Data-driven control design

linear and nonlinear systems

Lecture 1

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Models and control

> *If physics is the science of understanding the physical environment, then* **control theory** *may be viewed as* **the science of modifying that environment** *[...] Control theory does not deal directly with physical reality but with* **mathematical models**.

Rudolf Kalman, Control Theory, *Encyclopædia Britannica*

$$\begin{aligned} x^+/\dot{x} &= Ax + Bu \\ y &= Cx + Du \end{aligned} \qquad \begin{aligned} x^+/\dot{x} &= f(x,u) \\ y &= h(x,u) \end{aligned}$$

Data-driven control

To offset the lack of “known” models by the use of data

Using data through the lenses of control theory

Control when the dynamics is "unknown"

If the model is unknown, there are a few approaches

System identification from data + control of the identified system

- G. Pillonetto et al. "Kernel methods in system identification, machine learning and function estimation: A survey". Automatica, 50(3):657–682, 2014.

Direct data-based control design

- M.C. Campi, A. Lecchini, and S.M. Savaresi. "Virtual reference feedback tuning: a direct method for the design of feedback controllers". Automatica, 38(8):1337-1346, 2002.

These lectures "Direct" design of controllers from data for "unknown" systems

Direct because the method returns controllers via data-dependent SDPs

The system is "unknown" but some priors are available

Data are collected to infer information about the dynamics

The method
- works with perturbed data of low complexity
- provides analytical guarantees of correctness
- is based on basic tools of automatic control

Control when the dynamics is unknown

These lectures "Direct" design of controllers from data for "unknown" systems

Lec 1	Linear systems Unperturbed data of low complexity
Lec 2	Perturbed data
Lec 3	A first glimpse at nonlinear control system design: Lyapunov's indirect method
Lec 4	Nonlinear control system design via approximate and exact feedback linearization
Lec 5	Advanced topics: contraction and tracking problems

The lectures will present a personal perspective and will focus on a few selected papers (listed at the end of the lectures). A broader overview and a discussion of related work are provided in those papers.

Outline Lecture 1

We will study 2 (data-driven) control problems

- Full measurements Stabilization of linear systems via static state feedback
- Partial measurements Stabilization of linear systems via dynamic output feedback

To introduce the main ideas, in Lecture 1 we consider the ideal case of unperturbed (noise-free) data and linear systems.

Before diving into the control design, we introduce the dataset and a concept that is at the core of these lectures.

What we do not cover

- Linear Quadratic Regulation, robust invariance, model reference control, output feedback control with noisy data (linear systems)
- Bilinear, Polynomial and Lur'e systems (nonlinear systems)
- Many other topics.

Dynamical control systems

We focus our attention on systems of the form

$$x^+ = Ax + Bu$$

- ▷ $x \in \mathbb{R}^n$ (state) and $u \in \mathbb{R}^m$ (control)
- ▷ $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$ are unknown matrices

At this stage we do not impose any property on the system. Whenever a particular property is needed, we introduce it.

Focus on discrete-time systems (but we will also briefly remark on continuous-time systems later on)

Dataset

Information about the system's dynamics is obtained from a T-long dataset of input/state samples collected during (multiple) experiment(s)

$$\mathbb{D} := \{u(k), x(k)\}_{k=0}^{T-1} \cup \{x(T)\}$$

where the samples satisfy

$$x(k+1) = Ax(k) + Bu(k), \quad k = 0, \ldots, T-1$$

Persistence of excitation

The approach to the design of controllers from data is inspired by nonparametric data-dependent representations of the unknown dynamics. To recall the origin of such a representation, we recall a notion of persistently exciting signals, which is useful to generate "rich" data.

Definition The sequence of input values $u : [0, T-1] \cap \mathbb{Z} \to \mathbb{R}^m$

$$u(0), u(1), \ldots, u(T-1)$$

is persistently exciting (PE) of order L if the Hankel matrix associated to it

$$U_0 = \begin{bmatrix} u(0) & u(1) & \ldots & u(T-L) \\ u(1) & u(2) & \ldots & u(T-L+1) \\ \vdots & \vdots & \ddots & \vdots \\ u(L-1) & u(L) & \ldots & u(T-1) \end{bmatrix}$$

has full rank mL.

PE requires sufficiently long input sequences: $T \geq (m+1)L - 1$

Generating PE signals

```
% File Gen_PE_inputs.m
clear all
close all
rng(1);

% Order of the PE input
L=3;

% Dimension of the input space
m=2;

% Length of the  input sequence
T=(m+1)*L-1;

% Generating the input sequence u on [0,T-1] taking values in the interval
% [-0.5,0.5]^m in the form of an m x T matrix [u(0) u(1) .... u(T-1)]
magnitude=0.5;
aux=zeros(m,T);
aux(:)=magnitude;
u(1:m,1:T)=(2*magnitude).*rand(m,T)-aux;
```

```
% Arranging the samples in the Hankel matrix U0 on [0,T-1]
for j=1:T-L+1
    for i=1:L
        U0((i-1)*m+1:(i-1)*m+m,j)=u(1:m, j+i-1);
    end
end

% If rank(U0)= m*L then the sequence u(0),...u(T-1) is PE of order L

if rank(U0) == m*L
disp('input sequence is PE');
end
```

$L = 3, m = 2$ $(T = 8)$

$$u_{[0,T-1]} = \begin{bmatrix} -0.0830 & -0.4999 & -0.3532 & -0.3137 & -0.1032 & -0.0808 & -0.2955 & -0.4726 \\ 0.2203 & -0.1977 & -0.4077 & -0.1544 & 0.0388 & 0.1852 & 0.3781 & 0.1705 \end{bmatrix}$$

$$U_0 = \left[\begin{array}{cccccc} -0.0830 & -0.4999 & -0.3532 & -0.3137 & -0.1032 & -0.0808 \\ 0.2203 & -0.1977 & -0.4077 & -0.1544 & 0.0388 & 0.1852 \\ \hline -0.4999 & -0.3532 & -0.3137 & -0.1032 & -0.0808 & -0.2955 \\ -0.1977 & -0.4077 & -0.1544 & 0.0388 & 0.1852 & 0.3781 \\ \hline -0.3532 & -0.3137 & -0.1032 & -0.0808 & -0.2955 & -0.4726 \\ -0.4077 & -0.1544 & 0.0388 & 0.1852 & 0.3781 & 0.1705 \end{array}\right]$$

More simply, PE inputs can be generated analytically.

Example 1 Consider to generate a sequence of scalar ($m = 1$) inputs $\{u(k)\}_{k=0}^{T-1}$ that is PE of order $L = 3$ ($T \geq 5$).
We build the Hankel matrix

$$U_0 = \begin{bmatrix} u(0) & u(1) & u(2) & \dots & u(T-3) \\ u(1) & u(2) & u(3) & \dots & u(T-2) \\ u(2) & u(3) & u(4) & \dots & u(T-1) \end{bmatrix}$$

and we would like to design the samples to render U_0 a full-row rank matrix. The choice

$$u(0) = 0, u(1) = 0, u(2) = 1, u(3) = 0, \dots, u(T-1) = 0$$

returns the matrix

$$U_0 = \begin{bmatrix} 0 & 0 & 1 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ 1 & 0 & 0 & \dots & 0 \end{bmatrix}$$

with the desired property.

<u>Example 2</u> Consider to generate a sequence of inputs $\{u(k)\}_{k=0}^{T-1}$, $u(k) \in \mathbb{R}^2$ $(m = 2)$, which is PE of order $L = 3$ $(T \geq 8)$.
We build the Hankel matrix

$$U_0 = \begin{bmatrix} u(0) & u(1) & u(2) & u(3) & u(4) & u(5) & \dots & u(T-3) \\ u(1) & u(2) & u(3) & u(4) & u(5) & u(6) & \dots & u(T-2) \\ u(2) & u(3) & u(4) & u(5) & u(6) & u(7) & \dots & u(T-1) \end{bmatrix}$$

and we would like to design the samples to render U_0 a full-row rank matrix. As $m = 2$, the strategy is to render the submatrix made of the first 6 rows/columns nonsingular. The choice

$$u(0) = \left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right], u(1) = \left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right], u(2) = \left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right], u(3) = \left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right], u(4) = \left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right]$$

returns the matrix

$$U_0 = \begin{bmatrix} \left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right] & \left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right] & \left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right] & \left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right] & \left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right] & u(5) & \dots & u(T-3) \\ \left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right] & \left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right] & \left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right] & \left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right] & u(5) & u(6) & \dots & u(T-2) \\ \left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right] & \left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right] & \left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right] & u(5) & u(6) & u(7) & \dots & u(T-1) \end{bmatrix}$$

$$U_0 = \begin{bmatrix} \left[\begin{smallmatrix}0\\0\end{smallmatrix}\right] & \left[\begin{smallmatrix}0\\0\end{smallmatrix}\right] & \left[\begin{smallmatrix}1\\0\end{smallmatrix}\right] & \left[\begin{smallmatrix}0\\0\end{smallmatrix}\right] & \left[\begin{smallmatrix}0\\0\end{smallmatrix}\right] & u(5) & \dots & u(T-3) \\ \left[\begin{smallmatrix}0\\0\end{smallmatrix}\right] & \left[\begin{smallmatrix}1\\0\end{smallmatrix}\right] & \left[\begin{smallmatrix}0\\0\end{smallmatrix}\right] & \left[\begin{smallmatrix}0\\0\end{smallmatrix}\right] & u(5) & u(6) & \dots & u(T-2) \\ \left[\begin{smallmatrix}1\\0\end{smallmatrix}\right] & \left[\begin{smallmatrix}0\\0\end{smallmatrix}\right] & \left[\begin{smallmatrix}0\\0\end{smallmatrix}\right] & u(5) & u(6) & u(7) & \dots & u(T-1) \end{bmatrix}$$

To make the 4th column linearly independent from the previous 3, it is natural to design

$$u(5) = \left[\begin{smallmatrix}0\\1\end{smallmatrix}\right], u(6) = \left[\begin{smallmatrix}0\\0\end{smallmatrix}\right], \dots, u(T-1) = \left[\begin{smallmatrix}0\\0\end{smallmatrix}\right]$$

which returns

$$U_0 = \begin{bmatrix} \left[\begin{smallmatrix}0\\0\end{smallmatrix}\right] & \left[\begin{smallmatrix}0\\0\end{smallmatrix}\right] & \left[\begin{smallmatrix}1\\0\end{smallmatrix}\right] & \left[\begin{smallmatrix}0\\0\end{smallmatrix}\right] & \left[\begin{smallmatrix}0\\0\end{smallmatrix}\right] & \left[\begin{smallmatrix}0\\1\end{smallmatrix}\right] & \dots & \left[\begin{smallmatrix}0\\0\end{smallmatrix}\right] \\ \left[\begin{smallmatrix}0\\0\end{smallmatrix}\right] & \left[\begin{smallmatrix}1\\0\end{smallmatrix}\right] & \left[\begin{smallmatrix}0\\0\end{smallmatrix}\right] & \left[\begin{smallmatrix}0\\0\end{smallmatrix}\right] & \left[\begin{smallmatrix}0\\1\end{smallmatrix}\right] & \left[\begin{smallmatrix}0\\0\end{smallmatrix}\right] & \dots & \left[\begin{smallmatrix}0\\0\end{smallmatrix}\right] \\ \left[\begin{smallmatrix}1\\0\end{smallmatrix}\right] & \left[\begin{smallmatrix}0\\0\end{smallmatrix}\right] & \left[\begin{smallmatrix}0\\0\end{smallmatrix}\right] & \left[\begin{smallmatrix}0\\1\end{smallmatrix}\right] & \left[\begin{smallmatrix}0\\0\end{smallmatrix}\right] & \left[\begin{smallmatrix}0\\0\end{smallmatrix}\right] & \dots & \left[\begin{smallmatrix}0\\0\end{smallmatrix}\right] \end{bmatrix}$$

with the desired property (the 6×6 submatrix made of the first rows/columns is I_6 after elementary row/column manipulations).

This design can be applied to every m-dimensional input space and every PE order L and returns a sparse input sequence.

The Fundamental Lemma

A PE input applied to a linear reachable* system produces data that are sufficiently rich.

*A system is reachable if and only if rank $\begin{bmatrix} B & AB & \dots & A^{n-1}B \end{bmatrix} = n$

Lemma Let system

$$x(k+1) = Ax(k) + Bu(k)$$

be <u>reachable</u>. For any $t \geq 1$,

$$u_{[0,T-1]} \text{ PE of order } n+t \quad \Rightarrow \quad \text{rank} \begin{bmatrix} U_0 \\ X_0 \end{bmatrix} = n + tm$$

where the matrix U_0 consists of the samples of the input sequence $u_{[0,T-1]} = \{u(0), u(1), \dots, u(T-1)\}$

$$U_0 = \begin{bmatrix} u(0) & u(1) & \dots & u(T-t) \\ u(1) & u(2) & \dots & u(T-t+1) \\ \vdots & \vdots & \ddots & \vdots \\ u(t-1) & u(t) & \dots & u(T-1) \end{bmatrix}$$

and the matrix X_0 consists of the samples of the state response $x(k+1) = Ax(k) + Bu(k)$, $k = 0, 1, \dots, T-t$, to the input sequence $u_{[0,T-1]}$

$$X_0 = \begin{bmatrix} x(0) & x(1) & \dots & x(T-t) \end{bmatrix}$$

J.C. Willems, P. Rapisarda, I. Markovsky, B.L. De Moor. "A note on persistency of excitation." Systems & Control Letters, 54, 4, 325–329, 2005.

