Optimization-based motion planning

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Modena

Lecture objectives

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  5. Safety constraints. Static obstacles

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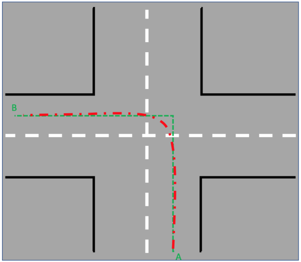

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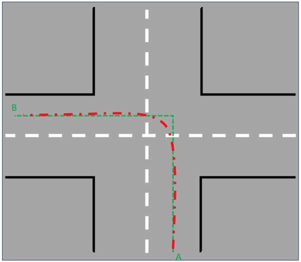

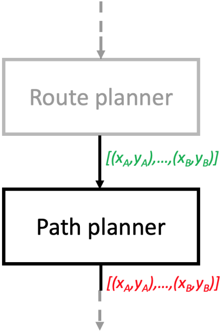

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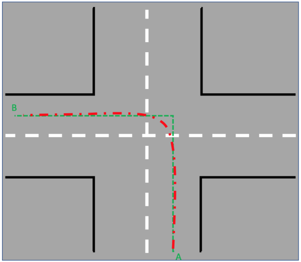

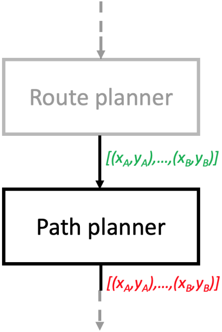

The cost should then be designed such that the *planned path* minimizes, in some sense, the deviation from the *reference path*.

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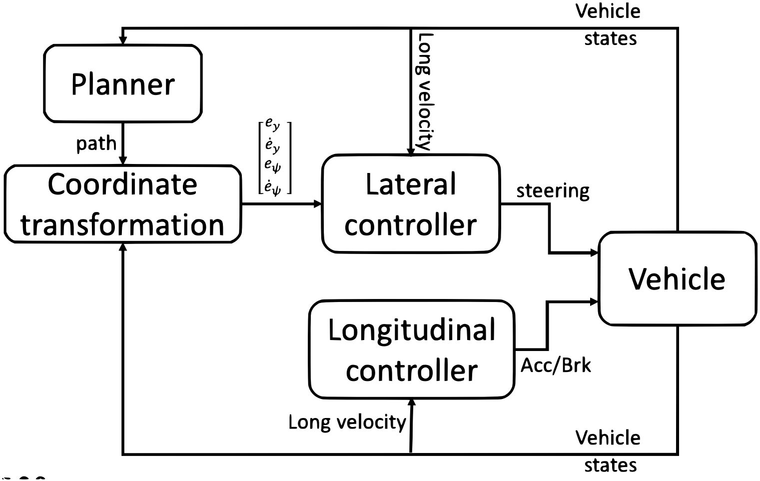
Architecture



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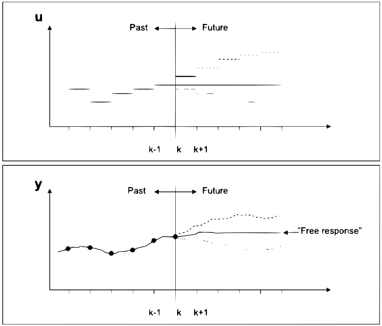

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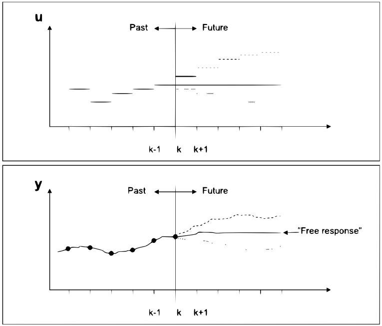

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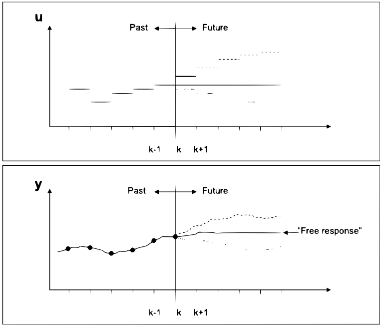

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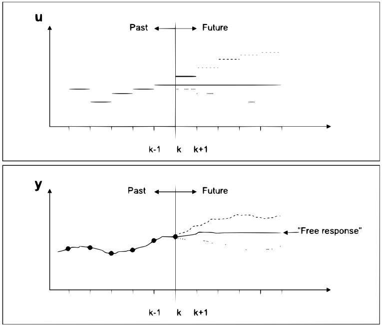
3. Apply the first element in the control sequence to the process, discard the rest of the sequence, and return to step 1.



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5. Increment time $k := k+1$ and go to 1.

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Space vs. Time-domain problem formulations

We have introduced vehicle models in the form

$$\dot{x} = f(x, u), \tag{3}$$

where the state and inputs are functions of time.

The vehicle model can be rewritten as

$$\frac{dx}{ds} = \frac{dt}{ds}\frac{dx}{dt} = \frac{1}{V_s} f(x(s), u(s)), \tag{4}$$

where s and V_s are the traveled distance and the vehicle speed over the path.

Time-domain. With (3) as prediction model, the problem is formulated over a finite-time horizon. Cost, reference and constraints needs to be defined w.r.t. the time variable.

Space-domain. With (3) as prediction model, the problem is formulated over a finite-distance ahead. Cost, reference and constraints needs to be defined w.r.t. the traveled distance s.

