1(z) = z^T P z$. Also, the system Σ_2 is readily seen to be passive, with a positive definite and proper storage function given by $V_2(\tilde{\theta}) = \gamma^{-1}\tilde{\theta}^T\tilde{\theta}$. By virtue of Proposition B.4.5, the feedback interconnection between Σ_1 and Σ_2 shown in Figure C.2 is passive with respect to the input/output pair (v, y_2), and when $v = 0$ the state trajectories of Σ_1 converge to the origin by virtue of Proposition B.4.6.

C.1.4 Model-Reference Adaptive Control of Scalar Linear Systems

As a second example, consider the SISO linear system defined by the I/O representation

$$\bar{y}(s) = \frac{b}{s+a}\bar{u}(s)$$

or, equivalently, by the state-space realization

$$\dot{y} = -ay + bu\,,\ y(0) = y_0 \tag{C.9}$$

with $y, u \in \mathbb{R}$. It is assumed that the parameter vector $\theta = \mathrm{col}(a, b)$ is unknown; however, the sign of b is known. In particular, without loss of generality, we let $b \geq b_0$ for some $b_0 > 0$. Note that the system (C.9) has unitary relative degree.

The problem we want to address is the following: Given an exponentially stable *reference model* of the form

$$\dot{y}_m = -a_m y_m + b_m u_r \,, \; y_m(0) = 0 \tag{C.10}$$

where $a_m, b_m > 0$ and $u_r(\cdot) \in \mathcal{L}^\infty_{[0,\infty)}$, find a control law for (C.9) to achieve *asymptotic model matching* between the two systems, that is, to let $\lim_{t\to\infty} |y(t) - y_m(t)| = 0$, regardless of the unknown value of the model parameter vector θ. To solve the problem, we appeal once again to the *certainty equivalence principle*, and first devise the solution under the assumption that θ is known. To this end, we postulate the following structure for the controller

$$u = k_1 y + k_2 r \tag{C.11}$$

which is comprised of a feedback and a feedforward term, and derive *matching conditions* relating the vector of controller gains, $k = \mathrm{col}(k_1, k_2)$, with θ to ensure fulfillment of the control objectives. To this end, the dynamics of the model matching error $e := y - y_m$ is easily derived as

$$\dot{e} = -a_m e + (bk_1 + a_m - a)y + (bk_2 - b_m)r \tag{C.12}$$

Consequently, setting

$$k_1 = k_1^* := \frac{a - a_m}{b} \,, \qquad k_2 = k_2^* := \frac{b_m}{b} \tag{C.13}$$

yields the converging dynamics $\dot{e} = -a_m e$, hence the solution to the asymptotic model matching problem. The identities (C.13) are precisely the matching conditions mentioned above. The second step is to replace the fixed gains in the certainty equivalence controller (C.11) with *tunable gains*, $\hat{k} = \mathrm{col}(\hat{k}_1, \hat{k}_2)$ and propose, in place of (C.11), the dynamic controller

$$\begin{aligned} \dot{\hat{k}} &= \tau \\ u &= \hat{k}_1 y + \hat{k}_2 r \end{aligned} \tag{C.14}$$

where $\tau \in \mathbb{R}^2$ is an update law to be determined. This yields the formulation of the asymptotic model matching problem as an adaptive control problem, commonly known as the *Model Reference Adaptive Control (MRAC)* problem. Two strategies may be pursued: In the first one, *direct* adaptation of the tunable gain vector $\hat{k}$ is sought, on the basis of the minimization of a quadratic functional of the model matching error (or, as we will see, to enforce stability of the ensuing error system). This is referred to as *direct MRAC*. The second strategy consists in obtaining an estimate $\hat{\theta} = \mathrm{col}(\hat{a}, \hat{b})$ of the plant parameter vector θ through on-line system identification techniques, and then computing the tunable gains from the matching conditions, that is, by letting

$$\hat{k}_1(\hat{\theta}) := \frac{\hat{a} - a_m}{\hat{b}} \,, \qquad \hat{k}_2(\hat{\theta}) := \frac{b_m}{\hat{b}} \tag{C.15}$$

This approach is referred to as *indirect MRAC*. Note that in the indirect approach one needs to bound the estimate $\hat{b}(t)$ away from the singularity at $\hat{b} = 0$. This is usually accomplished by means of projection techniques, where the assumption made previously that $b \geq b_0 > 0$ becomes instrumental.

Direct Approach

Using the matching conditions, one readily obtains for the closed-loop system

$$\begin{aligned}\dot{e} &= -ay + b(\hat{k}_1 - k_1^* + k_1^*)y + b(\hat{k}_2 - k_2^* + k_2^*)r + a_m y_m - b_m r \\ &= -a_m e + b(\hat{k}_1 - k_1^*)y + b(\hat{k}_2 - k_2^*)r \\ &= -a_m e + b\phi^T(t,e)\tilde{k} \qquad \text{(C.16)}\end{aligned}$$

where $\tilde{k} := \hat{k} - k^*$ is the *parameter estimate error*, and $\phi^T(t,e) := \begin{pmatrix} e + y_m(t) & r(t) \end{pmatrix}$ is a *known regressor*. Note that the dependence of the regressor on the reference signal, $r(\cdot)$, and the output of the reference model, $y_m(\cdot)$, has been regarded as an explicit dependence on time. Since $b > 0$, the function

$$V(e,\tilde{k}) := \tfrac{1}{2}e^2 + \tfrac{1}{2}b\gamma^{-1}\tilde{k}^T\tilde{k}$$

where $\gamma > 0$ is a gain parameter, is a Lyapunov function candidate for the closed-loop system. Evaluation of the derivative of V along the vector field of the closed-loop system yields (recall that $\dot{\tilde{k}} = \dot{\hat{k}}$)

$$\dot{V} = -a_m e + \tfrac{b}{\gamma}\left[\tau + \gamma\phi(t,e)e\right]$$

leading to the obvious choice

$$\tau = -\gamma\phi(t,e)e$$

for the update law. Application of La Salle/ Yoshizawa Theorem (Theorem B.3.18), yields global uniform stability of $(e,\tilde{k}) = (0,0)$, boundedness of all trajectories, and asympttotic convergence of $e(t)$ to zero.

Note that, at this point, we do not have enough tools yet to ascertain whether the origin $(e,\tilde{k}) = (0,0)$ is a uniformly asymptotically stable equilibrium, which is not ruled out by the conclusions of La Salle/ Yoshizawa Theorem. The following examples show that the possibility of achieving uniform asymptotic stability of the origin of the *error system*

$$\begin{aligned}\dot{e} &= -a_m e + b\phi^T(t,e)\tilde{k} \\ \dot{\tilde{k}} &= -\gamma\phi(t,e)e \qquad \text{(C.17)}\end{aligned}$$

depends indeed on the properties of the reference signal $r(\cdot)$.

Example 1: The case of constant reference signals. Consider the case $r(t) = r_0 = \text{const}$, and – for the sake of simplicity – let $a_m = 1$, $b_m = 1$ in the reference model (C.10). Letting $\tilde{y}_m := y_m - r_0$ one obtains the closed-loop error system in the form

$$\begin{aligned}\dot{\tilde{y}}_m &= -\tilde{y}_m \\ \dot{e} &= -a_m e + b\phi^T(\tilde{y}_m, e, r_0)\tilde{k} \\ \dot{\tilde{k}} &= -\gamma\phi(\tilde{y}_m, e, r_0)e \qquad \text{(C.18)}\end{aligned}$$

where

$$\phi^T(\tilde{y}_m, e, r_0) := \begin{pmatrix} e + \tilde{y}_m + r_0 & r_0 \end{pmatrix}$$

is the regressor. Note that the overall system is autonomous, hence one can use La Salle's invariance principle instead of La Salle's / Yoshizawa theorem to assess the asymptotic properties of its solutions. Note also that r_0 is regarded as a constant parameter. It is easily seen that the origin $(\tilde{y}, e, \tilde{k}) = (0, 0, 0)$, albeit stable in the sense of Lyapunov, is not uniformly attractive, hence not uniformly asymptotically stable. This is a simple consequence of the fact that the origin is not an isolated equilibrium point. As a matter of fact, the system possess an equilibrium manifold (subspace) given by

$$\mathcal{M} = \left\{ (\tilde{y}, e, \tilde{k}) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^2 : \tilde{y} = 0,\ e = 0,\ \tilde{k}_1 = -\tilde{k}_2 \right\}$$