Example

A partially known model ($n = 2$, $m = 1$, reachable system)

$u_{[0,T-1]}$ PE of order $L = n + t = 3$ ($n = 2$, $t = 1$), with $T = L(m+1) - 1 \geq 5$ ($T = 5$)

$$u_{[0,T-1]} = \begin{bmatrix} -0.0166 & 0.0441 & -0.1000 & -0.0395 & -0.0706 \end{bmatrix}$$

We "experimentally" determine the matrix ($U_0 \in \mathbb{R}^{m \times T-t+1}$, $X_0 \in \mathbb{R}^{n \times T-t+1}$)

$$\begin{bmatrix} U_0 \\ \hline X_0 \end{bmatrix} = \begin{bmatrix} -0.0166 & 0.0441 & -0.1000 & -0.0395 & -0.0706 \\ \hline -0.0815 & -0.0962 & -0.1132 & -0.1337 & -0.1577 \\ -0.0627 & -0.0451 & -0.0043 & -0.0438 & -0.0406 \end{bmatrix}$$

where

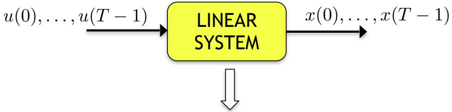
$$X_0 = \begin{bmatrix} x(0) & x(1) & x(2) & x(3) & x(4) \end{bmatrix}$$

contains the state response of the system from the initial condition $x(0)$ to the input $u_{[0,4]}$. As predicted, $\begin{bmatrix} U_0 \\ X_0 \end{bmatrix}$ has rank $n + tm = 3$.

```
%Bertinoro_Lect1_Exmpl_Data_Rich

clear all
close all
rng(1);

%% System

n = 2;
m = 1;

A=[ 1.178 0.001;...
     -0.051 0.661];

B= [0.004;...
       0.467];

T = 5; % (m+1)*L-1; % length of a trajectory; L=n+t; t=1
```

```
% Generate random values for the inputs and the initial state
% from the uniform distribution on the interval [-mag, mag].

mag = 0.1;

U0  = (2*mag).*rand(m,T)-mag; % as t=1, U0 is m x T
x   = (2*mag).*rand(n,1)-mag;

X   = x;

for i=1:T
    x=A*x+B*U0(:,i);
    X   = [X x];
end

X0  = X(:,1:end-1); % as t=1, X0 is n x T
X1  = X(:,2:end);

if rank([U0 ; X0]) == m+n
disp('data are sufficiently rich');
end
```

Profound implications for control



$$\begin{bmatrix} U_0 \\ \hline X_0 \end{bmatrix} = \begin{bmatrix} u(0) & \dots & u(T-t) \\ \vdots & \ddots & \vdots \\ u(t-1) & \dots & u(T-1) \\ \hline x(0) & \dots & x(T-t) \\ \vdots & \ddots & \vdots \\ x(t-1) & \dots & x(T-1) \end{bmatrix}$$

$$\begin{bmatrix} \bar{u}(0) \\ \vdots \\ \bar{u}(t-1) \\ \hline \bar{x}(0) \\ \vdots \\ \bar{x}(t-1) \end{bmatrix} = \begin{bmatrix} U_0 \\ \hline X_0 \end{bmatrix} g$$

For a reachable linear system
(i) Let $u(0), \dots, u(T-1)$ be PE of order $n+t$, $t \geq 1$, then any t-long input/state trajectory of the system $(\bar{u}_{[0,t-1]}, \bar{x}_{[0,t-1]})$ can be expressed as

$$\begin{bmatrix} \bar{u}_{[0,t-1]} \\ \bar{x}_{[0,t-1]} \end{bmatrix} = \begin{bmatrix} U_0 \\ X_0 \end{bmatrix} g$$

where $g \in \mathbb{R}^{T-t+1}$.
(ii) Any linear combination of the columns of the matrix of data, i.e.,

$$\begin{bmatrix} U_0 \\ \hline X_0 \end{bmatrix} g,$$

is a t-long input-state trajectory of the system.

Relating closed-loop trajectories with data

Consider Item (i) in the special case $t = 1$. Then

$$\begin{bmatrix} \bar{u}_{[0,t-1]} \\ \bar{x}_{[0,t-1]} \end{bmatrix} = \begin{bmatrix} U_0 \\ X_0 \end{bmatrix} g \quad \text{becomes} \quad \begin{bmatrix} \bar{u}(0) \\ \bar{x}(0) \end{bmatrix} = \begin{bmatrix} U_0 \\ X_0 \end{bmatrix} g$$

Given a $K \in \mathbb{R}^{m\times n}$, consider n 1-long input/state trajectories

$$\begin{bmatrix} \bar{u}(0) \\ \bar{x}(0) \end{bmatrix} = \begin{bmatrix} K\bar{x}(0) \\ \bar{x}(0) \end{bmatrix}, \quad \bar{x}(0) = e_i, \quad i = 1, 2, \ldots, n$$

where e_i is the i-th vector of the canonical basis of $\mathbb{R}^n$.

Then

$$\begin{bmatrix} K \\ I_n \end{bmatrix} \begin{bmatrix} e_1 & \ldots & e_n \end{bmatrix} = \begin{bmatrix} U_0 \\ X_0 \end{bmatrix} \begin{bmatrix} g_1 & \ldots & g_n \end{bmatrix}$$

that is,

$$\begin{bmatrix} K \\ I_n \end{bmatrix} = \begin{bmatrix} U_0 \\ X_0 \end{bmatrix} G$$

Stabilization of linear systems

Data-dependent representations

Consider the dataset

$$\mathbb{D} = \{u(k), x(k)\}_{k=0}^{T}, \quad x(k+1) = Ax(k) + Bu(k), \quad k = 0, \ldots, T-1$$

and store it into matrices U_0, X_0, X_1 defined as

$$\begin{aligned} U_0 &:= \begin{bmatrix} u(0) & u(1) & \cdots & u(T-1) \end{bmatrix} \\ X_0 &:= \begin{bmatrix} x(0) & x(1) & \cdots & x(T-1) \end{bmatrix} \\ X_1 &:= \begin{bmatrix} x(1) & x(2) & \cdots & x(T) \end{bmatrix} \end{aligned}$$

which satisfy the identity

$$\underbrace{\begin{bmatrix} x(1) & x(2) & \ldots & x(T) \end{bmatrix}}_{X_1} = A \underbrace{\begin{bmatrix} x(0) & x(1) & \ldots & x(T-1) \end{bmatrix}}_{X_0} + B \underbrace{\begin{bmatrix} u(0) & u(1) & \ldots & u(T-1) \end{bmatrix}}_{U_0}$$

$$X_1 = AX_0 + BU_0$$

Data-dependent representations

Consider a full-state feedback $u = Kx$ and the resulting closed-loop system $x^+ = (A + BK)x$

Consider any matrices $K \in \mathbb{R}^{m\times n}$, $G \in \mathbb{R}^{T\times n}$ such that

$$\begin{bmatrix} K \\ I_n \end{bmatrix} = \begin{bmatrix} U_0 \\ X_0 \end{bmatrix} G$$

where

$$\begin{aligned} U_0 &= \begin{bmatrix} u(0) & u(1) & \dots & u(T-1) \end{bmatrix} \\ X_0 &= \begin{bmatrix} x(0) & x(1) & \dots & x(T-1) \end{bmatrix} \end{aligned} \qquad X_1 = AX_0 + BU_0$$

The matrix $A + BK$ of the closed-loop system $x^+ = (A + BK)x$ is arranged as

$$\begin{aligned} & A + BK \\ = \; & \begin{bmatrix} B & A \end{bmatrix} \begin{bmatrix} K \\ I_n \end{bmatrix} \\ \overset{\begin{bmatrix} K \\ I_n \end{bmatrix} = \begin{bmatrix} U_0 \\ X_0 \end{bmatrix} G}{=} \; & \begin{bmatrix} B & A \end{bmatrix} \begin{bmatrix} U_0 \\ X_0 \end{bmatrix} G \\ \overset{X_1 = \begin{bmatrix} B & A \end{bmatrix} \begin{bmatrix} U_0 \\ X_0 \end{bmatrix}}{=} \; & X_1 G \end{aligned}$$

Data-based parametrization of the closed-loop system

Theorem Consider the system $x^+ = Ax + Bu$. Consider any matrices $K \in \mathbb{R}^{m\times n}$, $G \in \mathbb{R}^{T\times n}$ such that

$$\begin{bmatrix} K \\ I_n \end{bmatrix} = \begin{bmatrix} U_0 \\ X_0 \end{bmatrix} G$$

Then the closed-loop system $x^+ = (A + BK)x$ has the following equivalent representation

$$x^+ = X_1 G x$$

- The representation depends on data U_0, X_0, X_1 and design variables G
- The design of the controller is shifted from K to G and in the process the system's matrices are replaced by data.
- If the system is reachable and the input PE of order $n+1$, rank $\begin{bmatrix} U_0 \\ X_0 \end{bmatrix} = n + m$ and matrices $K \in \mathbb{R}^{m\times n}$, $G \in \mathbb{R}^{T\times n}$ such that $\begin{bmatrix} K \\ I_n \end{bmatrix} = \begin{bmatrix} U_0 \\ X_0 \end{bmatrix} G$ exist.

C. De Persis, P. Tesi. "Formulas for data-driven control: stabilization, optimality, robustness". IEEE Transactions on Automatic Control, 65, 3, 909–924, 2020.

Linear Matrix Inequalities and Semidefinite Programs

LMIs

A **linear matrix inequality** (LMI) is an expression of the form

$$F(y) := F_0 + F_1 y_1 + \ldots + F_N y_N \prec 0$$

where

- $F\colon \mathbb{R}^N \to \mathbb{S}^{M\times M}$ is an affine function
- $y = \begin{bmatrix} y_1 \ldots y_N \end{bmatrix}^\top \in \mathbb{R}^N$ is the variable
- $F_0, F_1, \ldots, F_N$ are **symmetric** matrices
- $F(y) \prec 0$ means that $F(y)$ is **negative definite**

Note that since F is affine, it can be written as $F(y) = F_0 + T(y)$, with $T\colon \mathbb{R}^N \to \mathbb{S}^{M\times M}$ a linear function.

Solving an LMI means finding $y \in \mathbb{R}^N$ that makes $F(y) \prec 0$ or establishing that such y does not exist.

A non-strict LMI is a linear matrix inequality of the form $F(y) \preceq 0$

C. Scherer and S. Weiland, "Linear matrix inequalities in control". Notes for a course of the Dutch Institute of Systems and Control, 2004.

Functions of matrix variables as LMIs

LMIs often appear as **functions of matrix variables**, that is in the form

$$\hat{F}(Y) \prec 0 \qquad Y \in \mathbb{R}^{N_1 \times N_2} \text{ matrix variable}$$

where $\hat{F}(Y) = \hat{T}(Y) + \hat{F}_0$ and $\hat{T}(Y)$ linear.

Example Discrete-time Lyapunov matrix inequality $\hat{F}(Y) = \hat{T}(Y) = A^\top Y A - Y$, where $A \in \mathbb{R}^{n\times n}$ is a given matrix and $Y \in \mathbb{S}^{n\times n}$ is the decision variable ($N_1 = N_2 = n$).

This is a special case of $F(y) = F_0 + F_1 y_1 + \ldots + F_N y_N \prec 0$. Let $E_1, \ldots, E_n$ be a basis of $\mathbb{R}^{N_1 \times N_2}$ and let

$$Y = \sum_j y_j E_j, \qquad y_j \in \mathbb{R}$$

Then

$$0 \succ \hat{F}(Y) = \hat{F}_0 + \hat{T}\Big(\sum_j y_j E_j\Big) = \underbrace{\hat{F}_0}_{=:F_0} + \sum_j y_j \underbrace{\hat{T}(E_j)}_{=:F_j}$$

Example (continued) ($n = 2$) Fix the basis $E_1 = \left[\begin{smallmatrix}1 & 0\\ 0 & 0\end{smallmatrix}\right], E_2 = \left[\begin{smallmatrix}0 & 0\\ 0 & 1\end{smallmatrix}\right], E_3 = \left[\begin{smallmatrix}0 & 1\\ 1 & 0\end{smallmatrix}\right]$. Then $Y = \left[\begin{smallmatrix}y_1 & y_3\\ y_3 & y_2\end{smallmatrix}\right] = y_1 E_1 + y_2 E_2 + y_3 E_3$. Hence

$$A^\top Y A - Y = y_1(A^\top E_1 A - E_1) + y_2(A^\top E_2 A - E_2) + y_3(A^\top E_3 A - E_3) = y_1 F_1 + y_2 F_2 + y_3 F_3$$

Systems of LMIs

A system of LMIs

$$\begin{cases} F^{(1)}(y) \prec 0 \\ F^{(2)}(y) \prec 0 \\ \vdots \\ F^{(p)}(y) \prec 0 \end{cases}$$

is still an LMI, because it is equivalent to

$$\begin{bmatrix} F^{(1)}(y) & 0 & \dots & 0 \\ 0 & F^{(2)}(y) & \dots & 0 \\ \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & F^{(p)}(y) \end{bmatrix} \prec 0$$

which in turn is equivalent to

$$\underbrace{\begin{bmatrix} F_0^{(1)} & 0 & \dots & 0 \\ 0 & F_0^{(2)} & \dots & 0 \\ \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & F_0^{(p)} \end{bmatrix}}_{F_0} + \sum_{j=1}^{N} y_j \underbrace{\begin{bmatrix} F_j^{(1)} & 0 & \dots & 0 \\ 0 & F_j^{(2)} & \dots & 0 \\ \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & F_j^{(p)} \end{bmatrix}}_{F_j} \prec 0$$

Feasibility and optimization

LMI are studied in connection with the following two problems

- Feasibility whether or not there exists $y \in \mathbb{R}^N$ such that $F(y) \prec 0$
- Optimization Given a function $f : \mathcal{S} \to \mathbb{R}$, where $\mathcal{S} = \{y \in \mathbb{R}^N : F(y) \prec 0\}$, an optimization problem with LMI constraints is $\inf_{y \in \mathcal{S}} f(y)$.

