Motion planning problem formulation

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Lecture objectives

- Motion planning problem statement

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- Trajectory and path planning problems

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- Randomized methods

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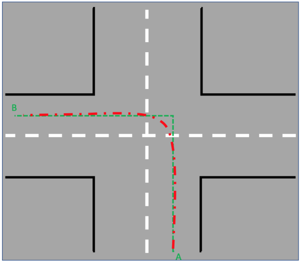

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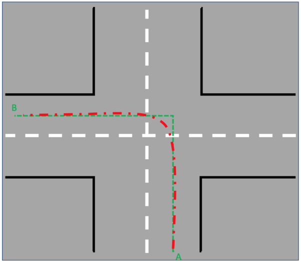

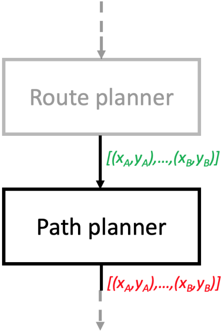

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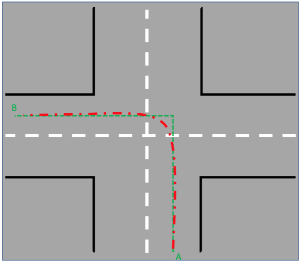

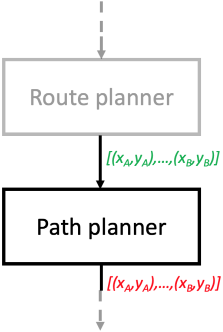

The cost should then be designed such that the *planned path* minimizes, in some sense, the deviation from the *reference path*.

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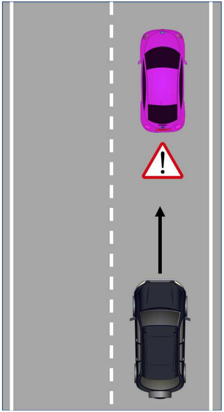

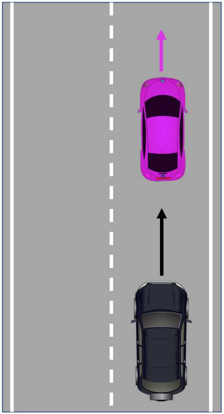

Randomized methods[a]

Problem formulation. Given a *configuration workspace* $C = C_{free} \bigcup C_{obs}$ and the points $x_{start},\ x_{goal} \in C_{free}$, find a function $f : [0, 1] \to C_{free}$, such that $f(0) = x_{start}$, $f(1) = x_{goal}$.

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3. simple (uniform) sampling policies may leave narrow regions poorly explored.

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The tree can grow faster if the tree grows from both x_{start} to x_{goal}.

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Actuator limitations and design constraints are not included in the force generation. The forces f_g, f_o could be limited.