Example 2: The case of sinusoidal reference signals. Consider this time the reference signal $r(t) = \cos(\omega_0 t)$, $\omega > 0$. Let

$$y_m^\star(t) := \int_{-\infty}^{t} e^{\tau - t} \cos(\omega_0 \tau) \mathrm{d}\tau = \frac{\cos(\omega_0 t) + \omega_0 \sin(\omega_0 t)}{1 + \omega_0^2}$$

be the steady-state solution of the reference model, where, once again, it has been assumed that $a_m = 1$ and $b_m = 1$. The change of coordinates $\tilde{y}_m := y_m - y_m^\star$ yields the error system

$$\begin{aligned} \dot{\tilde{y}}_m &= -\tilde{y}_m \\ \dot{e} &= -a_m e + b\phi^T(t, \tilde{y}_m, e)\tilde{k} \\ \dot{\tilde{k}} &= -\gamma\phi(t, \tilde{y}_m, e)e \end{aligned} \tag{C.19}$$

where

$$\phi^T(t, \tilde{y}_m, e) := \begin{pmatrix} e + \tilde{y}_m + y_m^\star(t) & r(t) \end{pmatrix}$$

is the new regressor. Note that both the reference signal and the steady-state of the reference model can be generated by the autonomous linear system (a so-called *exosystem*)

$$\begin{pmatrix} \dot{w}_1 \\ \dot{w}_2 \end{pmatrix} = \begin{pmatrix} 0 & \omega_0 \\ -\omega_0 & 0 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}, \qquad \begin{pmatrix} r \\ y_m^\star \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \frac{\omega_0}{\omega_0^2+1} & \frac{1}{\omega_0^2+1} \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \tag{C.20}$$

with initial condition $w_1(0) = 0$, $w_2(0) = 1$. Note also that the equilibrium $(w_1, w_2) = (0, 0)$ of the exosystem is stable in the sense of Lyapunov, all trajectories of the exosystem are bounded, and that the state matrix is skew-symmetric. As a result, La Salle's invariance principle applies to the closed-loop error system augmented with the exosystem, with Lyapunov function candidate given by

$$V(w, \tilde{y}_m, e, \tilde{k}) := w^T w + \tfrac{1}{2}\tilde{y}_m^2 + \tfrac{1}{2}e^2 + \tfrac{1}{2}b\gamma^{-1}\tilde{k}^T\tilde{k}$$

where $w = (w_1, w_2)$. A simple analysis shows that the trajectories of the closed-loop system (C.19)–(C.20) converge to the largest invariant set $\mathcal{M} \subset \mathbb{R}^2 \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^2$ contained in the set

$$\mathcal{E} := \left\{ (w, \tilde{y}_m, e, \tilde{k}) \in \mathbb{R}^2 \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^2 : \tilde{y}_m = 0,\ e = 0 \right\}$$

This invariant set is obviously comprised of the trajectory $w(t) = (r(t), y_m^\star(t))$ and trajectories $\tilde{k}(t) = \tilde{k}^\star = \text{const}$ satisfying

$$\begin{pmatrix} \frac{\cos(\omega_0 t) + \omega_0 \sin(\omega_0 t)}{1 + \omega_0^2} & \cos(\omega_0 t) \end{pmatrix} \begin{pmatrix} \tilde{k}_1^\star \\ \tilde{k}_2^\star \end{pmatrix} = 0$$

for all $t \in \mathbb{R}$. Differentiation of both sided of the above identity yields the system of equations

$$Q(t)\tilde{k}^\star = 0 \qquad \forall\, t \in \mathbb{R}$$

where

$$Q(t) := \begin{pmatrix} \dfrac{\cos(\omega_0 t) + \omega_0 \sin(\omega_0 t)}{1+\omega_0^2} & \cos(\omega_0 t) \\ \dfrac{\omega_0^2 \cos(\omega_0 t) - \omega_0 \sin(\omega_0 t)}{1_+\omega_0^2} & -\omega_0 \sin(\omega_0 t) \end{pmatrix}$$

Since $\det Q(t) = -\omega_0^2/(1+\omega_0^2)$, it is concluded that, necessarily, $\tilde{k}^\star = 0$. As a result, the equilibrium $(\tilde{y}, e, \tilde{k}) = (0,0,0)$ of the time-varying system (C.19) is uniformly (globally) stable in the sense of Lyapunov and globally attractive. Unfortunately, we are not still in the position to conclude that the equilibrium is globally uniformly asymptotically stable, as *uniform* (global) attractivity has not been determined. However, it is noted that asymptotic convergence of the parameter estimates $\hat{k}(t)$ to the "true values" $k^\star$ has been established.

Indirect Approach

In the indirect approach, we use a *model estimator* of the form

$$\dot{\hat{y}} = -\hat{a}y + \hat{b}u + \ell(y - \hat{y}) \tag{C.21}$$

where $\ell > 0$ is the output injection gain. Next, define the *model estimation error* $\tilde{y} := \hat{y} - y$, with dynamics

$$\dot{\tilde{y}} = -\ell\tilde{y} - (\hat{a} - a)y + (\hat{b} - b)r \tag{C.22}$$

and the *estimated model mismatch error* $\hat{e} := \hat{y} - y_m$, with dynamics

$$\dot{\hat{e}} = -a_m\hat{e} - \ell\tilde{y} + (a_m - \hat{a})y - b_m r + \hat{b}u$$

Applying the certainty-equivalence control

$$u = \hat{k}_1(\hat{\theta})y + \hat{k}_2(\hat{\theta})r \tag{C.23}$$

where the tunable gains are given in (C.15), yields

$$\dot{\hat{e}} = -a_m\hat{e} - \ell\tilde{y} \tag{C.24}$$

Next, the equation of the $\tilde{y}$-dynamics (C.22) is written in the more compact form

$$\dot{\tilde{y}} = -\ell\tilde{y} + \psi^T(t, \tilde{y}, \hat{e})\tilde{\theta} \tag{C.25}$$

where $\tilde{\theta} := \hat{\theta} - \theta$ is the *parameter estimate error*, and $\psi^T(t, \tilde{y}, \hat{e}) := \begin{pmatrix} \tilde{y} - \hat{e} - y_m(t) & r(t) \end{pmatrix}$ is a *known regressor*. Following a similar reasoning as in the direct approach, consider the Lyapunov function candidate

$$W(\hat{e}, \tilde{y}, \tilde{\theta}) := \tfrac{1}{2}\hat{e}^2 + \tfrac{\lambda}{2}\left(\tilde{y}^2 + \gamma^{-1}\tilde{\theta}^T\tilde{\theta}\right)$$

where $\lambda > 0$ is a scaling factor to be determined. Scaling the term in parenthesis in the Lyapunov function candiate above allows one to take into account the coupling between the

$\hat{e}$- and the $\tilde{y}$-subsystems in (C.24). Evaluation of the derivative of W along the vector field of the system one obtains

$$\dot{W} = -a_m\hat{e}^2 - \ell\hat{e}\tilde{y} + \lambda\left(-\ell\tilde{y}^2 + \tilde{y}\psi^T(t,\tilde{y},\hat{e})\tilde{\theta} + \gamma^{-1}\tilde{\theta}\dot{\hat{\theta}}\right)$$

Choosing

$$\dot{\hat{\theta}} = -\gamma\psi(t,\tilde{y},\hat{e})\tilde{y}$$

for the update law yields

$$\dot{W} = -a_m\hat{e}^2 - \ell\hat{e}\tilde{y} - \lambda\ell\tilde{y}^2 \tag{C.26}$$

The selection $\lambda > \ell/(4a_m)$ ensures that the quadratic form on the right-hand side of (C.26) is negative definite. As a consequence, application of La Salle/ Yoshizawa Theorem yields boundedness of all trajectories and asymptotic regulation of both $\hat{e}(t)$ and $\tilde{y}(t)$, *if one can show that the control* (C.23) *is well-defined*, for instance, if one can ensure that $\hat{b}(t) \geq b_0$ for all $t \geq 0$. As we will see later in this chapter, this goal can be easily accomplished (at least for this simple example) by projecting the estimate $\hat{b}(t)$ onto the convex set $\mathcal{R} := \{\hat{b} \geq b_0\}$.

Comparing side-by-side the two controllers (and ignoring for the time being the issue of possible singularity of $\hat{b}$ in the indirect approach) yields

$$\text{direct:}\ \begin{cases} \dot{\hat{k}}_1 = -\gamma(y - y_m)y \\ \dot{\hat{k}}_2 = -\gamma(y - y_m)r \\ u = \hat{k}_1 y + \hat{k}_2 r \end{cases} \qquad \text{indirect:}\ \begin{cases} \dot{\hat{y}} = -\hat{a}y + \hat{b}u + \ell(y - \hat{y}) \\ \dot{\hat{a}} = \gamma(\hat{y} - y)y \\ \dot{\hat{b}} = -\gamma(\hat{y} - y)u \\ u = \dfrac{\hat{a} - a_m}{\hat{b}}y + \dfrac{b_m}{\hat{b}}r \end{cases} \tag{C.27}$$

where $\gamma > 0$ and $\ell > 0$ are the adaptation and the observer gains, respectively. It is clear that the indirect approach is more complex, as it involves a controller of higher dimesionality and requires an additional controller gain to be selected (the gain ℓ). This may be a disadvantage when the dimansion of the plant model is large, as the order of an indirect controller increases roughly by a factor of two with respect to its direct counterpart. Nonetheless, the indirect approach presents a clear advantage over the direct approach in the presence of bounded control inputs. Specifically, consider again the plant model (C.9), and assume that the control input is saturated, that is,

$$\dot{y} = -ay + b\,\mathrm{sat}u\,,\ y(0) = y_0 \tag{C.28}$$

In this case, the direct design proceeds by ignoring the presence of the saturation function, essentially regaring the effect of input saturations as an unmeasurable disturbance. As a result, the ensuing direct controller is the same as the controller on the left in (C.27). Conversely, using the indirect approach one has the luxury of providing to the parameter estimator a model of the plant that incorporates the effect of the saturation. This task is achieved by replacing the controller on the right of (C.27) with the modified controller

$$\text{indirect (modified):}\ \begin{cases} \dot{\hat{y}} = -\hat{a}y + \hat{b}\,\mathrm{sat}u + \ell(y - \hat{y}) \\ \dot{\hat{a}} = \gamma(\hat{y} - y)y \\ \dot{\hat{b}} = -\gamma(\hat{y} - y)\mathrm{sat}u \\ u = \dfrac{\hat{a} - a_m}{\hat{b}}y + \dfrac{b_m}{\hat{b}}r \end{cases}$$

where, again, the issue of non-singularity of $\hat{b}$ has been set aside. It can be verified that this modification has a beneficial effect, similar to that of an anti-windup modification, as it prevents the adaptation law from reacting erroneously to the occurrence of input saturation (see the Matlab-Simulink example provided in the file repository.)