An LMI defines a convex set, i.e., the set $\{y : F(y) \prec 0\}$ is a convex set, hence checking the feasibility of an LMI or optimizing a convex function over a constraint defined by an LMI is a **convex optimization problem**

Minimizing linear objective functions over symmetric semidefinite matrix variables belongs to the realm of semidefinite programming for which effective numerical methods and software are available.

Here to illustrate some examples we use CVX.

Schur complement

Schur complement is a powerful tool to linearize nonlinear inequalities.

Consider the LMI

$$F(y) = \begin{bmatrix} F_{11}(y) & F_{12}(y) \\ F_{21}(y) & F_{22}(y) \end{bmatrix} \prec 0$$

where $F\colon \mathbb{R}^N \to \mathbb{S}^{M\times M}$ is an affine function. Then*

$$\begin{gathered} F(y) \prec 0 \\ \Updownarrow \\ \begin{cases} F_{11}(y) \prec 0 \\ F_{22}(y) - F_{21}(y)[F_{11}(y)]^{-1}F_{12}(y) \prec 0 \end{cases} \\ \Updownarrow \\ \begin{cases} F_{22}(y) \prec 0 \\ F_{11}(y) - F_{12}(y)[F_{22}(y)]^{-1}F_{21}(y) \prec 0 \end{cases} \end{gathered}$$

*The proof is based on the factorizations

$$\begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix} = \begin{cases} \begin{bmatrix} I & 0 \\ F_{21}F_{11}^{-1} & I \end{bmatrix} \begin{bmatrix} F_{11} & 0 \\ 0 & F_{22} - F_{21}F_{11}^{-1}F_{12} \end{bmatrix} \begin{bmatrix} I & F_{11}^{-1}F_{12} \\ 0 & I \end{bmatrix} & \text{if } F_{11} \text{ is invertible} \\ \begin{bmatrix} I & F_{12}F_{22}^{-1} \\ 0 & I \end{bmatrix} \begin{bmatrix} F_{11} - F_{12}F_{22}^{-1}F_{21} & 0 \\ 0 & F_{22} \end{bmatrix} \begin{bmatrix} I & 0 \\ F_{22}^{-1}F_{21} & I \end{bmatrix} & \text{if } F_{22} \text{ is invertible} \end{cases}$$

Schur complement

The Schur complement also applies to functions of matrix variables, and we will be using it mostly in this form.

Consider the LMI

$$\hat{F}(Y) = \begin{bmatrix} \hat{F}_{11}(Y) & \hat{F}_{12}(Y) \\ \hat{F}_{21}(Y) & \hat{F}_{22}(Y) \end{bmatrix} \prec 0$$

where $\hat{F}\colon \mathbb{R}^{N_1 \times N_2} \to \mathbb{S}^{M \times M}$ is an affine function. Then*

$$\hat{F}(Y) \prec 0$$
$$\Updownarrow$$
$$\begin{cases} \hat{F}_{11}(Y) \prec 0 \\ \hat{F}_{22}(Y) - \hat{F}_{21}(Y)[\hat{F}_{11}(Y)]^{-1}\hat{F}_{12}(Y) \prec 0 \end{cases}$$
$$\Updownarrow$$
$$\begin{cases} \hat{F}_{22}(Y) \prec 0 \\ \hat{F}_{11}(Y) - \hat{F}_{12}(Y)[\hat{F}_{22}(Y)]^{-1}\hat{F}_{21}(Y) \prec 0 \end{cases}$$

*The proof is based on the same factorizations considered before.

Schur complement

We will use the Schur complement also with nonstrict inequalities.

Consider the LMI

$$\hat{F}(Y) = \begin{bmatrix} \hat{F}_{11}(Y) & \hat{F}_{12}(Y) \\ \hat{F}_{21}(Y) & \hat{F}_{22}(Y) \end{bmatrix} \preceq 0$$

where $\hat{F} \colon \mathbb{R}^{N_1 \times N_2} \to \mathbb{S}^{M \times M}$ is an affine function.

If $\hat{F}_{11}(Y) \prec 0$, then

$$\hat{F}(Y) \preceq 0 \Leftrightarrow \hat{F}_{22}(Y) - \hat{F}_{21}(Y)[\hat{F}_{11}(Y)]^{-1}\hat{F}_{12}(Y) \preceq 0$$

If $\hat{F}_{22}(Y) \prec 0$, then

$$\hat{F}(Y) \preceq 0 \Leftrightarrow \hat{F}_{11}(Y) - \hat{F}_{12}(Y)[\hat{F}_{22}(Y)]^{-1}\hat{F}_{21}(Y) \preceq 0$$

Data-based stabilization

Direct data-driven stabilization

Problem (Stabilization) Based on the dataset $\mathbb{D}$

$$\begin{aligned} \text{find} \quad & K, P = P^\top \succ 0 \\ \text{such that} \quad & (A+BK)P(A+BK)^\top - P \prec 0 \end{aligned}$$

- The stabilization problem is solvable if and only if $u = Kx$ makes the origin a globally exponentially stable equilibrium for the closed-loop system $x^+ = (A+BK)^\top x$
- The stabilization problem is solvable if and only if all the eigenvalues of $(A+BK)^\top$ have magnitude strictly smaller than 1.
- As the eigenvalues of $A+BK$ and $(A+BK)^\top$ coincide, the stabilization problem is solvable if and only if $u = Kx$ makes the origin a globally exponentially stable equilibrium for $x^+ = (A+BK)x$

As A, B are unknown, to find a solution to the problem the idea is to work with X_1G instead of $A+BK$ under the condition for which $X_1G = A+BK$

A formula for direct data-driven stabilization

For any K, G such that $\begin{bmatrix} K \\ I_n \end{bmatrix} = \begin{bmatrix} U_0 \\ X_0 \end{bmatrix} G$, we have $A + BK = X_1 G$

Theorem Consider a system $x^+ = Ax + Bu$, which generates the dataset $\mathbb{D}$ from which the matrices U_0, X_1, X_0 are obtained. Consider the decision variables

$$P \in \mathbb{S}^{n \times n}, \; Y \in \mathbb{R}^{T \times n}$$

and the following SDP

$$X_0 Y = P \tag{1a}$$

$$\begin{bmatrix} -P & X_1 Y \\ Y^\top X_1^\top & -P \end{bmatrix} \prec 0 \tag{1b}$$

If it is feasible then the control gain

$$K = U_0 Y P^{-1}$$

solves the stabilization problem.

Let (1) be feasible. Constraint (1b) guarantees $P \succ 0$. Hence P is invertible. Constraint (1a) can be equivalently written as

$$\text{(1a) } X_0 Y = P \Leftrightarrow X_0 Y P^{-1} = I_n,$$

Perform the change of variable $G := YP^{-1}$, to obtain $X_0 G = I_n$.

By the same change of variable, the control gain

$$K = U_0 Y P^{-1}$$

can be written as $K = U_0 G$

Hence, $\begin{bmatrix} K \\ I_n \end{bmatrix} = \begin{bmatrix} U_0 \\ X_0 \end{bmatrix} G$. This returns the data-dependent representation of the closed-loop system, i.e., $A + BK = X_1 G$.

Consider constraint (1b) $\begin{bmatrix} -P & X_1 Y \\ Y^\top X_1^\top & -P \end{bmatrix} \prec 0$. By Schur complement, the inequality is equivalent to $P \succ 0$ and $-P + X_1 Y P^{-1} Y^\top X_1^\top \prec 0$. Rewrite the last inequality as $-P + X_1 Y P^{-1} P P^{-1} Y^\top X_1^\top \prec 0$. Bearing in mind the change of variable $G = YP^{-1}$, the latter can be written as $-P + X_1 G P G^\top X_1^\top \prec 0$, or, by the identity $A + BK = X_1 G$, as

$$P \succ 0, \quad (A + BK)P(A + BK)^\top - P \prec 0$$

A few comments

- Simple solution: data-dependent Lyapunov matrix inequality
- The data-based problem is solvable via efficient numerical algorithms (cvx)
- It only requires a finite number of data collected in one-shot low sample-complexity experiments
- Number of samples For $X_0 Y = P$ to be feasible, it is necessary that $X_0 \in \mathbb{R}^{n \times T}$ has full row rank, i.e., $T \geq n$.
- If the system is high-dimensional and unstable, then collecting data in one-shot experiment of length T might not be viable and one can use multiple dataset of shorter length

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What we will do next.

- An example that can be solved by hand.
- "Sufficiently rich" data gave several advantages.
- Parametrization of all stabilizing state feedback gains.
- Feasibility of the LMI.
- An example solved by software for convex optimization.
- The case of continuous-time systems.

Example

Consider the system

$$x^+ = Ax + Bu,$$

with $x, u \in \mathbb{R}$ and the dataset $\mathbb{D} = \{u(0), x(0), x(1)\}$ $(T = 1)$, where

$$x(0) = -2, u(0) = 3, x(1) = -1$$

In this case, $X_0 = x(0)$, $U_0 = u(0)$, $X_1 = x(1)$.

The decision variables $P \in \mathbb{S}^{n\times n}$, $Y \in \mathbb{R}^{T\times n}$ are both scalars $(n = 1,\ T = 1)$. Condition (1)

$$\begin{array}{l} X_0 Y = P \\ \begin{bmatrix} -P & X_1 Y \\ Y^\top X_1^\top & -P \end{bmatrix} \prec 0 \end{array} \quad \text{becomes} \quad \begin{array}{l} -2Y = P \\ \begin{bmatrix} -P & -Y \\ -Y & -P \end{bmatrix} \prec 0 \end{array}$$

which is equivalent to

$$\begin{cases} -2Y = P \\ P > 0 \\ -P + P^{-1}Y^2 < 0 \end{cases} \Leftrightarrow \begin{cases} -2Y = P \\ P > 0 \\ -P^2 + Y^2 < 0 \end{cases} \Leftrightarrow \begin{cases} -2Y = P \\ P > 0 \\ -3Y^2 < 0 \end{cases}$$

Hence, any Y, P such that $\underline{-2Y = P,\ P > 0}$ is a solution of the system of (in)equalities above. The controller solving the stabilisation problem is

$$K = U_0 Y P^{-1} = 3(-\frac{P}{2})P^{-1} = -\frac{3}{2}$$

By construction, K makes the closed-loop matrix $A + BK$ Schur stable for all $(A, B) \in \mathbb{R}^2$ that satisfy $X_1 = AX_0 + BU_0$, that is,

$$-1 < A + BK < 1 \text{ for all } A, B \text{ such that } -1 = -2A + 3B$$

By eliminating A from $-1 = -2A + 3B$, the above is equivalent to

$$-1 < \frac{3}{2}B + \frac{1}{2} + BK < 1 \text{ for all } B \in \mathbb{R}$$

By replacing $K = -\frac{3}{2}$ the condition above is trivially satisfied, confirming that K is the stabilising gain for all A, B that satisfy $X_1 = AX_0 + BU_0$. In fact it can be shown that K in this case is unique.

Note that the set of all A, B that satisfy $X_1 = AX_0 + BU_0$, that is the set of all A, B that satisfy $-1 = -2A + 3B$ is a line.

Data-based parameterization of all stabilizing controllers

Under the assumption of sufficiently rich data, i.e., $\begin{bmatrix} U_0 \\ X_0 \end{bmatrix}$ has full row rank, then one can parametrize via data all the controllers that solve the stabilization problem.

Corollary Assume that $\begin{bmatrix} U_0 \\ X_0 \end{bmatrix}$ has full row rank. Any control gain $K \in \mathbb{R}^{m \times n}$ that solves the stabilization problem must be of the form

$$K = U_0 Y P^{-1}$$

where Y, P are a solution of

$$X_0 Y = P \tag{2a}$$

$$\begin{bmatrix} -P & X_1 Y \\ Y^\top X_1^\top & -P \end{bmatrix} \prec 0 \tag{2b}$$

As K is stabilizing, $A + BK$ is Schur stable, that is, equivalently, there exists $P \succ 0$ such that $(A + BK)P(A + BK)^\top - P \prec 0$.

As $\left[\begin{smallmatrix} U_0 \\ X_0 \end{smallmatrix}\right]$ has full row rank, by Rouché-Capelli theorem there must exist G such that

$$\begin{bmatrix} K \\ I_n \end{bmatrix} = \begin{bmatrix} U_0 \\ X_0 \end{bmatrix} G$$

Hence, $K = U_0 G$, $I_n = X_0 G$ and $A + BK = X_1 G$. The latter implies that the Lyapunov inequality can be equivalently rewritten as

$$P \succ 0, \quad X_1 G P (X_1 G)^\top - P \prec 0$$

Proceedings as before, one performs the change of variable $Y := GP$ and the Lyapunov inequality above is equivalently rewritten as

$$\begin{bmatrix} -P & X_1 Y \\ Y^\top X_1^\top & -P \end{bmatrix} \prec 0$$

The identities $K = U_0 G$, $I_n = X_0 G$ expressed in the variables Y, P return $K = U_0 Y P^{-1}$, $I_n = X_0 Y P^{-1}$.