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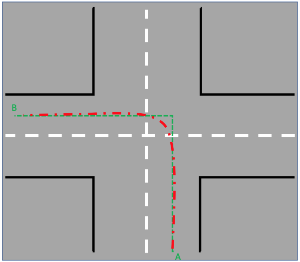

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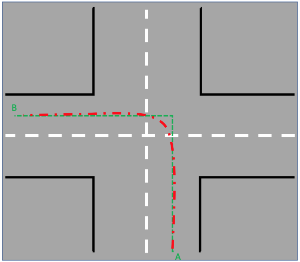

Quadratic cost. Squared, 2-norm of the distance from the path

$$V_N(x(0), u(0:N-1)) = e_x(N)^\top P_f e_x(N) + \sum_{i=0}^{N-1} e_x^\top(i) Q e_x(N)(i) + e_u(i)^\top(i) R e_u(i).$$

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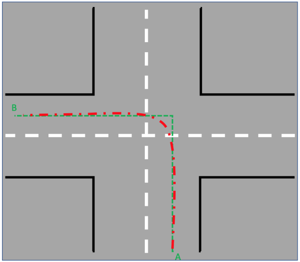

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Strongly penalizes large deviation from the path.

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The minimization of V_N results into a linear cost.

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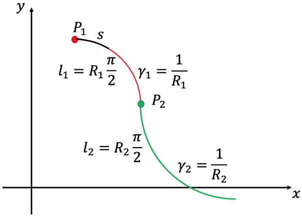

$$\begin{aligned} x(s) &= P_1^x + R_1 \sin \tfrac{s}{R_1}, \\ y(s) &= P_1^y - R_1\left(1 - \cos \tfrac{s}{R_1}\right), && 0 \le s \le l_1 \\ \psi(s) &= -\tfrac{s}{R_1}, \end{aligned}$$

$$\begin{aligned} x(s) &= P_2^x + R_2\left(1 - \cos \tfrac{s-l_1}{R_2}\right), \\ y(s) &= P_2^y + R_2 \sin \tfrac{s-l_1}{R_2}, && l_1 \le s \le l_2 \\ \psi(s) &= -\tfrac{\pi}{2} + \tfrac{s-l_1}{R_1}, \end{aligned}$$

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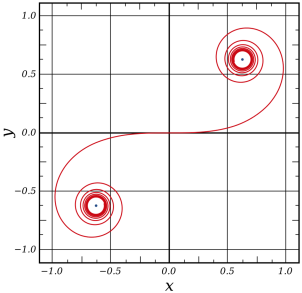

Path sampling

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The additional control input v is used to avoid aggressive maneuvers due to obstacles that may lead to large tracking errors.

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\dot{\psi} &= \frac{V\cos\beta}{l_f + l_r}\left(\tan\delta_f - \tan\delta_r\right), \\
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In case a bicycle model is used to plan the motion, motion planning and control can be lumped together into a single task.

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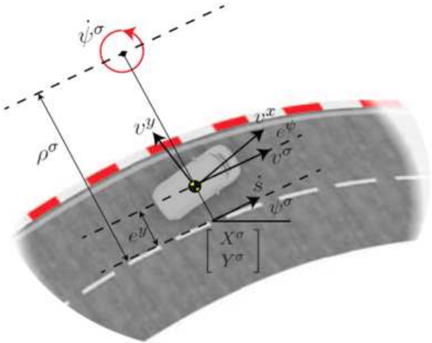

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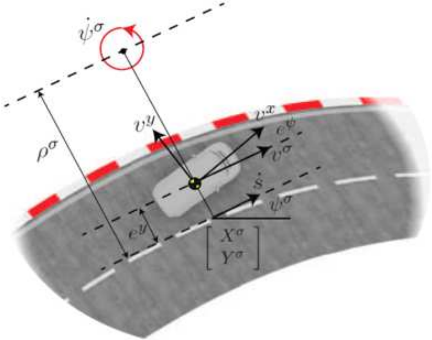

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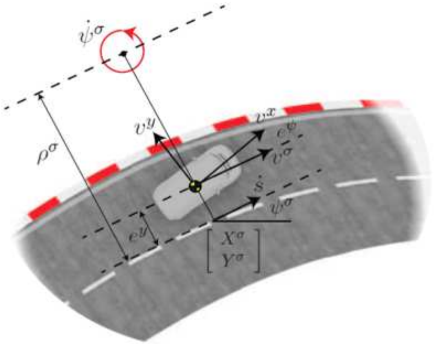

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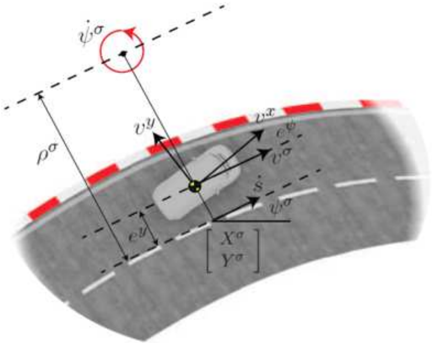

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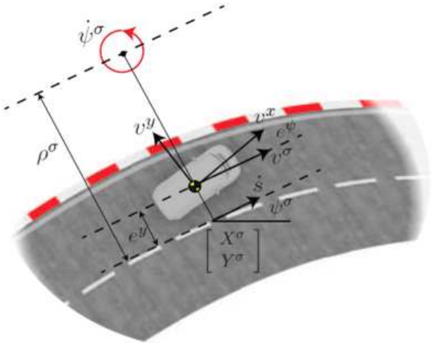

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Note that, all variables are to be expressed in the space domain (they are function of s).

Safety constraints. Static obstacles

By rewriting the vehicle model in the space domain, where the coordinates describe the vehicle position and orientation w.r.t. the reference path, the safety constraints simply become

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collision avoidance constraints are imposed by

$$e_y(s) \in \left[-\frac{3L_w(s)}{2},\ \frac{L_w(s)}{s}\right] \setminus \mathcal{X}^{obs},\ s \in [s^{obs,min},\ s^{obs,max}]$$