C.2 The Standard Adaptive Control Problem

It was shown in the previous sections that several adaptive control problems share a common formulation in which one needs to study the stability of the interconnection between a strictly passive and a passive system. We will refer to this particular setup as the *standard adaptive control problem*, or simply as the *standard problem*. Namely, we will analyze the stability of the equilibrium at the origin of a nonlinear time varying system of the form

$$\begin{aligned} \dot{x}_1 &= Ax_1 + B\phi^T(t,x)x_2 \\ \dot{x}_2 &= -\gamma\phi(t,x)Cx_1 \end{aligned} \tag{C.29}$$

where $x = \mathrm{col}(x_1, x_2) \in \mathbb{R}^{n_1+n_2}$ and γ is a positive constant. The vector field $\phi : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \to \mathbb{R}^{n_2}$, where $n = n_1 + n_2$, defined by the mapping $(t, x) \mapsto \phi(t, x)$ is piecewise continuous in t for any fixed x, and locally Lipschitz in x uniformly in t. In particular, we are interested in determining under which conditions system (C.29) possesses a UGAS (and LES) equilibrium a the origin. As we have already seen, the case in which the vector field ϕ depends only on x_1 and the triplet (A, B, C) is strictly passive or SPR can be easily dealt with using La Salle's invariance principle. A similar situation applies when the dependence on time is due to signals which can be generated as trajectories of autonomous exogenous systems, as in this case La Salle's invariance principle also applies.

A more interesting situation occurs obviously when ϕ depends explicitly on time, hence (C.29) is non-autonomous. We begin with considering the situation in which ϕ depends on t but not on the state x, and thus (C.29) takes the form of a time-varying linear system. Specifically, we consider first the linear time-varying system

$$\begin{aligned} \dot{x}_1 &= Ax_1 + B\phi^T(t)x_2 \\ \dot{x}_2 &= -\gamma\phi(t)Cx_1 \end{aligned} \tag{C.30}$$

with the following standing assumptions:

Assumption C.2.1 *There exist $P = P^T > 0$ and $Q = Q^T > 0$ such that*

$$\begin{aligned} A^TP + PA &\leq -Q \\ PB &= C^T\,. \end{aligned}$$

Assumption C.2.2 *The function $\phi : \mathbb{R}_{\geq 0} \to \mathbb{R}^{n_1}$ is bounded and globally Lipschitz.*

Let $Q = Q_1^T Q_1$, denote with $A^a(\cdot)$, C^a, and P^a respectively the mappings

$$A^a(t) = \begin{pmatrix} A & B\phi^T(t) \\ -\gamma\phi(t)C & 0 \end{pmatrix}, \qquad C^a = \begin{pmatrix} Q_1 & 0 \end{pmatrix}, \qquad P^a = \begin{pmatrix} P & 0 \\ 0 & \gamma^{-1}I \end{pmatrix},$$

and endow system (C.30) with the output $y = C^a x$. Then, the following holds:

Proposition C.2.3 *The system* (C.30) *is globally exponentially stable if the pair* $(C^a, A^a(\cdot))$ *is uniformly completely observable.*

Proof. The result follows directly from Proposition B.3.20, using the Lyapunov function candidate $V(x) = x^T P^a x$. □

The main problem in applying Proposition C.2.3 to a given system (C.30) is to assess uniform complete observability of the pair $(C^a, A^a(\cdot))$. A direct evaluation of the observability gramian is a formidable task, as it requires the explicit computation of the transition matrix of $A^a(\cdot)$. A useful result is provided by the following lemma, which states that uniform complete observability is invariant under bounded output injection.

Lemma C.2.4 *Given bounded matrix-valued functions* $A : \mathbb{R}_{\geq 0} \to \mathbb{R}^{n\times n}$, $C : \mathbb{R}_{\geq 0} \to \mathbb{R}^{p\times n}$ *and* $N : \mathbb{R}_{\geq 0} \to \mathbb{R}^{n\times p}$, *the pair* $(C(\cdot), A(\cdot))$ *is uniformly completely observable if and only if so is the pair* $(C(\cdot), A(\cdot) - N(\cdot)C(\cdot))$.

Proof. See [28, Lemma 4.8.1]. □

The above result can be used to replace the computation of the observability gramian of the original system with that of the system under output injection, provided that the latter takes a simpler form. For our purposes, it suffices to use

$$N^a(t) = \begin{pmatrix} AQ_1^{-1} \\ -\gamma\phi(t)CQ_1^{-1} \end{pmatrix}$$

to obtain

$$A^a(t) - N^a(t)C^a = \begin{pmatrix} 0 & B\phi^T(t) \\ 0 & 0 \end{pmatrix}$$

for which the transition matrix can be easily computed as

$$\Phi(t,\tau) = \begin{pmatrix} I & B\sigma(t,\tau) \\ 0 & I \end{pmatrix}, \qquad \sigma(t,\tau) \triangleq \int_\tau^t \phi^T(s)ds\,.$$

It follows that the observability gramian of $(C^a(\cdot), A^a(\cdot) - N^a(\cdot)C^a(\cdot))$ reads as, after some manipulations,

$$W(t_1,t_2) = \int_{t_1}^{t_2} \begin{pmatrix} Q & QB\sigma(\tau,t_1) \\ \sigma^T(\tau,t_1)B^TQ & \sigma^T(\tau,t_1)B^TQB\sigma(\tau,t_1) \end{pmatrix} d\tau\,. \tag{C.31}$$

Proposition C.2.5 *Assume that the function* $\phi(\cdot)$ *is bounded and globally Lipschitz, and that there exist constants* $\kappa > 0$, $\delta > 0$ *such that*

$$\int_t^{t+\delta} \phi(\tau)\phi^T(\tau)d\tau \geq \kappa I\,, \qquad \forall\, t \geq 0\,. \tag{C.32}$$

Then, there exists $\mu > 0$ *such that*

$$W(t, t+\delta) \geq \mu I\,, \qquad \forall\, t \geq 0$$

where $W(\cdot,\cdot)$ *is the observability gramian in* (C.31).

Proof. See [28, Lemma 4.8.4]. □

The condition (C.32) is commonly referred to as a *persistence of excitation* (PE) condition. The PE condition plays a fundamental role in the analysis of the asymptotic properties of adaptive systems. In a nutshell, it guarantees that the time-varying signal $\phi(\cdot)$ yields enough couplings between the trajectories $x_1(\cdot)$ and $x_2(\cdot)$ of (C.30) to obtain uniform complete observability. The PE condition has been studied quite extensively in the adaptive control literature. For a comprehensive survey of the properties of PE signals and their role in control and system identification, the reader should consult [28], [38], [39], [40], and the recent paper [41], which provides a nice review of earlier results.

The PE property (C.32), used in conjunction with Assumption C.2.1 and Assumption C.2.2, yields a sufficient condition for global exponential stability of (C.30), established by means of Proposition B.3.20. The result is summarized as follows:

Theorem C.2.6 *Consider system* (C.30)*, and let assumptions C.2.1 and C.2.2 hold. Assume, in addition, that the function $\phi(\cdot)$ satisfies the PE condition* (C.32)*. Then, the origin is a globally exponentially stable equilibrium of* (C.30)*.*

Reverting back to the full nonlinear system (C.29), one may wonder to what extent the result of Theorem C.2.6 can be used to find conditions for global uniform asymptotic stability of the origin, as opposed to the much weaker form of stability implied by La Salle/Yoshizawa theorem. For this purpose, assume that Assumption C.2.1 holds for the triplet (A, B, C) in (C.29). As a result, by La Salle/Yoshizawa theorem, the origin is a uniformly globally stable equilibrium, and thus for any initial condition $(t_0, x_0) \in \mathbb{R} \times \mathbb{R}^n$ the corresponding trajectory $x(t; t_0, x_0)$ is bounded for all $t \geq t_0$. Let the parameterized family of functions

$$\tilde{\phi}_{(t_0,x_0)}(\cdot) \triangleq \phi(\cdot, x(\cdot\,; t_0, x_0)), \qquad (t_0, x_0) \in \mathbb{R} \times \mathbb{R}^n \tag{C.33}$$

be defined as the function $\phi(\cdot,\cdot)$ evaluated along the trajectories of (C.29), that is, as the mapping

$$t \mapsto \phi(t, x(t; t_0, x_0)), \qquad t \geq t_0$$

parameterized by the initial condition of (C.29). Note that since each *single trajectory* $x(t) \triangleq x(t; t_0, x_0)$ satisfies the differential equation

$$\begin{pmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{pmatrix} = \begin{pmatrix} A & B\tilde{\phi}^T_{(t_0,x_0)}(t) \\ -\gamma\tilde{\phi}_{(t_0,x_0)}(t)C & 0 \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}, \tag{C.34}$$

to each trajectory of (C.29) one can associate a linear time-varying system, which can in principle be used to study the asymptotic properties of that particular trajectory. In particular, the standing assumptions on $\phi(\cdot,\cdot)$ and boundedness of $x(\cdot\,; t_0, x_0)$ imply that the system (C.34) is well defined for for each pair (t_0, x_0). Note, however, that it is not possible to replace (C.29) with (C.34), and that any conclusion about the asymptotic behavior of $x(t)$ drawn from (C.34) will be valid only for that particular trajectory, unless additional conditions hold.