Feasibility of the SDP

The solution to the data-dependent stabilization problem rests on the feasibility of the SDP

$$X_0 Y = P$$
$$\begin{bmatrix} -P & X_1 Y \\ Y^\top X_1^\top & -P \end{bmatrix} \prec 0$$

Under which conditions is the SDP feasible?

If the matrix $\begin{bmatrix} U_0 \\ X_0 \end{bmatrix}$ has full row rank, by Rouché-Capelli theorem, for any K there exists G such that $\begin{bmatrix} U_0 \\ X_0 \end{bmatrix} G = \begin{bmatrix} K \\ I_n \end{bmatrix}$. This implies that $A + BK = X_1 G$.

Pick K such that $A + BK$ is Schur and fix G such that $\begin{bmatrix} U_0 \\ X_0 \end{bmatrix} G = \begin{bmatrix} K \\ I_n \end{bmatrix}$. Since $A + BK = X_1 G$ is Schur, there exists $P = P^\top \succ 0$ such that

$$X_1 G P G^\top X_1^\top - P \prec 0$$

Setting $GP =: Y$ and applying the Schur complement returns $\begin{bmatrix} -P & X_1 Y \\ Y^\top X_1^\top & -P \end{bmatrix} \prec 0$. Furthermore, $X_0 G = I_n$ implies $X_0 Y = P$, thus showing the feasibility of the SDP.

A technical result that is useful for the other lectures

We tackled the stabilization problem

Based on the dataset $\mathbb{D}$

$$\begin{aligned} \text{find} \quad & K, P = P^\top \succ 0 \\ \text{such that} \quad & (A + BK)P(A + BK)^\top - P \prec 0 \end{aligned}$$

and found out that the feasibility of $X_0 Y = P$, $\begin{bmatrix} -P & X_1 Y \\ Y^\top X_1^\top & -P \end{bmatrix} \prec 0$ returns a solution to the problem, with $K = U_0 Y P^{-1}$.

For some of the other lectures, it will be useful to also have a solution to this other problem

Based on the dataset $\mathbb{D}$

$$\begin{aligned} \text{find} \quad & K, P = P^\top \succ 0 \\ \text{such that} \quad & (A + BK)^\top P(A + BK) - P \prec 0 \end{aligned}$$

It can be shown that feasibility of $X_0 Y = Q$, $\begin{bmatrix} -Q & Y^\top X_1^\top \\ X_1 Y & -Q \end{bmatrix} \prec 0$ returns a solution to the problem, with $P = Q^{-1}$ and $K = U_0 Y Q^{-1}$.

Proof The proof proceeds as in the case of the previous result to show that $\begin{bmatrix} K \\ I_n \end{bmatrix} = \begin{bmatrix} U_0 \\ X_0 \end{bmatrix} YQ^{-1}$ implies $X_1 G = A + BK$, where $G = YQ^{-1}$. On the other hand, the manipulation of the constraint $\begin{bmatrix} -Q & Y^\top X_1^\top \\ X_1 Y & -Q \end{bmatrix} \prec 0$ goes in a slightly different way.

By Schur complement, the inequality is equivalent to

$$Q \succ 0 \text{ and } -Q + Y^\top X_1^\top Q^{-1} X_1 Y \prec 0.$$

Multiply the last inequality by Q^{-1} on both sides, to obtain

$$Q \succ 0 \text{ and } -Q^{-1} + Q^{-1} Y^\top X_1^\top Q^{-1} X_1 Y Q^{-1} \prec 0.$$

Bearing in mind the change of variable $G = YP^{-1}$, the latter can be written as $-P^{-1} + G^\top X_1^\top P^{-1} X_1 G \prec 0$, or, by the identity $A + BK = X_1 G$, as $P \succ 0$, $(A + BK)^\top P^{-1} (A + BK) - P^{-1} \prec 0$, as claimed.

Example (cont'd)

Data-based stabilization of the unknown dynamics

State response to PE input from experiment

$$X_0 = \begin{bmatrix} 0.4027 & 0.3478 & 0.3571 & 0.3216 & 0.2362 \\ 0.4448 & 1.1451 & 1.7499 & 2.3708 & 2.9301 \end{bmatrix}$$

$$X_1 = \begin{bmatrix} 0.3478 & 0.3571 & 0.3216 & 0.2362 & 0.1541 \\ 1.1451 & 1.7499 & 2.3708 & 2.9301 & 3.3409 \end{bmatrix}$$

Solve for Y, P the (nonstrict*) LMI

```
cvx_begin sdp
variable Y(T,n)
variable P(n,n) symmetric
[P-eye(n) X1*Y; Y'*X1' P]>=0;
P==X0*Y
cvx_end
```

* "The use of strict inequalities in CVX is strongly discouraged"

which returns

$$Y = \begin{bmatrix} 17.4905 & -12.3092 \\ -16.1889 & -5.2031 \\ -2.7196 & -1.4260 \\ 5.0981 & 3.3803 \\ -0.0311 & 11.2900 \end{bmatrix}$$

$$P = \begin{bmatrix} 2.0739 & -3.5219 \\ -3.5219 & 27.1664 \end{bmatrix}$$

S. Boyd. "Solving semidefinite programs using cvx," http://stanford.edu/class/ee363/notes/lmi-cvx.pdf

Replacing strict inequalities with weak ones

Replacing the strict inequality in (1) with the weak inequality results in no loss of generality because of the following

$$\text{(a)}\ \begin{bmatrix} -P & X_1Y \\ Y^\top X_1^\top & -P \end{bmatrix} \prec 0 \text{ is feasible} \iff \text{(b)}\ \begin{bmatrix} -P+I_n & X_1Y \\ Y^\top X_1^\top & -P \end{bmatrix} \preceq 0 \text{ is feasible}$$

By Schur complement, (a) holds if and only if $\begin{cases} P \succ 0 \\ -P + X_1YP^{-1}(X_1Y)^\top \prec 0 \end{cases}$

Set $Q := P - X_1YP^{-1}(X_1Y)^\top \succ 0$ and

$$\lambda := \min\{\lambda_{\min}(Q), \lambda_{\min}(P)\}, \quad \hat{P} := \frac{P}{\lambda}, \quad \hat{Y} := \frac{Y}{\lambda}$$

Then $\hat{P} = \frac{P}{\lambda} \succeq \frac{\lambda_{\min}(P)}{\lambda} I_n \succeq I_n$, and

$$0 \prec Q := P - X_1YP^{-1}(X_1Y)^\top \overset{\text{divide by } \lambda}{\iff} 0 \prec \frac{Q}{\lambda} := \hat{P} - X_1\hat{Y}\hat{P}^{-1}(X_1\hat{Y})^\top$$

from which $I_n \preceq \frac{Q}{\lambda} = \hat{P} - X_1\hat{Y}\hat{P}^{-1}(X_1\hat{Y})^\top$, i.e., by Schur complement, (b) holds with $Y \to \hat{Y}, P \to \hat{P}$.

Replacing the strict inequality in (1) with the weak inequality results in no loss of generality because of the following

(a) $\begin{bmatrix} -P & X_1 Y \\ Y^\top X_1^\top & -P \end{bmatrix} \prec 0$ is feasible $\Longleftrightarrow$ (b) $\begin{bmatrix} -P + I_n & X_1 Y \\ Y^\top X_1^\top & -P \end{bmatrix} \preceq 0$ is feasible

We will use the following version of the Schur complement: for any symmetric matrix $M = \begin{bmatrix} A & B \\ B^\top & C \end{bmatrix}$, if $C \prec 0$, then $M \preceq 0$ if and only if $A - BC^{-1}B^\top \preceq 0$.

If (b) holds, then $P \succeq I_n \succ 0$; hence, by the Schur complement recalled above, $\begin{bmatrix} -P + I_n & X_1 Y \\ Y^\top X_1^\top & -P \end{bmatrix} \preceq 0$ if and only if $-P + I_n + X_1 Y P^{-1} (X_1 Y)^\top \preceq 0$, which implies

$$-P + X_1 Y P^{-1} (X_1 Y)^\top \prec 0$$

The latter and $P \succ 0$ shown before imply that (a) holds.

The case of continuous-time systems

Input and state sampled trajectories Given a sampling time $T_s > 0$, let

$$U_0 = \begin{bmatrix} u_d(0) & u_d(T_s) & \dots & u_d((T-1)T_s) \end{bmatrix}$$
$$X_0 = \begin{bmatrix} x_d(0) & x_d(T_s) & \dots & x_d((T-1)T_s) \end{bmatrix}$$

Data-dependent representation of the closed-loop system As in the discrete-time case, $A + BK = X_1 G$ where

$$X_1 := \begin{bmatrix} \dot{x}_d(0) & \dot{x}_d(T_s) & \dots & \dot{x}_d((T-1)T_s) \end{bmatrix}$$

Lyapunov stabilization condition Any matrices Y, P satisfying

$$\begin{cases} X_1 Y + Y^\top X_1^\top \prec 0 \\ P = X_0 Y \succ 0 \end{cases}$$

are such that $K = U_0 Y P^{-1}$ is a stabilizing feedback gain for the continuous-time system

Main difference Derivatives of the state at the sampling times X_1 are required $\Longrightarrow$ Noisy data (Lecture 2-4)

The case of continuous-time systems

Alternative[1] Integral version of $\dot{x} = Ax + Bu$

$$\overbrace{x((k+1)T_s) - x(kT_s)}^{\xi(k)} = A \overbrace{\int_{kT_s}^{(k+1)T_s} x(t)dt}^{r(k)} + B \overbrace{\int_{kT_s}^{(k+1)T_s} u(t)dt}^{v(k)}$$

and work with the relation

$$\overbrace{\left[\xi(0) \dots \xi(T-1)\right]}^{\underline{X}_1} = A \overbrace{\left[r(0) \dots r(T-1)\right]}^{\underline{X}_0} + B \overbrace{\left[v(0) \dots v(T-1)\right]}^{\underline{U}_0}$$

Lyapunov stabilization condition Any matrices Y, P satisfying

$$\begin{cases} \underline{X}_1 Y + Y^\top \underline{X}_1^\top \prec 0 \\ P = \underline{X}_0 Y \succ 0 \end{cases}$$

is such that $K = \underline{U}_0 Y P^{-1}$ is a stabilizing feedback gain for the continuous-time system (and does not require state derivatives!)

[1] De Persis, Postoyan, Tesi. Event-triggered control from data. IEEE Transactions on Automatic Control, 69 (6), 2024

A bridge towards Lecture 2

- The derivations in Lecture 1 were based on the data-dependent closed-loop system representation

$$x(k+1) = X_1 G x(k) \quad \text{with} \quad \begin{bmatrix} K \\ I_n \end{bmatrix} = \begin{bmatrix} U_0 \\ X_0 \end{bmatrix} G$$

- Suppose now that the system's dynamics is affected by disturbances

$$x(k+1) = Ax(k) + Bu(k) + d(k)$$

  How does the system's representation change?
  Spoiler The presence of noise leads to a perturbed data-dependent representation

$$x(k+1) = (X_1 - D_0) G x(k) \quad \text{with} \quad D_0 = \begin{bmatrix} d(0) & \dots d(T-1) \end{bmatrix}$$

- How would you design a controller for the system above if D_0 is unknown? Which new assumptions would you introduce?

In the second part of this lecture, we will look at the output feedback stabilization problem (partial information).

The lack of a model discourages the use of an observer.

We will see how to overcome this obstacle to design dynamic output feedback controllers from data.

Partial information

Output feedback stabilization problem

Consider minimal (reachable and observable) MIMO space representation with A, B, C unknown matrices

$$\begin{aligned} x(k+1) &= Ax(k) + Bu(k) \quad & x(k) \in \mathbb{R}^n,\ u(k) \in \mathbb{R}^m \\ y(k) &= Cx(k) \quad & y(k) \in \mathbb{R}^p,\ k = 0, 1, 2, \ldots \end{aligned}$$

Design from data a dynamic output feedback controller

$$\begin{aligned} \chi(k+1) &= F\chi(k) + Gy(k) \\ u(k) &= H\chi(k) \end{aligned}$$

such that the equilibrium $(x, \chi) = (0, 0)$ is globally asymptotically stable for the closed-loop system

$$\begin{bmatrix} x(k+1) \\ \chi(k+1) \end{bmatrix} = \begin{bmatrix} A & BH \\ GC & F \end{bmatrix} \begin{bmatrix} x(k) \\ \chi(k) \end{bmatrix}$$

Output feedback stabilization problem - rationale

Minimal SISO space representation with output measurements

$$\begin{aligned} x(k+1) &= Ax(k)+Bu(k) \quad & x(k)\in\mathbb{R}^n,\ u(k)\in\mathbb{R}^m \\ y(k) &= Cx(k) \quad & y(k)\in\mathbb{R}^p,\ k=0,1,2,\dots \end{aligned}$$

Rationale Reduce the data-driven output feedback control design to the state feedback one.

We assume to know the observability index ℓ of the system, that is, the minimum integer $\ell \geq 1$ for which

$$\operatorname{rank}\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{\ell-1} \end{bmatrix} = n$$

Given the I/O sequence $\{u(k), y(k)\}_{k=0}^{\infty}$, we consider, for every $k \geq \ell$, a vector $\phi(k)$ of the past ℓ values of input and output samples

$$\phi(k) = \begin{bmatrix} y(k-\ell)^\top & y(k-\ell+1)^\top & \dots & y(k-1)^\top & u(k-\ell)^\top & u(k-\ell+1)^\top & \dots & u(k-1)^\top \end{bmatrix}^\top$$

Observe that $\phi(k)$ is a measured vector

Towards an auxiliary system

The sequence $\{\phi(k)\}_{k=\ell}^{\infty}$, defined starting from $\{u(k), y(k)\}_{k=0}^{\infty}$, where

$$\phi(k) = \begin{bmatrix} y(k-\ell)^\top & y(k-\ell+1)^\top & \dots & y(k-2)^\top & y(k-1)^\top & u(k-\ell)^\top & u(k-\ell+1)^\top & \dots & u(k-2)^\top & u(k-1)^\top \end{bmatrix}^\top,$$

satisfies the equation

$$\phi(k+1) = \left[\begin{array}{c} y(k-\ell+1) \\ y(k-\ell+2) \\ \vdots \\ y(k-1) \\ y(k) \\ \hline u(k-\ell+1) \\ u(k-\ell+2) \\ \vdots \\ u(k-1) \\ u(k) \end{array}\right] = \left[\begin{array}{ccccc|ccccc} 0 & I_p & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & I_p & 0 & 0 & \dots & 0 & 0 \\ \star & \star & \dots & \star & \star & \star & \star & \dots & \star & \star \\ \hline 0 & 0 & \dots & 0 & 0 & 0 & I_m & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & I_m \\ 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 \end{array}\right] \left[\begin{array}{c} y(k-\ell) \\ y(k-\ell-1) \\ \vdots \\ y(k-2) \\ y(k-1) \\ \hline u(k-\ell) \\ u(k-\ell-1) \\ \vdots \\ u(k-2) \\ u(k-1) \end{array}\right] + \left[\begin{array}{c} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ \hline 0 \\ 0 \\ \vdots \\ 0 \\ I_m \end{array}\right] u(k), \forall k \geq \ell$$

To complete the expression, we must compute the relation between $y(k)$ and $\phi(k), u(k)$.

We write first the expression of the output response $y(k)$ at time k obtained starting from the "initial state" $x(k-\ell)$ when the input sequence $u(k-\ell), u(k-\ell+1), \dots, u(k-1)$ is applied:

$$y(k) = CA^{\ell}x(k-\ell) + CA^{\ell-1}Bu(k-\ell) + CA^{\ell-2}Bu(k-\ell+1) + \dots + CABu(k-2) + CBu(k-1)$$

$$= CA^{\ell}x(k-\ell) + C\underbrace{\begin{bmatrix} A^{\ell-1}B & A^{\ell-2}B & \dots & AB & B \end{bmatrix}}_{=:\mathcal{R}_\ell}\begin{bmatrix} u(k-\ell) \\ u(k-\ell+1) \\ \vdots \\ u(k-2) \\ u(k-1) \end{bmatrix}$$

To eliminate $x(k-\ell)$, we express it through the sequence of past I/O sequences

$$\begin{bmatrix} y(k-\ell) \\ \vdots \\ y(k-1) \end{bmatrix} = \underbrace{\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{\ell-1} \end{bmatrix}}_{=:\mathcal{O}_\ell} x(k-\ell) + \underbrace{\begin{bmatrix} 0_{p\times m} & 0 & \dots & 0 & 0 \\ CB & 0_{p\times m} & \dots & 0 & 0 \\ CAB & CB & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ CA^{\ell-2}B & CA^{\ell-3}B & \dots & CB & 0_{p\times m} \end{bmatrix}}_{=:\mathcal{T}_\ell} \begin{bmatrix} u(k-\ell) \\ \vdots \\ u(k-1) \end{bmatrix}$$

By observability, $\mathcal{O}_\ell$ has a left inverse $\mathcal{O}_\ell^\dagger := (\mathcal{O}_\ell^\top \mathcal{O}_\ell)^{-1}\mathcal{O}_\ell^\top$, from which

$$x(k-\ell) = \mathcal{O}_\ell^\dagger \begin{bmatrix} y(k-\ell) \\ \vdots \\ y(k-1) \end{bmatrix} - \mathcal{O}_\ell^\dagger \mathcal{T}_\ell \begin{bmatrix} u(k-\ell) \\ \vdots \\ u(k-1) \end{bmatrix}$$

In turn we get the expression of $y(k)$ we were looking for.

$$\begin{aligned} y(k) = &\quad CA^\ell x(k-\ell) + C\mathcal{R}_\ell \begin{bmatrix} u(k-\ell) \\ \vdots \\ u(k-1) \end{bmatrix} \\ = &\quad CA^\ell \left(\mathcal{O}_\ell^\dagger \begin{bmatrix} y(k-\ell) \\ \vdots \\ y(k-1) \end{bmatrix} - \mathcal{O}_\ell^\dagger \mathcal{T}_\ell \begin{bmatrix} u(k-\ell) \\ \vdots \\ u(k-1) \end{bmatrix} \right) + C\mathcal{R}_\ell \begin{bmatrix} u(k-\ell) \\ \vdots \\ u(k-1) \end{bmatrix} \\ = &\quad \left[\begin{array}{c|c} CA^\ell \mathcal{O}_\ell^\dagger & C\mathcal{R}_\ell - CA^\ell \mathcal{O}_\ell^\dagger \mathcal{T}_\ell \end{array} \right] \left[\begin{array}{c} y(k-\ell) \\ \vdots \\ y(k-1) \\ \hline u(k-\ell) \\ \vdots \\ u(k-1) \end{array} \right] \end{aligned}$$

We turn our attention again to

$$\phi(k+1) = \left[\begin{array}{c} y(k-\ell+1) \\ y(k-\ell+2) \\ \vdots \\ y(k-1) \\ y(k) \\ \hline u(k-\ell+1) \\ u(k-\ell+2) \\ \vdots \\ u(k-1) \\ u(k) \end{array}\right] = \left[\begin{array}{ccccc|ccccc} 0 & I_p & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & I_p & 0 & 0 & \dots & 0 & 0 \\ \star & \star & \dots & \star & \star & \star & \star & \dots & \star & \star \\ \hline 0 & 0 & \dots & 0 & 0 & 0 & I_m & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & I_m \\ 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 \end{array}\right] \left[\begin{array}{c} y(k-\ell) \\ y(k-\ell+1) \\ \vdots \\ y(k-2) \\ y(k-1) \\ \hline u(k-\ell) \\ u(k-\ell+1) \\ \vdots \\ u(k-2) \\ u(k-1) \end{array}\right] + \left[\begin{array}{c} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ \hline 0 \\ 0 \\ \vdots \\ 0 \\ I_m \end{array}\right] u(k)$$

and replace the "$\star$"s with the expression computed before,which returns

$$\phi(k+1) = \left[\begin{array}{c} y(k-\ell+1) \\ y(k-\ell+2) \\ \vdots \\ y(k-1) \\ y(k) \\ \hline u(k-\ell+1) \\ u(k-\ell+2) \\ \vdots \\ u(k-1) \\ u(k) \end{array}\right] = \left[\begin{array}{ccccc|ccccc} 0 & I_p & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & I_p & 0 & 0 & \dots & 0 & 0 \\ \multicolumn{5}{c}{CA^\ell \mathcal{O}_\ell^\dagger} & \multicolumn{5}{c}{C\mathcal{R}_\ell - CA^\ell \mathcal{O}_\ell^\dagger \mathcal{T}_\ell} \\ \hline 0 & 0 & \dots & 0 & 0 & 0 & I_m & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & I_m \\ 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 \end{array}\right] \left[\begin{array}{c} y(k-\ell) \\ y(k-\ell+1) \\ \vdots \\ y(k-2) \\ y(k-1) \\ \hline u(k-\ell) \\ u(k-\ell+1) \\ \vdots \\ u(k-2) \\ u(k-1) \end{array}\right] + \left[\begin{array}{c} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ \hline 0 \\ 0 \\ \vdots \\ 0 \\ I_m \end{array}\right] u(k)$$

An auxiliary system

Starting from system $\begin{cases} x^+ = & Ax + Bu \\ y = & Cx, \end{cases}$ we construct the auxiliary system $\phi^+ = \mathcal{A}\phi + \mathcal{B}v$, where

$$
\mathcal{A} = \left[\begin{array}{ccccc|ccccc}
0 & I_p & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\
0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \dots & 0 & I_p & 0 & 0 & \dots & 0 & 0 \\
\multicolumn{5}{c|}{CA^\ell \mathcal{O}_\ell^\dagger} & \multicolumn{5}{c}{C\mathcal{R}_\ell - CA^\ell \mathcal{O}_\ell^\dagger \mathcal{T}_\ell} \\
\hline
0 & 0 & \dots & 0 & 0 & 0 & I_m & \dots & 0 & 0 \\
0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & I_m \\
0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0
\end{array}\right], \quad
\mathcal{B} = \left[\begin{array}{c}
0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ \hline 0 \\ 0 \\ \vdots \\ 0 \\ I_m
\end{array}\right]
$$

and $\mathcal{O}_\ell, \mathcal{R}_\ell, \mathcal{T}_\ell$ are the observability, reachability and Toeplitz matrix of order ℓ.

For any initial condition $x(0)$ and input sequence $\{u(k)\}_{k=0}^{\infty}$, there exist an initial condition $\phi(\ell)$ and an input sequence $\{v(k)\}_{k=\ell}^{\infty} = \{u(k)\}_{k=\ell}^{\infty}$ such that the solution of $\phi^+ = \mathcal{A}\phi + \mathcal{B}v$ satisfies

$$
\phi(k) = \left[\begin{array}{c}
y(k-\ell) \\ \vdots \\ y(k-1) \\ \hline u(k-\ell) \\ \vdots \\ u(k-1)
\end{array}\right] \quad \text{for all } k \geq \ell
$$

Another form of the auxiliary system

Before studying a key property of the auxiliary system $\phi^+ = \mathcal{A}\phi + \mathcal{B}v$, we give it another form, which will be useful in deriving the dynamic controller. The form is as follows:

$$\phi^+ = \mathcal{A}\phi + \mathcal{B}v = (\mathcal{F} + \mathcal{L}\mathcal{Z})\phi + \mathcal{B}v$$

where

$$\mathcal{F} = \left[\begin{array}{ccccc|ccccc} 0 & I_p & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & I_p & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ \hline 0 & 0 & \dots & 0 & 0 & 0 & I_m & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & I_m \\ 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 \end{array}\right], \mathcal{L} = \left[\begin{array}{c} 0 \\ 0 \\ \vdots \\ 0 \\ I_p \\ \hline 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{array}\right], \mathcal{Z} = \left[\begin{array}{c|c} CA^{\ell}\mathcal{O}_{\ell}^{\dagger} & C\mathcal{R}_{\ell} - CA^{\ell}\mathcal{O}_{\ell}^{\dagger}\mathcal{T}_{\ell} \end{array}\right]$$

A key property of the Auxiliary System

For (A, B, C) minimal, the pair $(\mathcal{A}, \mathcal{B})$ is reachable if and only if $p\ell = n$.

- "Lifting" the system $\begin{cases} x^+ = & Ax + Bu \\ y = & Cx \end{cases}$ to $\phi^+ = \mathcal{A}\phi + \mathcal{B}v$ preserves reachability iff $p\ell = n$.
- For SISO observable systems, the condition $pl = n$ is always satisfied.
- For SISO systems, the proof is based on the Key Reachability Lemma ("*The pair $(\mathcal{A}, \mathcal{B})$ above is reachable if and only the polynomials $z^n + a_n z^{n-1} \ldots + a_2 z + a_1$, $b_n z^{n-1} + \ldots + b_2 z + b_1$ defined by*

$$CA^\ell \mathcal{O}_\ell^\dagger = \begin{bmatrix} -a_1 & -a_2 & -a_3 & \ldots & -a_n \end{bmatrix}, \quad C\mathcal{R}_\ell - CA^\ell \mathcal{O}_\ell^\dagger \mathcal{T}_\ell = \begin{bmatrix} b_1 & b_2 & b_3 & \ldots & b_n \end{bmatrix}$$

are coprime") to conclude that $(\mathcal{A}, \mathcal{B})$ is reachable.
G.C. Goodwin, K.S. Sin. Adaptive Filtering Prediction and Control. Courier Corporation, 2014.
Here, we give a proof valid for MIMO systems.

The proof that, for (A, B, C) minimal,

$$(\mathcal{A}, \mathcal{B}) \text{ reachable} \iff p\ell = n$$

is based on the following technical lemma

For (A, B, C) minimal, let

- $\mathcal{R}(\mathcal{A}, \mathcal{B})$ be the reachability subspace of the pair $(\mathcal{A}, \mathcal{B})$;
- $H_\ell := \begin{bmatrix} \mathcal{O}_\ell & \mathcal{T}_\ell \\ 0_{m\ell \times n} & I_{m\ell} \end{bmatrix}$.

Then

$$\mathrm{Im} H_\ell = \mathcal{R}(\mathcal{A}, \mathcal{B}).$$

Preliminary observation By the structure of H_ℓ and observability of (A, C), H_ℓ has full-column rank, i.e. $\mathrm{rank}(H_\ell) = n + m\ell$.