Theorem C.2.7 *Assume that, in addition to Assumption C.2.1, the following conditions hold:*

i.) *For any* $(t_0, x_0) \in \mathbb{R} \times \mathbb{R}^n$*, the function* $\tilde{\phi}_{(t_0,x_0)}(\cdot)$ *is globally Lipschitz.*

ii.) *The function* $\tilde{\phi}_{(t_0,x_0)}(\cdot)$ *satisfies a PE condition, that is, for each pair* $(t_0, x_0) \in \mathbb{R} \times \mathbb{R}^n$ *there exist* $\kappa_0 > 0$ *and* $\delta_0 > 0$ *such that*

$$\int_t^{t+\delta_0} \tilde{\phi}_{(t_0,x_0)}(\tau)\tilde{\phi}^T_{(t_0,x_0)}(\tau)d\tau \geq \kappa_0 I\,, \qquad \forall\, t \geq t_0\,.$$

Then, the system (C.29) *is globally exponentially convergent (see Definition B.2.6).*

It is important to point out that Theorem C.2.7 does not imply neither exponential stability nor uniform asymptotic stability of the origin of (C.29). As a matter of fact, Theorem C.2.7 only improves on the results of La Salle/Yoshizawa theorem establishing convergence of $x(t)$ to the origin (as opposed to that of $x_1(t)$ alone,) but the convergence need not be uniform. Moreover, Theorem C.2.7 may be difficult to apply, as in order to check the conditions i.) and ii.) above, knowledge of the solution $x(\cdot\,; t_0, x_0)$ may be required.

C.2.8 Uniform Asymptotic Stability of Adaptive Systems

From the above discussion, it is clear that for the prototype system (C.29) persistence of excitation of the parameterized family of functions (C.33) plays an important role in extending convergence to the origin of the trajectory $x_1(t)$, implied by LaSalle/Yoshizawa theorem, to the whole trajectory $x(t) = (x_1(t), x_2(t))$. The result, stated formally in Theorem C.2.7, establishes "pointwise" convergence of each individual trajectory $x(t; t_0, x_0)$, interpreted as a parameterized family of functions indexed by the initial condition (t_0, x_0). A natural question to ask is whether such a convergence can be made uniform with respect to all (t_0, x_0) in any given set of the form $\mathbb{R}_{\geq 0} \times \bar{\mathcal{B}}_r$, so that the result of Theorem C.2.7 can be extended to yield global uniform asymptotic stability of the origin, versus mere exponential converge. Not surprisingly, the key to achieving this goal is an enhanced persistence of excitation property for the family of functions $\tilde{\phi}_{(t_0,x_0)}$ in (C.33), which holds uniformly with respect to (t_0, x_0). In particular, the following definition is introduced in [41]:

Definition C.2.9 *Assume that the system* (C.29) *is forward complete. The parameterized family of functions* $\tilde{\phi}_{(t_0,x_0)}$ *in* (C.33) *is said to be* uniformly persistently exciting *(u-PE) if for any* $r > 0$ *there exist* $\kappa > 0$ *and* $\delta > 0$ *such that for any* $(t_0, x_0) \in \mathbb{R}_{\geq 0} \times \bar{\mathcal{B}}_r$ *the corresponding trajectory* $x(t; t_0, x_0)$ *of* (C.29) *satisfies*

$$\int_t^{t+\delta} \tilde{\phi}_{(t_0,x_0)}(\tau)\tilde{\phi}^T_{(t_0,x_0)}(\tau)d\tau \geq \kappa I\,, \qquad \forall\, t \geq t_0\,.$$

Applying the definition of u-PE directly to a system of the form (C.29) appears to be of limited use, as one needs to know *a priori* the solutions of (C.29) to be able to check that the given conditions are satisfied. However, it is possible to infer the u-PE property without solving explicitly the differential equation if appropriate conditions on the solutions of (C.29) and on the vector field $\phi(t, x)$ hold.

Proposition C.2.10 *Let* $\phi(\cdot, x)$ *be piecewise continuous for each* $x \in \mathbb{R}^n$*, and let* $\phi(t, \cdot)$ *be locally Lipschitz uniformly in* t*. Consider a system of the form* (C.29)*, and assume that there exist:*

i. A number $\mu > 0$ such that for any initial condition $(t_0, x_0) \in \mathbb{R}_{\geq 0} \times \mathbb{R}^n$ the corresponding solution $x(\cdot; t_0, x_0)$ satisfies $\max\{\|x\|_\infty, \|x_1\|_2\} \leq \mu|x_0|$;

ii. A function $\bar{\phi} : \mathbb{R}_{\geq 0} \to \mathbb{R}^{n_2}$ which is bounded and satisfies the PE condition (C.32) *for some $\kappa > 0$ and some $\lambda > 0$.*

iii. A nondecreasing function $\psi : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ and nonnegative constants c_1 and c_2 satisfying $c_1 + c_2 > 0$ such that for any unitary vector $\xi \in \mathbb{R}^{n_2}$

$$|\phi_0^T(t,x)\,\xi| \geq [\,c_1 + c_2\psi(|x_2|)|x_2|\,]\;|\bar{\phi}^T(t)\xi| \tag{C.35}$$

where $\phi_0(t,x) = \phi(t,x)|_{x_1=0}$.

Then, the parameterized family of functions $\tilde{\phi}_{(t_0,x_0)}(\cdot)$ in (C.33) *is u-PE. Moreover, if* (C.35) *holds with $c_1 > 0$, the function $\bar{\phi}$ is not required to be bounded*

Proof. See [41, Prop.2]. □

A simplified version of the above result holds for the important case in which the vector field $\phi(t,x)$ does not depend on the component x_2, and the realization (A, B, C) is strictly passive.

Corollary C.2.11 *For the given system* (C.29)*, let Assumption C.2.1 hold. Assume that the vector field $\phi(t,x)$ does not depend on x_2, that is, let $\phi(t,x) = \phi(t,x_1)$. Then, if the function $\phi_0 : \mathbb{R}_{\geq 0} \to \mathbb{R}^{n_2}$ defined as $\phi_0(t) = \phi(t,0)$ is PE, then the parameterized family of functions $\tilde{\phi}_{(t_0,x_0)}(\cdot) \triangleq \phi(\cdot, x_1(\cdot\,; t_0, x_0))$ is u-PE.*

The concept of u-PE is instrumental in deriving a sufficient condition for global uniform asymptotic stability of system (C.29). Specifically, the following result can be proven using the arguments in [41, Theorem 1]:

Theorem C.2.12 *Consider the system* (C.29)*, where the vector field $\phi(t,x)$ is such that $\phi(\cdot, x)$ is bounded for each fixed $x \in \mathbb{R}^n$, and $\phi(t, \cdot)$ is locally Lipschitz uniformly in t. Let Assumption C.2.1 hold. If, in addition:*

i.) There exists a nondecreasing function $\rho : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ such that

$$\max\left\{\,\|\frac{\partial\phi}{\partial x}\|,\;\left|\frac{\partial\phi}{\partial t}\right|\,\right\} \leq \rho(|x|)$$

for all $(t,x) \in \mathbb{R}_{\geq 0} \times \mathbb{R}^n$;

ii.) The parameterized family of functions $\tilde{\phi}_{(t_0,x_0)}(\cdot)$ in (C.33) *is u-PE.*

Then, the origin is a uniformly globally asymptotically and locally exponentially stable equilibrium of system (C.29).

C.3 The Issue of Robustness

Consider again the standard adaptive control system (C.29), endowed with Assumptions C.2.1 and C.2.2. Let us consider the presence of external disturbance signals $d = \mathrm{col}(d_1, d_2)$, with $d_1(\cdot) \in \mathcal{L}^\infty(\mathbb{R}^{n_1})$ and $d_2(\cdot) \in \mathcal{L}^\infty(\mathbb{R}^{n_2})$ as follows

$$\begin{aligned} \dot{x}_1 &= Ax_1 + B\phi^T(t,x)x_2 + d_1(t) \\ \dot{x}_2 &= -\gamma\phi(t,x)Cx_1 + d_2(t) \end{aligned} \tag{C.36}$$

The aim of this section is to investigate the effect of bounded disturbances on the trajectories of system (C.36), in particular on the properties of boundedness and asymptotic regulation of $x_1(t)$, which are guaranteed by La Salle/Yoshizawa theoerm in absence of model perturbation. The first result is a direct consequence of the theorem of total stability (Theorem B.3.11), which is behind the *raison d'être* for uniform global asymptotic stability:

Corollary C.3.1 *Assume that the assumptions of Theorem C.2.12 hold for system* (C.36)*. Then, for any $\varepsilon > 0$ there exist $\delta_1 > 0$ and $\delta_2 > 0$ such that for all $t_0 \in \mathbb{R}$, all $x_0 \in \bar{\mathcal{B}}_{\delta_1} \subset \mathbb{R}^n$ and all $d(\cdot) \in \mathcal{L}^\infty(\mathbb{R}^n)$ such that $\|d\|_\infty \leq \delta_2$, the forward solution $x(t) := x(t; t_0, x_0, d_{[t_0,t)})$, $t \geq t_0$, of* (C.36) *satisfies $\|x(\cdot)\|_\infty \leq \varepsilon$.*

The above result establishes the property of *small-signal bounded-input bounded-state stabiliy* for the perturbed system (C.36), under the assumption of UGAS and LES of the equilibrium at the origin of the unforced system (C.29). It must be noted that the above result is local in nature (that is, it is only valid for "small" values of the $\mathcal{L}^\infty$-norm of the disturbance and the norm of the initial condition). The following example serves the purpose of clarifying this issue.

Example C.3.2 Consider again the one-dimensional direct MRAC problem of Section C.1.4, with the following assumptions:

1. The control input coefficient b is known (without loss of generality, let $b = 1$;)
2. The reference model is the identity operator, $y_m(t) = r(t)$, $t \geq 0$;
3. The reference signal is constant, $r(t) = r_0$, $t \geq 0$, where $r_0 \geq 0$.
4. The adaptation gain is selected as $\gamma = 1$;
5. The tracking error dynamics is affected by a constant disturbance, $d(t) = -d_0$, $t \geq 0$, where $d_0 \geq 0$.

Under these assumptions, the equations of the closed-loop system read as

$$\begin{aligned} \dot{e} &= -e + (e + r_0)\tilde{k} - d_0 \\ \dot{\tilde{k}} &= -(e + r_0)e \end{aligned} \tag{C.37}$$

where $\tilde{k} := \hat{k} - k^*$ is the estimation error, and $k^* = a - 1$ (refer to Section C.1.4.)

Let us consider first the case in which $r_0 > 0$. It is readily seen that (C.37) has a unique equilibrium point at $(e, \tilde{k}) = (0, d_0/r_0)$. Changing coordinates as $\theta := \tilde{k} - d_0/r_0$, system (C.37) is written as

$$\begin{aligned} \dot{e} &= -\left(1 - \frac{d_0}{r_0}\right) e + (e + r_0)\theta \\ \dot{\theta} &= -(e + r_0)e \end{aligned} \tag{C.38}$$

where the equilibrium point has been shifted to the origin, $(e, \theta) = (0, 0)$. The jacobian matrix of the vector field of the system evaluated at the origin reads as

$$A = \begin{pmatrix} \dfrac{d_0}{r_0} - 1 & r_0 \\ -r_0 & 0 \end{pmatrix}$$

and its characteristic polynomial is $p_A(\lambda) = \lambda^2 + (1 - d_0/r_0)\lambda + r_0^2$. Clearly, for $r_0 > d_0$ the equilibrium at the origin of (C.38) is LES, whereas for $0 < r_0 < d_0$ the equilibrium is unstable. To determine the global portrait of the solutions, consider first the case $r_0 > d_0$. Using the Lyapunov function candidate $V(e, \theta) = e^2 + \theta^2$, one obtains

$$\dot{V}(e, \theta) = -2\left(1 - \frac{d_0}{r_0}\right) e^2 \leq 0$$

Application of La Salle's invariance principle (notice that the system is autonomous) yields that the only invariant set contained in the set $\{(e, \theta) \in \mathbb{R}^2 : \dot{V}(e, \theta) = 0\}$ is the origin, hence the origin is a globally asymptotically and locally exponentially stable equilibrium. Clearly, this situation also includes the case in which $d_0 = 0$, where in this case $\theta = \tilde{k}$.

For the case $0 < r_0 < d_0$, let us consider the *backward solutions* of (C.38), which are obtained as the forward solutions of system

$$\begin{aligned} \dot{e} &= \left(1 - \frac{d_0}{r_0}\right) e - (e + r_0)\theta \\ \dot{\theta} &= (e + r_0)e \end{aligned} \tag{C.39}$$

Once again, using the Lyapunov function candidate $V(e, \theta) = e^2 + \theta^2$, one obtains

$$\dot{V}(e, \theta) = 2\left(1 - \frac{d_0}{r_0}\right) e^2 \leq 0$$

hence, using the same reasoning as before, one concludes that the origin is a globally asymptotically stable equilibrium of (C.39). Consequently, reverting back to system (C.38), it is concluded that for any $\varepsilon > 0$ and any $R > 0$, there exists $T_{\varepsilon,R} > 0$ such that for all initial conditions $x(0) := \mathrm{col}(e(0), \theta(0)) \in \bar{\mathcal{B}}_R$ the corresponding backward trajectory satisfies $x(t) := \mathrm{col}(e(t), \theta(t)) \in \bar{\mathcal{B}}_\varepsilon$ for all $t \leq -T_{\varepsilon,R}$. This implies that all forward trajectories of (C.38), except the one originating at $x(0) = 0$, satisfy $\lim_{t\to+\infty} |x(t)| = +\infty$.

Finally, for the case $0 < r_0 = d_0$, it is readily seen that the function $V(e, \theta)$ is a first integral of motion for the system (that is, $\dot{V}(e, \theta) = 0$ for all $(e, \theta) \in \mathbb{R}^2$), hence the solutions generate a family of closed orbits given by the level curves $V(e, \theta) = c$, $c \geq 0$.

To summarize the behavior of the solutions of (C.38) when $r_0 > 0$:

- For $r_0 > d_0$, the origin is GAS and LES;
- For $0 < r_0 < d_0$, the origin is unstable, and solutions originating away from the origin diverge as $t \to \infty$;
- For $0 < r_0 = d_0$, solutions are bounded, as solutions originating away from the origin describe a closed orbit.

It is clear that $\mu := 1 - d_0/r_0 \in (-\infty, \infty)$ plays the role of a *bifurcation parameter* for the system, with $\mu = 0$ corresponding to the critical case. It is also clear that the *stability margin* of the system (in the sense of robustness of the stability of the equilibrium at the origin with respect to the constant disturbance d_0) depends on r_0: the larger the value of r_0, the larger the disturbance that can be accommodated by the system. As $r_0 \to 0$, the system loses robustness to constant disturbances. In particular, when $r_0 = 0$ and $d_0 = 0$, system (C.38) possesses an equilibrium manifold $\mathcal{A} := \{(e, \tilde{k}) \in \mathbb{R}^2 : e = 0\}$ that is globally attractive but not stable in the sense of Lyapunov (see the discussion in Example C.1.3.) In this case, there is no robustness whatsoever, and even an infinitesimally small positive constant disturbance results in unbounded forward trajectories (note that when $r_0 = 0$ and $d_0 > 0$ the system does not have equilibrium points, hence no closed orbits either.)

The previous discussion has highlighted two important issues related to robustness of adaptive systems in the standard form (C.29) with respect to external disturbances:

- When the equilibrium $x = 0$ is UGAS and LES, there is robustness to "small enough" external disturbance signals, for solutions originating within a neighborhood of the origin, as provided by the theorem of total stability;
- In absence of a UGAS equilibrium at the origin (that is, when only the weaker properties provided by the La Salle/ Yoshizawa theorem hold) there is no guaranteed robustness to external disturbances.

Clearly, these issues make the application of adaptive control techniques less than ideal, especially in all those cases (which are indeed typical) when uniform persistence of excitation of the regressor can not be guaranteed. This lack of robustness to model perturbations has prompted the development of *robust update laws*, that is, modifications of the standard passivity-based update laws aiming at providing robustness to external disturbance of arbitrarily large magnitude. This will be the topic of the next section.

C.4 Robust Modifications of Passivity-based Update Laws

The aim of this section is to introduce three different strategies aimed at providing robustness of adaptive control systems to external bounded disturbances. For notational convenience, we write the standard adaptive control problem in the following form

$$\begin{aligned}
\dot{z} &= Az + B\phi^T(t, z, \hat{\theta})\tilde{\theta} + d_1 \\
\dot{\hat{\theta}} &= \tau + d_2 \\
e &= Cz
\end{aligned} \tag{C.40}$$

where $z \in \mathbb{R}^{n_1}$ comprise the state of the plant model and that of the controller, $\hat{\theta} \in \mathbb{R}^{n_2}$ is the vector of parameter estimates, $\tilde{\theta} := \hat{\theta} - \theta^*$ is the parameter estimate error, $d = \mathrm{col}(d_1, d_2) \in \mathbb{R}^{n_1+n_2}$ is an external disturbance, $e \in \mathbb{R}$ is the error to be regulated and $\tau \in \mathbb{R}^{n_2}$ is an update law to be designed. The regressor $\phi : \mathbb{R} \times \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \to \mathbb{R}^{n_2}$ defined by the mapping $(t, z, \hat{\theta}) \mapsto \phi(t, z, \hat{\theta})$ is continuous and bounded in t for any fixed z and $\hat{\theta}$, and locally Lipschitz in z and $\hat{\theta}$, uniformly in t. Furthermore, it is assumed that the triplet $\{C, A, B\}$ defines a strictly passive system with positive definite storage function $V_1(z) = z^T P z$ and negative definite supply rate $W(z) = -z^T Q z$. It has been shown in the previous sections that the passivity-based update law

$$\tau = -\gamma\phi(t, z, \hat{\theta})Cz \tag{C.41}$$

achieves boundedness of all trajectories and asymptotic regulation of $e(t)$ when $d = 0$, but does not ensure robustness (in the sense of bounded-input bounded-state behavior) to arbitrary disturbance signals $d \in \mathcal{L}^\infty(\mathbb{R}^n)$. To achieve the goal of ensuring bounded-input bounded-state behavior (and, possibly, preserving asymptotic regulation when $d = 0$), we will consider three modifications to the update law (C.41), namely *leakage*, *leakage with dead-zone*, and *parameter projection*.

C.4.1 Update Laws with Leakage

The first and simplest modification consists in adding a dissipation term (a so-called *leakage*) to the update law, namely to replace (C.41) with

$$\tau = -\gamma\phi(t, z, \hat{\theta})Cz - \sigma\gamma\hat{\theta} \tag{C.42}$$

where $\sigma > 0$ is a *small* gain parameter, resulting in the closed-loop system

$$\begin{aligned}\dot{z} &= Az + B\phi^T(t, z, \hat{\theta})\tilde{\theta} + d_1 \\ \dot{\tilde{\theta}} &= -\gamma\phi(t, z, \hat{\theta})Cz - \sigma\gamma\tilde{\theta} - \sigma\gamma\theta^* + d_2\end{aligned} \tag{C.43}$$

It is noted that the addition of the leakage term destroys the property of the closed-loop system possessing an equilibrium in $(z, \tilde{z}) = (0, 0)$ when $d = 0$, due to the presence of the constant term $-\sigma\gamma\theta^*$ on the equation of the $\tilde{\theta}$-dynamics. This is the reason why the gain of the leakage term should not be chosen too large in order to prevent an unduly deterioration of regulation performance.