$(\mathcal{A}, \mathcal{B})$ reachable $\Longrightarrow p\ell = n$ $(\mathcal{A}, \mathcal{B})$ reachable $\Longrightarrow \dim(\mathcal{R}(\mathcal{A}, \mathcal{B})) = (m+p)\ell$ $\Longrightarrow \dim(\mathrm{Im}(H_\ell)) = (m+p)\ell$. Note now that 1) H_ℓ is a $(m+p)\ell \times (n+m\ell)$ matrix; 2) $\mathrm{rank}(H_\ell) = (m+p)\ell$. We observed before that $\mathrm{rank}(H_\ell) = n + m\ell$, hence $n = p\ell$.

$p\ell = n \Longrightarrow (\mathcal{A}, \mathcal{B})$ reachable As H_ℓ is a $(m+p)\ell \times (n+m\ell)$ matrix and $\mathrm{rank}(H_\ell) = n + m\ell$, then $\dim(\mathcal{R}(\mathcal{A}, \mathcal{B})) = n + m\ell$. $p\ell = n$ implies $\dim(\mathcal{R}(\mathcal{A}, \mathcal{B})) = (p+m)\ell$. Hence, the pair $(\mathcal{A}, \mathcal{B}))$ is reachable.

Dataset

Information about the system's dynamics is obtained from a $T+1$-long dataset of input/output samples

$$\mathbb{D} := \{u(k), y(k)\}_{k=0}^{T}$$

collected from the system

$$\begin{cases} x^+ = & Ax + Bu \\ y = & Cx \end{cases}$$

We define the matrices of data

$$\Phi_1 := \left[\begin{array}{cccc} y(1) & y(2) & \dots & y(T-\ell+1) \\ \vdots & \vdots & & \vdots \\ y(\ell) & y(\ell+1) & \dots & y(T) \\ \hline u(1) & u(2) & \dots & u(T-\ell+1) \\ \vdots & \vdots & & \vdots \\ u(\ell) & u(\ell+1) & \dots & u(T) \end{array}\right] \in \mathbb{R}^{(p+m)\ell \times T-\ell+1}, \quad \Phi_0 := \left[\begin{array}{cccc} y(0) & y(1) & \dots & y(T-\ell) \\ \vdots & \vdots & & \vdots \\ y(\ell-1) & y(\ell) & \dots & y(T-1) \\ \hline u(0) & u(1) & \dots & u(T-\ell) \\ \vdots & \vdots & & \vdots \\ u(\ell-1) & u(\ell) & \dots & u(T-1) \end{array}\right]$$

$$U_0 := \left[\begin{array}{cccc} u(\ell) & u(\ell+1) & \dots & u(T) \end{array}\right] \in \mathbb{R}^{m \times T-\ell+1}$$

Bearing in mind the "lifted" dynamics $\phi^+ = \mathcal{A}\phi + \mathcal{B}v$, the matrices satisfy the identity

$$\Phi_1 = \mathcal{A}\Phi_0 + \mathcal{B}U_0$$

Controller

We focus on the feedback law

$$u(k) = \mathcal{K} \left[\begin{array}{c} y(k-\ell) \\ \vdots \\ y(k-1) \\ \hline u(k-\ell) \\ \vdots \\ u(k-1) \end{array}\right], \forall k \geq \ell$$

for some matrix $\mathcal{K}$ to be designed.

This corresponds to the dynamic controller $\begin{cases} \chi^+ = & \mathcal{F}\chi + \mathcal{L}y + \mathcal{B}u \\ u = & \mathcal{K}\chi \end{cases}$ where

$$\mathcal{F} = \left[\begin{array}{ccccc|ccccc} 0 & I_p & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & I_p & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ \hline 0 & 0 & \dots & 0 & 0 & 0 & I_m & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & I_m \\ 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 \end{array}\right], \mathcal{L} = \left[\begin{array}{c} 0 \\ 0 \\ \vdots \\ 0 \\ I_p \\ \hline 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{array}\right], \mathcal{B} = \left[\begin{array}{c} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ \hline 0 \\ 0 \\ \vdots \\ 0 \\ I_m \end{array}\right]$$

In fact, by the expression of the matrices $\mathcal{F}, \mathcal{L}, \mathcal{B}$,

$$\mathcal{F} = \left[\begin{array}{ccccc|ccccc} 0 & I_p & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & I_p & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ \hline 0 & 0 & \dots & 0 & 0 & 0 & I_m & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & I_m \\ 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 \end{array}\right], \mathcal{L} = \left[\begin{array}{c} 0 \\ 0 \\ \vdots \\ 0 \\ I_p \\ \hline 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{array}\right], \mathcal{B} = \left[\begin{array}{c} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ \hline 0 \\ 0 \\ \vdots \\ 0 \\ I_m \end{array}\right]$$

for any initial condition $\chi(0) \in \mathbb{R}^{2\ell}$, starting from time step ℓ, the state of the controller

$\begin{cases} \chi^+ = & \mathcal{F}\chi + \mathcal{L}y + \mathcal{B}u \\ u = & \mathcal{K}\chi \end{cases}$ satisfies $\chi(k) = \left[\begin{array}{c} y(k-\ell) \\ \vdots \\ y(k-1) \\ \hline u(k-\ell) \\ \vdots \\ u(k-1) \end{array}\right]$ for all $k \geq \ell$, thus providing the past

ℓ I/O samples required in $u(k) = \mathcal{K} \left[\begin{array}{c} y(k-\ell) \\ \vdots \\ y(k-1) \\ \hline u(k-\ell) \\ \vdots \\ u(k-1) \end{array}\right]$ for all $k \geq \ell$.

Direct data-driven output-feedback stabilization

<u>Problem</u> (Output-feedback stabilization) Consider the minimal system

$$\begin{cases} x^+ = & Ax + Bu \\ y = & Cx \end{cases}$$

Design a matrix $\mathcal{K}$ for the dynamic output feedback controller

$$\begin{cases} \chi^+ = & \mathcal{F}\chi + \mathcal{L}y + \mathcal{B}u \\ u = & \mathcal{K}\chi \end{cases}$$

such that the equilibrium $(x, \chi) = (0, 0)$ is globally asymptotically stable for the feedback interconnection

$$\begin{cases} x^+ = & Ax + B\mathcal{K}\chi \\ \chi^+ = & \mathcal{L}Cx + (\mathcal{F} + \mathcal{B}\mathcal{K})\chi \end{cases}$$

- ▷ We will design $\mathcal{K}$ by focusing on the auxiliary system $\phi^+ = \mathcal{A}\phi + \mathcal{B}v$ and looking for $v = \mathcal{K}\phi$ that globally asymptotically stabilizes $\phi^+ = (\mathcal{A} + \mathcal{B}\mathcal{K})\phi$, i.e. renders $\mathcal{A} + \mathcal{B}\mathcal{K}$ Schur. We refer to $\phi^+ = (\mathcal{A} + \mathcal{B}\mathcal{K})\phi$ as the auxiliary closed-loop system.
- ▷ We will then show that the controller $\begin{cases} \chi^+ = & \mathcal{F}\chi + \mathcal{L}y + \mathcal{B}u \\ u = & \mathcal{K}\chi \end{cases}$ with $\mathcal{K}$ as above solves the output feedback stabilization problem.

Data-based parametrization of the auxiliary closed-loop system

- To design $\mathcal{K}$ that renders $\mathcal{A}+\mathcal{B}\mathcal{K}$ Schur, we follow the previous path: data-dependent closed-loop representation followed by a convex program to stabilize such a representation.
- The data-dependent representation is obtained via the identity $\Phi_1 = \mathcal{A}\Phi_0 + \mathcal{B}U_0$ where

$$\Phi_1 := \begin{bmatrix} y(1) & y(2) & \dots & y(T-\ell+1) \\ \vdots & \vdots & & \vdots \\ y(\ell) & y(\ell+1) & \dots & y(T) \\ \hline u(1) & u(2) & \dots & u(T-\ell+1) \\ \vdots & \vdots & & \vdots \\ u(\ell) & u(\ell+1) & \dots & u(T) \end{bmatrix}, \Phi_0 := \begin{bmatrix} y(0) & y(1) & \dots & y(T-\ell) \\ \vdots & \vdots & & \vdots \\ y(\ell-1) & y(\ell) & \dots & y(T-1) \\ \hline u(0) & u(1) & \dots & u(T-\ell) \\ \vdots & \vdots & & \vdots \\ u(\ell-1) & u(\ell) & \dots & u(T-1) \end{bmatrix}, U_0 := \begin{bmatrix} u(\ell) & u(\ell+1) & \dots & u(T) \end{bmatrix}$$

Consider the system $\phi^+ = \mathcal{A}\phi + \mathcal{B}v$ in closed-loop with the feedback law $v = \mathcal{K}\phi$. Consider any matrices $\mathcal{K} \in \mathbb{R}^{m\times(p+m)\ell}$, $\mathcal{G} \in \mathbb{R}^{T-\ell+1\times(p+m)\ell}$ such that

$$\begin{bmatrix} \mathcal{K} \\ I_{(p+m)\ell} \end{bmatrix} = \begin{bmatrix} U_0 \\ \Phi_0 \end{bmatrix} \mathcal{G}$$

Then the closed-loop system $\phi^+ = (\mathcal{A}+\mathcal{B}\mathcal{K})\phi$ has the following equivalent representation

$$\phi^+ = \Phi_1 \mathcal{G} \phi$$

A formula for direct data-driven output feedback stabilization

For any $\mathcal{K}, \mathcal{G}$ such that $\begin{bmatrix} \mathcal{K} \\ I_{(p+m)\ell} \end{bmatrix} = \begin{bmatrix} U_0 \\ \Phi_0 \end{bmatrix} \mathcal{G}$, we have $\mathcal{A} + \mathcal{B}\mathcal{K} = \Phi_1 \mathcal{G}$

Theorem Consider the auxiliary system $\phi^+ = \mathcal{A}\phi + \mathcal{B}v$ and the matrices of data U_0, Φ_1, Φ_0 assembled from the dataset $\mathbb{D} = \{u(k), y(k)\}_{k=0}^{T}$ obtained from the minimal system $\begin{cases} x^+ = & Ax + Bu \\ y = & Cx \end{cases}$.

Consider the decision variables $\mathcal{P} \in \mathbb{S}^{(p+m)\ell \times (p+m)\ell}$, $\mathcal{Y} \in \mathbb{R}^{T-\ell+1 \times (p+m)\ell}$ and the following SDP

$$\Phi_0 \mathcal{Y} = \mathcal{P}$$

$$\begin{bmatrix} -\mathcal{P} & \Phi_1 \mathcal{Y} \\ \mathcal{Y}^\top \Phi_1^\top & -\mathcal{P} \end{bmatrix} \prec 0$$

If it is feasible then the control gain

$$\mathcal{K} = U_0 \mathcal{Y} \mathcal{P}^{-1}$$

is such that $(\mathcal{A} + \mathcal{B}\mathcal{K})\mathcal{P}(\mathcal{A} + \mathcal{B}\mathcal{K})^\top - \mathcal{P} \prec 0$, i.e. $\mathcal{A} + \mathcal{B}\mathcal{K}$ is Schur.

Stability of the closed-loop system

Closed-loop system

$$\begin{cases} x^+ = & Ax + Bu, \;\; y = Cx \\ \chi^+ = & \mathcal{F}\chi + \mathcal{L}y + \mathcal{B}u \\ u = & \mathcal{K}\chi \end{cases}$$

where $\mathcal{K}$ has been designed such that $\mathcal{A} + \mathcal{B}\mathcal{K}$ is Schur. We want to show that $(x, \chi) = (0, 0)$ is a globally asymptotically stable equilibrium for the system.

Reminder 1 For any initial condition $\chi(0) \in \mathbb{R}^{(p+m)\ell}$, the state of the controller

$$\begin{cases} \chi^+ = & \mathcal{F}\chi + \mathcal{L}y + \mathcal{B}u \\ u = & \mathcal{K}\chi \end{cases} \quad \text{satisfies } \chi(k) = \begin{bmatrix} y(k-\ell) \\ \vdots \\ y(k-1) \\ \hline u(k-\ell) \\ \vdots \\ u(k-1) \end{bmatrix} \text{ for all } k \geq \ell.$$

Reminder 2 For any initial condition $x(0)$ and input sequence $\{u(k)\}_{k=0}^{\infty}$, there exist an initial condition $\phi(\ell)$ and an input sequence $\{v(k)\}_{k=\ell}^{\infty} = \{u(k)\}_{k=\ell}^{\infty}$ such that the solution of $\phi^+ = \mathcal{A}\phi + \mathcal{B}v$ satisfies $\phi(k) = \begin{bmatrix} y(k-\ell) \\ \cdots \\ y(k-1) \\ \hline u(k-\ell) \\ \cdots \\ u(k-1) \end{bmatrix}$ for all $k \geq \ell$.

Hence, if $\{v(k)\}_{k=\ell}^{\infty} = \{u(k)\}_{k=\ell}^{\infty}$, then $\chi(k) = \phi(k)$ for all $k \geq \ell$.

Here $\{u(k)\}_{k=\ell}^{\infty} = \{\mathcal{K}\chi(k)\}_{k=\ell}^{\infty}$, hence $\{v(k)\}_{k=\ell}^{\infty} = \{u(k)\}_{k=\ell}^{\infty}$ implies that $\{v(k)\}_{k=\ell}^{\infty} = \{\mathcal{K}\chi(k)\}_{k=\ell}^{\infty}$ and $\phi(k)$ coincides with the solution of $\phi(k+1) = (\mathcal{A} + \mathcal{B}\mathcal{K})\phi(k)$ for all $k \geq \ell$.

Bearing in mind that $\chi(k) = \phi(k)$ for all $k \geq \ell$, we conclude that $\chi(k) \overset{k\to+\infty}{\longrightarrow} 0$.

As $\chi(k) = \begin{bmatrix} y(k-\ell) \\ \vdots \\ y(k-1) \\ \hline u(k-\ell) \\ \vdots \\ u(k-1) \end{bmatrix}$, also $\begin{bmatrix} y(k-\ell) \\ \vdots \\ y(k-1) \\ \hline u(k-\ell) \\ \vdots \\ u(k-1) \end{bmatrix} \overset{k\to+\infty}{\longrightarrow} 0$.

Previously, we computed that $y(k) = \left[\ CA^{\ell}\mathcal{O}_{\ell}^{\dagger} \mid C\mathcal{R}_{\ell} - CA^{\ell}\mathcal{O}_{\ell}^{\dagger}\mathcal{T}_{\ell}\ \right] \begin{bmatrix} y(k-\ell) \\ \vdots \\ y(k-1) \\ \hline u(k-\ell) \\ \vdots \\ u(k-1) \end{bmatrix}$. Similarly,

we can derive that $x(k) = \left[\ A^{\ell}\mathcal{O}_{\ell}^{L} \mid \mathcal{R}_{\ell} - A^{\ell}\mathcal{O}_{\ell}^{L}\mathcal{T}_{\ell}\ \right] \begin{bmatrix} y(k-\ell) \\ \vdots \\ y(k-1) \\ \hline u(k-\ell) \\ \vdots \\ u(k-1) \end{bmatrix}$. Hence, $x(k) \overset{k\to+\infty}{\longrightarrow} 0$.

For LTI systems, attractivity implies stability. Hence, we have shown that $(x, \chi) = (0, 0)$ is a globally asymptotically stable equilibrium for the closed-loop system.

Recap - A procedure to design output feedback controllers

Priors A, B, C minimal, the observability index is known, and $p\ell = n$.

Acquire the dataset $\mathbb{D} = \{u(k), y(k)\}_{k=0}^{T}$ and form the matrices of data

$$\Phi_1 := \left[\begin{array}{cccc} y(1) & y(2) & \dots & y(T-\ell+1) \\ \vdots & \vdots & & \vdots \\ y(\ell) & y(\ell+1) & \dots & y(T) \\ \hline u(1) & u(2) & \dots & u(T-\ell+1) \\ \vdots & \vdots & & \vdots \\ u(\ell) & u(\ell+1) & \dots & u(T) \end{array}\right], \Phi_0 := \left[\begin{array}{cccc} y(0) & y(1) & \dots & y(T-\ell) \\ \vdots & \vdots & & \vdots \\ y(\ell-1) & y(\ell) & \dots & y(T-1) \\ \hline u(0) & u(1) & \dots & u(T-\ell) \\ \vdots & \vdots & & \vdots \\ u(\ell-1) & u(\ell) & \dots & u(T-1) \end{array}\right], U_0 := \begin{bmatrix} u(\ell) & u(\ell+1) & \dots & u(T) \end{bmatrix}$$

Consider the decision variables $\mathcal{P} \in \mathbb{S}^{(p+m)\ell \times (p+m)}$, $\mathcal{Y} \in \mathbb{R}^{T-\ell+1 \times (p+m)\ell \times (p+m)}$ and the SDP

$$\Phi_0 \mathcal{Y} = \mathcal{P}, \quad \begin{bmatrix} -\mathcal{P} & \Phi_1 \mathcal{Y} \\ \mathcal{Y}^\top \Phi_1^\top & -\mathcal{P} \end{bmatrix} \prec 0$$

If feasible, then design $\mathcal{K} = U_0 \mathcal{Y} \mathcal{P}^{-1}$.

Consider the known matrices $\mathcal{F}, \mathcal{L}, \mathcal{B}$ (see slide 41). The output feedback controller

$$\begin{cases} \chi^+ = & (\mathcal{F} + \mathcal{B}\mathcal{K})\chi + \mathcal{L}y \\ u = & \mathcal{K}\chi \end{cases}$$

globally exponentially stabilizes the equilibrium $(x, \chi) = (0, 0)$ of

$$\begin{cases} x^+ = & Ax + Bu, & y = Cx \\ \chi^+ = & \mathcal{F}\chi + \mathcal{L}y + \mathcal{B}u, & u = \mathcal{K}\chi \end{cases}$$

Comment 1 If the $(m+p)\ell + m \times T - \ell + 1$ matrix $\begin{bmatrix} U_0 \\ \Phi_0 \end{bmatrix}$ has full row rank, then the SDP

$$\Phi_0 \mathcal{Y} = \mathcal{P}, \quad \begin{bmatrix} -\mathcal{P} & \Phi_1 \mathcal{Y} \\ \mathcal{Y}^\top \Phi_1^\top & -\mathcal{P} \end{bmatrix} \prec 0$$

is feasible.

Comment 2 Under the standing assumptions (A, B, C is minimal, the observability index ℓ is known and $u_{[0,T-1]}$ is PE of order $L = (m+p)\ell + 1$) it holds that

$$\begin{bmatrix} U_0 \\ \Phi_0 \end{bmatrix} \qquad \text{has full row rank.}$$

Example – Output feedback stabilization of a mechanical system

Consider the SISO system

$$\begin{aligned}\dot{x} &= \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\gamma & 0 & \gamma & 0 \\ 0 & 0 & 0 & 1 \\ \gamma & 0 & -\gamma & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} u \\ y &= \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix} x\end{aligned}$$

representing two carts mechanically coupled by a spring with unknown stiffness γ (data are collected assuming that $\gamma = 1$). The output is the position of one of the carts and the input is a force applied to the other cart.

System is discretized using a sampling time of 1sec to obtain $x^+ = Ax + Bu$, $y = Cx$, where

$$A = \begin{bmatrix} 0.5780 & 0.8492 & 0.4220 & 0.1508 \\ -0.6985 & 0.5780 & 0.6985 & 0.4220 \\ 0.4220 & 0.1508 & 0.5780 & 0.8492 \\ 0.6985 & 0.4220 & -0.6985 & 0.5780 \end{bmatrix}, \quad B = \begin{bmatrix} 0.4610 \\ 0.8492 \\ 0.0390 \\ 0.1508 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix}$$

The system is reachable and observable. As the system is SISO, $\ell = n$ and $p\ell = n$.

Data are generated from random initial conditions applying a random input sequence of length $T \geq m(L+1)-1 = m(2\ell+2)-1 = 9$ (we take $T = 12$). The dataset

$$u_{[0,T]} = \begin{bmatrix} -0.8456 & -0.5727 & -0.5587 & 0.1784 & -0.1969 & 0.5864 & -0.8519 & 0.8003 & -1.5094 & 0.8759 & -0.2428 & 0.1668 & -1.9654 \end{bmatrix}$$
$$y_{[0,T]} = \begin{bmatrix} -0.758 & -1.509 & -1.252 & -1.304 & -2.921 & -4.892 & -5.414 & -5.008 & -5.839 & -8.040 & -9.702 & -10.047 & -10.330 \end{bmatrix}$$

is arranged in the matrices

$$\Phi_1 := \left[\begin{array}{cccc} y(1) & y(2) & \dots & y(T-\ell+1) \\ \vdots & \vdots & & \vdots \\ y(\ell) & y(\ell+1) & \dots & y(T) \\ \hline u(1) & u(2) & \dots & u(T-\ell+1) \\ \vdots & \vdots & & \vdots \\ u(\ell) & u(\ell+1) & \dots & u(T) \end{array}\right], \Phi_0 := \left[\begin{array}{cccc} y(0) & y(1) & \dots & y(T-\ell) \\ \vdots & \vdots & & \vdots \\ y(\ell-1) & y(\ell) & \dots & y(T-1) \\ \hline u(0) & u(1) & \dots & u(T-\ell) \\ \vdots & \vdots & & \vdots \\ u(\ell-1) & u(\ell) & \dots & u(T-1) \end{array}\right], \; U_0 := \begin{bmatrix} u(\ell) & u(\ell+1) & \dots & u(T) \end{bmatrix}$$

These are used in the formulation of the SDP

$$\Phi_0 \mathcal{Y} = \mathcal{P}, \quad \begin{bmatrix} -\mathcal{P} & \Phi_1 \mathcal{Y} \\ \mathcal{Y}^\top \Phi_1^\top & -\mathcal{P} \end{bmatrix} \prec 0.$$

With the dataset above, the SDP is feasible (it can be checked that $\begin{bmatrix} U_0 \\ \Phi_0 \end{bmatrix}$ has full row rank).

We obtain the "controller gain"

$$\mathcal{K} = U_0 \mathcal{Y} \mathcal{P}^{-1} = \begin{bmatrix} 1.3529 & -1.7460 & 1.5509 & -1.6854 & -0.0496 & -0.5617 & -1.0801 & -1.0371 \end{bmatrix}$$

which stabilizes the auxiliary system $\phi^+ = \mathcal{A}\phi + \mathcal{B}v$, i.e., it makes $\mathcal{A} + \mathcal{B}\mathcal{K}$ Schur, where

$$\mathcal{A} = \left[\begin{array}{cccc|cccc} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ \multicolumn{4}{c|}{CA^n\mathcal{O}_n^\dagger} & \multicolumn{4}{c}{C\mathcal{R}_n - CA^n\mathcal{O}_n^\dagger\mathcal{T}_n} \\ \hline 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right], \quad \mathcal{B} = \left[\begin{array}{c} 0 \\ 0 \\ 0 \\ 0 \\ \hline 0 \\ 0 \\ 0 \\ 1 \end{array}\right], \quad n = 4$$

Recall that $CA^n\mathcal{O}_n^\dagger, C\mathcal{R}_n - CA^n\mathcal{O}_n^\dagger\mathcal{T}_n$ are unknown.

<u>Sanity check</u> The eigenvalues (modulus) of $\mathcal{A} + \mathcal{B}\mathcal{K}$ are

$$\{0.8896, 0.8896, 0.5863, 0.5863, 0.5235, 0.3028, 0.3028, 0.2406\}.$$

The output feedback controller

$$\begin{cases} \chi^+ = & (\mathcal{F} + \mathcal{B}\mathcal{K})\chi + \mathcal{L}y \\ u = & \mathcal{K}\chi \end{cases}$$

is

$$\begin{cases} \chi^+ = & \left(\left[\begin{array}{cccc|cccc} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right] + \left[\begin{array}{c} 0 \\ 0 \\ 0 \\ 0 \\ \hline 0 \\ 0 \\ 0 \\ 1 \end{array} \right] \mathcal{K} \right) \chi + \left[\begin{array}{c} 0 \\ 0 \\ 0 \\ 1 \\ \hline 0 \\ 0 \\ 0 \\ 0 \end{array} \right] y \\ u = & \mathcal{K}\chi, \quad \mathcal{K} = \begin{bmatrix} 1.3529 & -1.7460 & 1.5509 & -1.6854 & -0.0496 & -0.5617 & -1.0801 & -1.0371 \end{bmatrix} \end{cases}$$

The matrix of the closed-loop system

$$\begin{bmatrix} x^+ \\ \chi^+ \end{bmatrix} = \begin{bmatrix} A & B\mathcal{K} \\ \mathcal{L}C & \mathcal{F} + \mathcal{B}\mathcal{K} \end{bmatrix} \begin{bmatrix} x \\ \chi \end{bmatrix}$$

has eigenvalues (modulus)

$$\{\underline{0.8896}, \underline{0.8896}, \underline{0.2406}, \underline{0.3028}, \underline{0.3028}, \underline{0.5863}, \underline{0.5863}, 0.0005, 0.0005, 0.0005, 0.0005, \underline{0.5235}\},$$