Stability Analysis

As customary, consider the Lyapunov function candidate

$$V(z, \tilde{\theta}) = \tfrac{1}{2}z^T P z + \tfrac{1}{2\gamma}\tilde{\theta}^T\tilde{\theta} \tag{C.44}$$

and evaluate its derivative along the vector field of (C.43) to obtain

$$\begin{aligned}\dot{V}(z, \tilde{\theta}) &= -\tfrac{1}{2}z^T Q z + z^T P d_1 - \sigma\tilde{\theta}^T\tilde{\theta} - \sigma\tilde{\theta}^T\theta^* + \gamma^{-1}\tilde{\theta}^T d_2 \\ &\leq -\frac{\lambda_{\min}}{2}|z|^2 - \sigma|\tilde{\theta}|^2 + |z||P||d_1| + \gamma^{-1}|\tilde{\theta}||d_2| + \sigma|\tilde{\theta}||\theta^*|\end{aligned} \tag{C.45}$$

where $\lambda_{\min} > 0$ is the smallest eigenvalue of Q. Letting $x := \mathrm{col}(z, \tilde{\theta})$, one obtains (with a minor abuse of notation)

$$\dot{V}(x) \le -\lambda_0|x|^2 + \mu_0|x||d| + \sigma|x||\theta^*| \tag{C.46}$$

where $\lambda_0 := \min\{\lambda_{\min}/2, \sigma\}$ and $\mu_0 := |P| + \gamma^{-1}$. Using Young's inequality[4] in the expression

$$-\lambda_0|x|^2 + \sqrt{\tfrac{\lambda_0}{2}}|x|\sqrt{\tfrac{2}{\lambda_0}}\mu_0|d| + \sqrt{\tfrac{\lambda_0}{2}}|x|\sqrt{\tfrac{2}{\lambda_0}}\sigma|\theta^*|$$

which is equivalent to the right-hand side of (C.46), one obtains

$$\dot{V}(x) \le -\tfrac{\lambda_0}{2}|x|^2 + \tfrac{\mu_0^2}{\lambda_0}|d|^2 + \tfrac{\sigma^2}{\lambda_0}|\theta^*|^2 \tag{C.47}$$

Defining the class-$\mathcal{N}$ function $\chi(\cdot)$ as

$$\chi(s) = \sqrt{\frac{2\mu_0^2}{\lambda_0^2}s^2 + \frac{2\sigma^2}{\lambda_0^2}|\theta^*|^2}$$

from (C.47) one obtains

$$|x| > \chi(|d|) \implies \dot{V}(x) < 0$$

therefore, by Theorem B.3.13 the perturbed system (C.43) has the GUUB property when $d(\cdot) \in \mathcal{L}^\infty$.

C.4.2 Update Laws with Leakage and Dead-zone Modification

As mentioned, the leakage modification to the passivity-based update law has the undesired effect of destroying the equilibrium at the origin of the closed-loop system in the coordinates $(z, \tilde{\theta})$ in absence of the disturbance. To remedy the situation, a further modification is introduced via the use of a dead-zone function that "switches off" the leakage when the estimation error is inside a given compact set.

To begin, we need a preliminary assumption:

Assumption C.4.3 *The parameter vector θ^* ranges over the interior a known compact and convex set, $\Theta \subset \mathbb{R}^{n_2}$, that is, $\theta^* \in \operatorname{int}\Theta$.*

Fix a number $\ell > 0$ such that

$$\ell > \max_{\theta\in\Theta}\{|\theta_1|, |\theta_2|, \ldots, |\theta_{n_2}|\}$$

and consider the *decentralized multivariable dead-zone function* $\mathbf{dz}_\ell(\cdot) : \mathbb{R}^{n_2} \to \mathbb{R}^{n_2}$, defined as

$$\mathbf{dz}_\ell(\vartheta) = \begin{pmatrix} \mathrm{dz}_\ell(\vartheta_1) \\ \mathrm{dz}_\ell(\vartheta_2) \\ \vdots \\ \mathrm{dz}_\ell(\vartheta_{n_2}) \end{pmatrix}, \qquad \mathrm{dz}_\ell(\vartheta_i) = \vartheta_i - \ell\,\mathrm{sat}\frac{\vartheta_i}{\ell}, \qquad \mathrm{sat}\vartheta_i = \begin{cases} -1 & \vartheta_i \le -1 \\ \vartheta_i & |\vartheta_i| < 1 \\ 1 & \vartheta_i \ge 1 \end{cases}$$

The decentralized dead-zone (hereby simply referred to as "dead-zone") with the given choice of the level ℓ has the following properties:

[4] Given $a \ge 0$ and $b \ge 0$, $ab \le a^2/2 + b^2/2$.

- For all $\vartheta \in \mathbb{R}^{n_2}$ and all $\theta \in \Theta$

$$\vartheta^T \mathbf{dz}_\ell(\vartheta + \theta) \geq 0 \tag{C.48}$$

- There exist constants $c_1 > 0$ and $c_2 > 0$ such that for all $\vartheta \in \mathbb{R}^{n_2}$ satisfying $|\vartheta| \geq c_1$ and all $\theta \in \Theta$

$$\vartheta^T \mathbf{dz}_\ell(\vartheta + \theta) \geq c_2 |\vartheta|^2 \tag{C.49}$$

Note also that $\mathbf{dz}_\ell(\theta) = 0$ for all $\theta \in \Theta$. The *leakage with dead-zone modification* of the passivity-based update law (C.41) is defined as

$$\tau = -\gamma \phi(t, z, \hat{\theta}) C z - \sigma \gamma \, \mathbf{dz}_\ell(\hat{\theta}) \,, \qquad \sigma > 0 \tag{C.50}$$

resulting in the closed-loop system

$$\begin{aligned} \dot{z} &= Az + B\phi^T(t, z, \hat{\theta})\tilde{\theta} + d_1 \\ \dot{\tilde{\theta}} &= -\gamma \phi(t, z, \hat{\theta}) C z - \sigma \gamma \, \mathbf{dz}_\ell(\tilde{\theta} + \theta^*) + d_2 \end{aligned} \tag{C.51}$$

Note that, as opposed to the standard leakage modification, when $d = 0$ the system preserves the equilibrium at $(z, \tilde{\theta}) = (0, 0)$, due to the fact that $\theta^* \in \Theta$ by assumption.

Stability Analysis

Consider the Lyapunov function candidate (C.44), and evaluate its derivative along the vector field of (C.51) to obtain

$$\begin{aligned} \dot{V}(z, \tilde{\theta}) &= -\tfrac{1}{2} z^T Q z + z^T P d_1 - \sigma \tilde{\theta}^T \mathbf{dz}_\ell(\tilde{\theta} + \theta^*) + \gamma^{-1} \tilde{\theta}^T d_2 & \text{(C.52)} \\ &\leq -\frac{\lambda_{\min}}{2} |z|^2 + |z||P||d_1| + \gamma^{-1} |\tilde{\theta}||d_2| - \sigma \tilde{\theta}^T \mathbf{dz}_\ell(\tilde{\theta} + \theta^*) & \text{(C.53)} \end{aligned}$$

where $\lambda_{\min} > 0$ is the smallest eigenvalue of Q. As before, let $x := \mathrm{col}(z, \tilde{\theta})$. First, consider the case $|\tilde{\theta}| \leq c_1$, which together with (C.48) and (C.53) implies

$$\dot{V}(z, \tilde{\theta}) \leq -\frac{\lambda_{\min}}{2} |z|^2 + |z||P||d_1| + \frac{c_1}{\gamma} |d_2|$$

Applying Young's inequality to the right-hand side of the above inequality, and using the fact that $|d_i| \leq |d|$, $i = 1, 2$, one obtains

$$\dot{V}(z, \tilde{\theta}) \leq -\frac{\lambda_{\min}}{4} |z|^2 + \frac{|P|^2}{\lambda_{\min}} |d|^2 + \frac{c_1}{\gamma} |d| \tag{C.54}$$

Defining the class-$\mathcal{K}_\infty$ function $\chi_1(\cdot)$ as follows

$$\chi_1(s) = \sqrt{\frac{4|P|^2}{\lambda_{\min}^2} s^2 + \frac{4c_1}{\lambda_{\min}\gamma} s}$$

one obtains, from (C.54) and the assumption $|\tilde{\theta}| \leq c_1$,

$$|z| > \chi_1(|d|) \;\text{ and }\; |\tilde{\theta}| \leq c_1 \implies \dot{V}(z, \tilde{\theta}) < 0 \tag{C.55}$$

Assume now that $|\tilde{\theta}| > c_1$. Using (C.49), the right-hand side of (C.52) can be bounded as

$$\begin{aligned}\dot{V}(z,\tilde{\theta}) &\leq -\frac{\lambda_{\min}}{2}|z|^2 + |z||P||d_1| - c_2\sigma|\tilde{\theta}|^2 + \gamma^{-1}|\tilde{\theta}||d_2| \\ &\leq -\frac{\lambda_{\min}}{4}|z|^2 + \frac{|P|^2}{\lambda_{\min}}|d_1|^2 - \frac{c_2\sigma}{2}|\tilde{\theta}|^2 + \frac{1}{2c_2\sigma\gamma^2}|d_2|^2 \end{aligned} \tag{C.56}$$

where we have made again use of Young's inequality. Letting $x := \mathrm{col}(z, \tilde{\theta})$, one obtains

$$\dot{V}(x) \leq -\lambda_0|x|^2 + \mu_0|d|^2 \tag{C.57}$$

where $\lambda_0 := \min\{\lambda_{\min}/4, c_2\sigma/2\}$ and $\mu_0 := |P|^2/\lambda_{\min} + (2c_2\sigma\gamma^2)^{-1}$. As a result, defining the class-$\mathcal{K}_\infty$ function $\chi_2(\cdot)$ as

$$\chi_2(s) = \sqrt{\frac{\mu_0}{\lambda_0}}s$$

one obtains

$$|x| > \chi_2(|d|) \ \text{ and } \ |\tilde{\theta}| > c_1 \implies \dot{V}(x) < 0 \tag{C.58}$$

Next, we combine the two conditions (C.55) and (C.58) into a single one involving a class-$\mathcal{N}$ function. Let the class-$\mathcal{N}$ function $\chi(\cdot)$ be defined as

$$\chi(s) = \sqrt{c_1^2 + \chi_1^2(s) + \chi_2^2(s)}$$

and notice that $|x| > \chi(|d|)$ implies $|x| > \chi_2(|d|)$, and that $|x| > \chi(|d|)$ implies $|x|^2 > c_1^2 + \chi_1^2(|d|)$. In particular, when $|\tilde{\theta}| \leq c_1$ one obtains

$$c_1^2 + \chi_1^2(|d|) < |x|^2 \implies c_1^2 + \chi_1^2(|d|) < |z|^2 + |\tilde{\theta}|^2 \leq |z|^2 + c_1^2 \implies \chi_1^2(|d|) < |z|^2$$

hence

$$|\tilde{\theta}| \leq c_1 \ \text{ and } \ |x| > \chi(|d|) \implies |\tilde{\theta}| \leq c_1 \ \text{ and } \ |z| > \chi_1(|d|) \implies \dot{V}(x) < 0$$

Conversely,

$$|\tilde{\theta}| > c_1 \ \text{ and } \ |x| > \chi(|d|) \implies |\tilde{\theta}| > c_1 \ \text{ and } \ |z| > \chi_2(|d|) \implies \dot{V}(x) < 0$$

therefore, by Theorem B.3.13 the perturbed system (C.51) has the GUUB property when $d(\cdot) \in \mathcal{L}^\infty$.

C.4.4 Update Laws with Parameter Projection

The last modification of the standard passivity-based update law presented in this section is applicable to those cases in which the disturbance affects only the z-dynamics of system (C.40), that is, when $d_2 = 0$. As in the previous section, it is assumed that Assumption C.4.3 holds. Note that convexity of the parameter set Θ is a strict requirement, along with compactness. In this regard, we pose an additional requirement:

Assumption C.4.5 *The set Θ is given by*

$$\Theta = \{\theta \in \mathbb{R}^{n_2} : \Pi(\theta) \leq 0\}$$

where $\Pi(\cdot) : \mathbb{R}^{n_2} \to \mathbb{R}$ is a convex and differentiable function.

Denote with $\nabla\Pi(\cdot)$ the gradient of $\Pi(\cdot)$, that is $\nabla\Pi(\theta) = \left(\frac{\partial\Pi}{\partial\theta_1}(\theta) \quad \frac{\partial\Pi}{\partial\theta_2}(\theta) \quad \cdots \quad \frac{\partial\Pi}{\partial\theta_{n_2}}(\theta)\right)^T$, and define the *projection operator onto* Θ as follows:

$$\underset{\hat\theta\in\Theta}{\mathrm{Proj}}\{\tau\} = \begin{cases} \tau & \text{if } \hat\theta \in \mathrm{int}\,\Theta \text{ or } \{\hat\theta\in\partial\Theta \text{ and } \nabla\Pi^T(\hat\theta)\tau \le 0\} \\ \left(I - \dfrac{\nabla\Pi(\hat\theta)\nabla\Pi^T(\hat\theta)}{|\nabla\Pi(\hat\theta)|^2}\right)\tau & \text{if } \hat\theta\in\partial\Theta \text{ and } \nabla\Pi^T(\hat\theta)\tau > 0 \end{cases}$$

The dynamics of the parameter vector estimate is selected as

$$\dot{\hat\theta} = \underset{\hat\theta\in\Theta}{\mathrm{Proj}}\{\tau\}\,, \quad \hat\theta(0)\in\mathrm{int}\,\Theta \tag{C.59}$$

where τ is the passivity-based update law (C.41) resulting in the closed-loop system[5]

$$\begin{aligned} \dot z &= Az + B\phi^T(t,z,\hat\theta)\tilde\theta + d_1 \\ \dot{\hat\theta} &= \underset{\hat\theta\in\Theta}{\mathrm{Proj}}\left\{-\gamma\phi(t,z,\hat\theta)Cz\right\} \end{aligned} \tag{C.60}$$

The use of parameter projection ensures the following properties:

Proposition C.4.6 *The set* Θ *is forward invariant under the flow of* (C.59)*.*

Proof. At each point $\hat\theta\in\partial\Theta$

$$\nabla\Pi^T(\hat\theta)\underset{\hat\theta\in\Theta}{\mathrm{Proj}}\{\tau\} = \begin{cases} \nabla\Pi^T(\hat\theta)\tau & \text{if } \nabla\Pi^T(\hat\theta)\tau\le 0\} \\ \nabla\Pi^T(\hat\theta)\left(I - \dfrac{\nabla\Pi(\hat\theta)\nabla\Pi^T(\hat\theta)}{|\nabla\Pi(\hat\theta)|^2}\right)\tau & \text{if } \nabla\Pi^T(\hat\theta)\tau > 0 \end{cases}$$

Clearly, if $\nabla\Pi^T(\hat\theta)\tau \le 0$, then $\nabla\Pi^T(\hat\theta)\,\mathrm{Proj}_{\hat\theta\in\Theta}\{\tau\} \le 0$ as well. Conversely, assume $\nabla\Pi^T(\hat\theta)\tau > 0$, and decompose τ along the direction of the vector $\nabla\Pi(\hat\theta)$ and a given basis of the tangent plane to $\partial\Theta$ at $\hat\theta$, that is, let

$$\tau = \alpha\nabla\Pi(\hat\theta) + \psi$$

for some $\alpha > 0$ and $\psi\in\{\mathrm{span}\nabla\Pi(\hat\theta)\}^\perp$. Then

$$\begin{aligned} \nabla\Pi^T(\hat\theta)\underset{\hat\theta\in\Theta}{\mathrm{Proj}}\{\tau\} &= \nabla\Pi^T(\hat\theta)\left(\tau - \frac{\nabla\Pi^T(\hat\theta)\tau}{|\nabla\Pi(\hat\theta)|^2}\nabla\Pi(\hat\theta)\right) \\ &= \nabla\Pi^T(\hat\theta)\left(\alpha\nabla\Pi(\hat\theta) + \psi - \alpha\frac{\nabla\Pi^T(\hat\theta)\nabla\Pi(\hat\theta)}{|\nabla\Pi(\hat\theta)|^2}\nabla\Pi(\hat\theta) - \frac{\nabla\Pi^T(\hat\theta)\psi}{|\nabla\Pi(\hat\theta)|^2}\nabla\Pi(\hat\theta)\right) \\ &= \nabla\Pi^T(\hat\theta)\left(\alpha\nabla\Pi(\hat\theta) + \psi - \alpha\nabla\Pi(\hat\theta)\right) \\ &= 0 \end{aligned}$$

As a consequence, $\nabla\Pi^T(\hat\theta)\,\mathrm{Proj}_{\hat\theta\in\Theta}\{\tau\}\le 0$ at each point $\hat\theta\in\partial\Theta$, hence the vector field of system (C.59) points inward along the boundary of Θ. □

[5]Recall that, by assumption, $d_2 = 0$.

Proposition C.4.7 *Let* $\tilde{\theta} := \hat{\theta} - \theta^*$*. Then,* $\tilde{\theta}^T \operatorname{Proj}_{\hat{\theta}\in\Theta}\{\tau\} \le \tilde{\theta}^T\tau$ *for all* $\tilde{\theta} \in \mathbb{R}^{n_2}$ *and all* $\theta^* \in \operatorname{int}\Theta$*.*

Proof. According to the definition of $\operatorname{Proj}_{\hat{\theta}\in\Theta}\{\tau\}$, we only need to prove the proposition in the case where $\hat{\theta} \in \partial\Theta$ and $\nabla\Pi^T(\hat{\theta})\tau > 0$. From the convexity of the function $\Pi(\cdot)$ and the fact that $\theta^* \in \operatorname{int}\Theta$, it follows that

$$\tilde{\theta}^T\nabla\Pi(\hat{\theta}) = (\hat{\theta} - \theta^*)^T\nabla\Pi(\hat{\theta}) \ge 0 \quad \forall\,\hat{\theta} \in \partial\Theta$$

Consequently, if $\hat{\theta} \in \partial\Theta$ and $\nabla\Pi^T(\hat{\theta})\tau > 0$

$$\tilde{\theta}^T \operatorname*{Proj}_{\hat{\theta}\in\Theta}\{\tau\} \le \tilde{\theta}^T\tau = \tilde{\theta}^T\tau - \frac{\tilde{\theta}^T\nabla\Pi(\hat{\theta})\;\nabla\Pi^T(\hat{\theta})\tau}{|\nabla\Pi(\hat{\theta})|^2} \le \tilde{\theta}^T\tau$$

□

Stability Analysis

Evaluation of the Lyapunov function candidate (C.44) along the vector field of the closed-loop system (C.60) yields

$$\begin{aligned}\dot{V}(z,\tilde{\theta}) &= -\tfrac{1}{2}z^TQz + z^TPB\phi^T(t,z,\hat{\theta}) + z^TPd_1 + \tilde{\theta}^T \operatorname*{Proj}_{\hat{\theta}\in\Theta}\left\{-\gamma\phi(t,z,\hat{\theta})Cz\right\} \\ &\le -\tfrac{1}{2}z^TQz + z^TPd_1 \\ &\le -\frac{\lambda_{\min}}{2}|z|^2 + |z||P||d_1| \end{aligned} \tag{C.61}$$

where we have made use of Proposition C.4.7. Adding and subtracting the term $\lambda_{\min}|\tilde{\theta}|^2/2$ to the right-hand side of the last inequality in (C.61), and recalling that the solution of (C.59) satisfies $\hat{\theta}(t) \in \Theta$ for all $t \ge 0$, one obtains

$$\begin{aligned}\dot{V}(z,\tilde{\theta}) &\le -\frac{\lambda_{\min}}{2}|z|^2 - \frac{\lambda_{\min}}{2}|\tilde{\theta}|^2 + |z||P||d_1| + \frac{\lambda_{\min}}{2}|\tilde{\theta}|^2 \\ &\le -\frac{\lambda_{\min}}{2}|x|^2 + |x||P||d_1| + \frac{\lambda_{\min}}{2}\mu^2 \end{aligned} \tag{C.62}$$

where $\mu = 2\max_{\theta\in\Theta}|\theta|$. Application of Young's inequality yields

$$\dot{V}(x) \le -\frac{\lambda_{\min}}{4}|x|^2 + \frac{|P|^2}{\lambda_{\min}}|d_1|^2 + \frac{\lambda_{\min}}{2}\mu^2$$

Consequently, defining the class-$\mathcal{N}$ function

$$\chi(s) = \sqrt{\frac{4|P|^2}{\lambda_{\min}^2}s^2 + 2\mu^2}$$

one obtains

$$|x| > \chi(|d_1|) \implies \dot{V}(x) < 0$$

therefore, by Theorem B.3.13 the perturbed system (C.60) has the GUUB property when $d_1(\cdot) \in \mathcal{L}^\infty$.

Bibliography

[1] B. L. Stevens and F. L. Lewis. *Aircraft Control and Simulation*. John Wiley and Sons, 2nd edition, 2003.

[2] L. Zaccarian. Dynamic allocation for input redundant control systems. *Automatica*, 45(6):1431–1438, June 2009.

[3] A. Serrani. Output regulation for over-actuated linear systems via inverse model allocation. In *Proceedings of the 2012 IEEE Conference on Decision and Control*, Maui, HI, 2012.

[4] L. Zaccarian and A. R. Teel. *Modern Anti-windup Synthesis: Control Augmentation for Actuator Saturation*. Princeton University Press, Princeton, NJ, 2011.

[5] Daniel Y. Abramovitch and Gene F. Franklin. On the Stability of Adaptive Pole-Placement Controllers with a Saturating Actuator. *IEEE Transactions on Automatic Control*, 35(3):303–306, 1990.

[6] SP Karason and AM Annaswamy. Adaptive Control in the Presence of Input Constraints. *IEEE Transactions on Automatic Control*, 39(11):2325–2330, 1994.

[7] AM Annaswamy and J.E. Wong. Adaptive control in the presence of saturation non-linearity. *International Journal of Adaptive Control and Signal Processing*, 11(1):3–19, 1997.

[8] E N Jonhson and A J Calise. Limited authority adaptive flight control for reusable launch vehicles. *Journal of Guidance, Control, and Dynamics*, 26(6):906–913, 2003.

[9] E Lavretsky and N Hovakimyan. Stable adaptation in the presence of input constraints. *Systems & Control Letters*, 56(11-12):722–729, nov 2007.

[10] Alexander Leonessa, W.M. Haddad, Tomohisa Hayakawa, and Y. Morel. Adaptive control for nonlinear uncertain systems with actuator amplitude and rate saturation constraints. *International Journal of Adaptive Control and Signal Processing*, 23(1):73–96, 2009.

[11] J. Farrell, M. Sharma, and M. Polycarpou. Backstepping-Based Flight Control with Adaptive Function Approximation. *Journal of Guidance, Control, and Dynamics*, 28(6):1089–1102, 2005.

[12] L. Sonneveldt, Q. P. Chu, and J. A. Mulder. Nonlinear flight control design using constrained adaptive backstepping. *Journal of Guidance, Control, and Dynamics*, 30(2):322–336, 2007.

[13] P. Tsiotras. Stabilization and optimality results for the attitude control problem. *Journal of Guidance, Control, and Dynamics*, 19(4):772–779, July 1996.

[14] A.R. Teel. A nonlinear small gain theorem for the analysis of control systems with saturation. *IEEE Transactions on Automatic Control*, 41(9):1256–1270, 1996.

[15] E.D. Sontag. Smooth stabilization implies coprime factorization. *IEEE Transactions on Automatic Control*, 34(4):435–443, 1989.

[16] C. M. Kellett. A compendium of comparison function results. *Mathematics of Control, Signals, and Systems*, 26(3):339–374, 2014.

[17] R. Baggi, E. Franco, and A. Serrani. Dynamic control allocation for a class of over-actuated aircraft. In *Proceedings of the 2020 AIAA Science and Technology Forum and Exposition*, 2020.

[18] R. Baggi. *Design, Development and Testing of Dynamic Adaptive Control Allocation for a Class of Over-Actuated Aircraft.* Ph.D. Dissertation, Department of Mechanical Engineering, University of California, Riverside, CA, 2020.

[19] R. Baggi, A. Serrani, and E. Franco. Hierarchical dynamic control allocation for over-actuated aircraft: Methodology and flight tests on a scaled-down model. In *Proceedings of the 6th IEEE Conference on Control Technology and Applications*, Trieste, Italy, 2022.

[20] H.K. Khalil. *Nonlinear Systems.* Prentice Hall, Upper Saddle River, NJ, 3rd edition, 2002.

[21] M. Vidyasagar. *Nonlinear Systems Analysis.* Classics in Applied Mathematics. SIAM, 2nd edition, 2002.

[22] Y. Lin, E. D. Sontag, and Y. Wang. A smooth converse Lyapunov theorem for robust stability. *SIAM Journal on Control and Optimization*, 34(1):124–60–, 1996.

[23] A. R. Teel and L. Praly. A smooth Lyapunov function from a class-$\mathcal{KL}$ estimate involving two positive semidefinite functions. *ESAIM. Control, Optimisation and Calculus of Variations*, 5:313–67–, 2000.

[24] J.L. Massera. Contributions to stability theory. *Annals of Mathematics. Second Series*, 64:182–206, 1956.

[25] A. Isidori. *Nonlinear Control Systems II.* Springer-Verlag, New York, NY, 1999.

[26] J. Kurzweil. On the inversion of Lyapunov's second theorem on stability of motion. *Ann. Math. Soc. Trans. Ser.2*, 24:19–77, 1956.

[27] V.I. Zubov. *Methods of A.M. Lyapunov and their application.* Noordhoff, Groningen, The Netherlands, 1964.

[28] P.A. Ioannou and J. Sun. *Robust Adaptive Control.* Prentice Hall, Englewood Cliffs, NJ, 1996.

[29] A. R. Teel. Asymptotic convergence from $\mathcal{L}_p$ stability. *IEEE Transactions on Automatic Control*, 44(11):2169–70–, 1999.

[30] T. Yoshizawa. *Stability theory by Ljapunov's second method*, volume 9. The Mathematical Society of Japan., Tokyo, Japan, 1966.

[31] B.D.O. Anderson and J.B. Moore. New results in linear system stability. *SIAM J. Control*, 7:398–414, 1969.

[32] M. Ikeda, H. Maeda, and S. Kodama. Estimation and feedback in linear time-varying systems: a deterministic theory. *SIAM J. Control*, 13(2):304–26, 1975.

[33] L. M. Silverman and B. D. O. Anderson. Controllability, observability and stability of linear systems. *SIAM J. Control*, 6(1):121–130, 1968.

[34] V.M. Matrosov. On the stability of motion. *J. Appl. Math. Mech.*, 26:1337–1353, 1962.

[35] N. Rouche and J. Mawhin. *Ordinary differential equations. Stability and periodic solutions.* Pitman, London, UK, 1980.

[36] A. Loria, E. Panteley, D. Popovic, and A. R. Teel. A nested Matrosov theorem and persistency of excitation for uniform convergence in stable nonautonomous systems. *IEEE Transactions on Automatic Control*, 50(2):183–198, 2005.

[37] A. Van Der Schaft. *L_2-Gain and Passivity Techniques in Nonlinear Control.* Springer-Verlag, New York, NY, 1996.

[38] L. Ljung. *System identification: Theory for the user.* Prentice Hall, Upper Saddle River, NJ, 2nd edition, 1999.

[39] K.S. Narendra and A.M. Annaswamy. *Stable adaptive systems.* Prentice Hall,, 1989.

[40] S. Sastry and M. Bodson. *Adaptive Control: Stability, Convergence and Robustness.* Prentice Hall, Upper Saddle River, NJ, 1989.

[41] E. Panteley, A. Loria, and A. Teel. Relaxed persistency of excitation for uniform asymptotic stability. *IEEE Transactions on Automatic Control*, 46(12):1874–1886, 2001. ID: 7157470.