hence, it is Schur stable.

```
% Bertinoro_output_feedback
clear all
close all
rng(1)

%% system

n = 4;  % dimension state, which we assume to be known
m = 1;  % dimension input
p = 1;  % dimension output
T = 12; % number of samples
J = T-n;

gam = 1;

A = [0 1 0 0;-gam 0 gam 0;0 0 0 1;gam 0 -gam 0];
B = [0;1;0;0];
C = [0 0 1 0];

STcont    = ss(A,B,C,0);
STdisc    = c2d(STcont,1);
[A,B,C,D] = ssdata(STdisc);
```

```
%% Minimality test

OB = obsv(A,C);
if rank(OB) < n
    disp('system not observable');
    return
end
CO = ctrb(A,B);
if rank(CO) < n
    disp('system not controllable');
    return
end

%% i/o representation

[num,den] = ss2tf(A,B,C,0);
A_coeff   = den(:,2:end);
B_coeff   = num(:,2:end);
A_coeff   = fliplr(A_coeff);
B_coeff   = fliplr(B_coeff);
```

```
%% auxiliary system

A_cal                      = zeros(2*n,2*n);
app2                       = eye(n-1);
A_cal(1:n-1,2:n)           = app2;
A_cal(n+1:2*n-1,n+2:2*n) = app2;
A_cal(n,:)                 = [-A_coeff B_coeff];

F_cal                      = zeros(2*n,2*n);
app2                       = eye(n-1);
F_cal(1:n-1,2:n)           = app2;
F_cal(n+1:2*n-1,n+2:2*n) = app2;

B_big         = zeros(2*n,1);
B_big(end,1) = 1;
B_cal         = B_big;
C_big = [-A_coeff B_coeff];

L_big         = zeros(2*n,1);
L_big(n,1)    = 1;
L_cal         = L_big;

A_cal2 = F_cal+L_cal*[-A_coeff B_coeff]; % same as A_cal above
```

```
CO_big_sys = ctrb(A_cal,B_cal);
if rank(CO_big_sys) < 2*n
    disp('System A_cal, B_cal not reachable');
    return
end

%% data acquisition

X = zeros(n,T);   % storage, corresponds to X_{0,T}
U = randn(m,T+1);   % storage, corresponds to U_{0,1,T}
Y = zeros(m,T);   % storage, corresponds to Y_{0,1,T}

x = randn(n,1);  % initial conditions

for i =1:T+1
    u = U(:,i);
    X(:,i) = x;
    Y(:,i) = C*x;
    x = A*x+B*u;
end

M = zeros(2*n,J+1); % to construct matrices Phi0, Phi1
```

```
for i =1:n
    M(i,:) = Y(1,i:i+J);
end
for i =1:n
    M(n+i,:) = U(1,i:i+J);
end

Phi0  = M;
U0 = U(1,n+1:n+J+1);

N = [U0;Phi0];

if rank(N) < 2*n+1
    disp('PE condition failed');
    return
end

Phi_aux = [Y(1,J+2:J+n+1)'; U(1,J+2:J+n+1)'];
Phi1  = [Phi0(:,2:end) Phi_aux];

%% test on the identity A_cal*Phi0+B_cal*U0 = Phi1
if norm(A_cal*Phi0+B_cal*U0 - Phi1) > 1e-5
    disp('numerical problems');
    return
end
```

```
%% controller design (using CVX)

cvx_begin sdp
    variable Q(J+1,2*n)
    variable P(2*n,2*n) symmetric
    [P-eye(2*n), Phi1*Q; transpose(Phi1*Q), P] >= 0;
    Phi0*Q==P;
cvx_end

K_cal = U0*Q/P;

A_closed_loop_aux=A_cal+B_cal*K_cal;
disp('Aux system closed-loop eigenvalues (modulus)'); disp(abs(eig(A_closed_loop_aux)));

A_closed_loop=[A B*K_cal;  L_cal*C F_cal+B_cal*K_cal];
disp('Closed-loop system eigenvalues (modulus)'); disp(abs(eig(A_closed_loop)));
```

Some final comments

- The whole construction requires a few priors, the most demanding of which is arguably the knowledge of the observability index ℓ. In the case of SISO observable systems, this boils down to the knowledge of the number of states ($\ell = n$). This is either available from physical principles or can be obtained from techniques processing the input-output data, as in e.g. subspace identification, without requiring the whole procedure to identify the system's model.
- What if $p\ell \neq n$? By observability, $p\ell \geq n$, hence the case of interest is $p\ell > n$. In this case, we can augment the system with an artificial one of our choice connected in parallel with the actual system and aim at having $p\ell = n_{\text{aug}}$.
- The arguments can be extended to deal with the case of noisy output measurements, but it is outside the scope of these lectures.
- Dealing with the output feedback stabilization problem for continuous-time systems is more challenging than dealing with the state feedback problem.
- A similar construction can be extended to nonlinear systems that are uniformly observable.

Summary Lecture 1

Lecture 1

- Data-driven stabilization of linear systems via full state static feedback
- Data-driven stabilization of linear systems via output dynamic feedback.

- Lecture 2 discusses how the design of a state feedback controller can be extended in the presence of perturbed measurements
- Lectures 3-5 discusses extensions to nonlinear systems

De Persis, Tesi. "Formulas for data-driven control: stabilization, optimality and robustness". IEEE Transactions on Automatic Control, 65 (3), 909-924, 2020.
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Rotulo, De Persis, Tesi. "Online learning of data-driven controllers for unknown switched linear systems". Automatica 145, 110519, 2022.
Bisoffi, De Persis, Tesi. "Controller design for robust invariance from noisy data." IEEE Transactions on Automatic Control 68 (1), 636-643, 2022.
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Additional material

Optimality

Optimality - Linear Quadratic Regulation

LQR problem Assume (A, B) reachable. Consider the problem of minimizing

$$J_\infty(x_0, u) := \sum_{k=0}^{\infty} (x(k)Qx(k) + u(k)^\top Ru(k)), \quad Q \succ 0, R \succ 0$$

over the set of input sequences $u\colon \mathbb{Z}_{\geq 0} \to \mathbb{R}^m$ for which the solution $x\colon \mathbb{Z}_{\geq 0} \to \mathbb{R}^n$ to $x(k+1) = Ax(k) + Bu(k)$, $x(0) = x_0$, satisfies $\lim_{k\to\infty} x(k) = 0$.

There exists a unique optimal controller given by

$$u_\star := K_\star x, \quad K_\star := -(R + B^\top PB)^{-1}B^\top PA$$

where $P \succ 0$ is the unique solution of the DARE

$$A^\top PA - P - A^\top PB(R + B^\top PB)^{-1}B^\top PA + Q = 0$$

that renders the matrix $A - B(R + B^\top PB)^{-1}B^\top PA$ Schur stable. Moreover, the optimal cost is $x_0^\top Px_0$.

Importance of data-driven LQR

- Infinite-horizon LQR is the prime example of challenges encountered in data-driven optimal control (effect of noise, deviation from optimality)
- Of interest to both the data-driven control and machine learning community

A reformulation of LQR: computing $K_\star$ via SDP

For the system

$$\begin{aligned} x(k+1) &= Ax(k) + Bu(k) + \xi(k) \\ z(k) &= \begin{bmatrix} Q^{1/2} & 0 \\ 0 & R^{1/2} \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} \end{aligned} \qquad (\star)$$

design K that

- makes $A + BK$ Schur stable
- minimizes the sum of the squares of the energy of the output responses to the impulse inputs of the closed-loop system

$$\begin{aligned} x(k+1) &= (A+BK)x(k) + \xi(k) \\ z(k) &= \begin{bmatrix} Q^{1/2} \\ R^{1/2}K \end{bmatrix} x(k) \end{aligned}$$

Impulse response

Consider the Schur stable closed-loop system

$$x(k+1) = \underbrace{(A+BK)}_{A_c} x(k) + \underbrace{I_n}_{B_c} \cdot \xi(k)$$

$$z(k) = \underbrace{\begin{bmatrix} Q^{1/2} \\ R^{1/2}K \end{bmatrix}}_{C_c} x(k)$$

and compute the output energy of the impulse responses of the system.

- Let $z^{(j)}$ be the response to the impulse $e_j\delta(k)$, with e_j the j-th vector of the canonical basis of $\mathbb{R}^n$ and $\delta(k)$ the discrete-time impulse

$$z^{(j)}(k) = \begin{cases} 0 & k = 0 \\ C_c A_c^{k-1} e_j & k > 0 \end{cases}$$

- Let $\|z^{(j)}\|_2^2$ denote its energy (the series is summable because A_c is Schur)

$$\sum_{k=0}^{\infty} \|z^{(j)}(k)\|^2 = \sum_{k=0}^{\infty} e_j^\top (A_c^\top)^k C_c^\top C_c A_c^k e_j = \sum_{k=0}^{\infty} \text{trace}(C_c A_c^k e_j e_j^\top (A_c^\top)^k C_c^\top)$$

  Then

$$\begin{aligned} \sum_{j=1}^{n} \|z^{(j)}\|_2^2 &= \text{trace}\Big(\sum_{k=0}^{\infty} C_c A_c^k B_c B_c^\top (A_c^\top)^k C_c^\top\Big) \\ &= \text{trace}\Big(\sum_{k=0}^{\infty} B_c^\top (A_c^\top)^k C_c^\top C_c A_c^k B_c\Big) \end{aligned}$$

From $\sum_{j=1}^{n} \|z^{(j)}\|_2^2 = \text{trace}\Big(\sum_{k=0}^{\infty} C_c A_c^k B_c B_c^\top (A_c^\top)^k\Big) = \text{trace}\Big(C_c\Big(\sum_{k=0}^{\infty} A_c^k B_c B_c^\top (A_c^\top)^k\Big) C_c^\top\Big)$,
if one sets

$$P := \sum_{k=0}^{\infty} A_c^k B_c B_c^\top (A_c^\top)^k$$

one realizes that P, the controllability gramian, is the (unique) positive semidefinite matrix satisfying

$$A_c P A_c^\top - P + B_c B_c^\top = (A + BK)P(A + BK)^\top - P + I = 0$$

The last equation and $P \succeq 0$ implies that

$$P = (A + BK)P(A + BK)^\top + I \succeq I$$

Finally

$$\begin{aligned} \sum_{j=1}^{n} \|z^{(j)}\|_2^2 = & \quad \text{trace}\Big(C_c P C_c^\top\Big) \\ = & \quad \text{trace}\left(\begin{bmatrix} Q^{1/2} \\ R^{1/2}K \end{bmatrix} P \begin{bmatrix} Q^{1/2} \\ R^{1/2}K \end{bmatrix}^\top\right) = \text{trace}(QP) + \text{trace}(R^{1/2} K P K^\top R^{1/2}) \end{aligned}$$

In summary The sum of the squares of the energy of the output responses to the impulse inputs of the Schur stable system

$$\begin{aligned} x(k+1) &= \underbrace{(A+BK)}_{A_c} x(k) + \underbrace{I_n}_{B_c} \cdot \xi(k) \\ z(k) &= \underbrace{\begin{bmatrix} Q^{1/2} \\ R^{1/2}K \end{bmatrix}}_{C_c} x(k) \end{aligned}$$

is given by

$$\sum_{j=1}^{n} \|z^{(j)}\|_2^2 = \text{trace}(QP) + \text{trace}(R^{1/2}KPK^\top R^{1/2})$$

with P the unique matrix satisfying

$$\begin{aligned} &(A+BK)P(A+BK)^\top - P + I_n = 0 \\ &P \succeq I_n \end{aligned}$$

The $\mathcal{H}_2$-norm minimization problem

$\mathcal{H}_2$-norm By the discrete-time version of Parseval's theorem

$$\sum_{j=1}^{n} \|z^{(j)}\|_2^2 = \|\mathcal{T}(K)\|_2^2$$

where $\|\mathcal{T}(K)\|_2^2$ is the $\mathcal{H}_2$-norm* of the transfer function $\mathcal{T}(K)$ of the Schur stable system

$$x(k+1) = \underbrace{(A+BK)}_{A_c} x(k) + \underbrace{I_n}_{B_c} \cdot \xi(k)$$

$$z(k) = \underbrace{\begin{bmatrix} Q^{1/2} \\ R^{1/2}K \end{bmatrix}}_{C_c} x(k)$$

$^\star \|\mathcal{T}(K)\|_2^2 := \frac{1}{2\pi} \int_0^{2\pi} \text{trace}\Big(\mathcal{T}(\mathrm{e}^{i\theta})^* \mathcal{T}(\mathrm{e}^{i\theta})\Big) d\theta$ where $\mathcal{T}(\mathrm{e}^{i\theta}) := \mathcal{T}(K)|_{z=\mathrm{e}^{i\theta}}$

The state feedback controller that minimizes $\|\mathcal{T}(K)\|_2^2$, i.e., that solves

$$\begin{aligned} \min_{K,P} \quad & \text{trace}(QP) + \text{trace}(R^{1/2}KPK^\top R^{1/2}) \\ \text{subject to} \quad & \begin{cases} (A+BK)P(A+BK)^\top - P + I_n = 0 \\ P \succeq I_n \end{cases} \end{aligned}$$

is unique and coincides with the solution to the LQR problem, i.e., $K = K_\star$ (Chen-Francis, *Optimal sampled-data control system*, Section 6.4).

A semidefinite program for solving the $\mathcal{H}_2$-norm minimization problem

The previous arguments suggest the following convex relaxation of the $\mathcal{H}_2$-norm minimization problem

$$\begin{aligned} &\min_{K,P,L} \ \text{trace}\,(QP) + \text{trace}\,(L) \\ &\text{subject to} \\ &\begin{cases} (A+BK)P(A+BK)^\top - P + I_n \preceq 0 \\ P \succeq I_n \\ L - R^{1/2}KPK^\top R^{1/2} \succeq 0 \end{cases} \end{aligned}$$

where the equality constraint is relaxed to an inequality and the constraint

$$R^{1/2}KWK^\top R^{1/2} \preceq L$$

is introduced to remove

$$\text{trace}(R^{1/2}KWK^\top R^{1/2})$$

from the cost function and replace it with the linear term trace(L).

By (Feron *et al.*, Proposition 1), under the given assumptions, the problem above is well-posed, ie. the feasible set is compact or empty. As the feasible set is non-empty, then the feasible set is compact.

E. Feron, V. Balakrishnan, S. Boyd, L. El Ghaoui, "Numerical methods for H_2 related problems," in 1992 American Control Conference, pp. 2921–2922.

A data-dependent solution to the LQR

The $\mathcal{H}_2$-norm minimization problem and its convex relaxation

$$\begin{array}{ll} \min_{K,P} & \text{trace}(QP) + \text{trace}(R^{1/2}KPK^\top R^{1/2}) \\ \text{subject to} & \begin{cases} (A+BK)P(A+BK)^\top - P + I_n = 0 \\ P \succeq I_n \end{cases} \end{array}$$

$$\begin{array}{ll} \min_{K,P,L} & \text{trace}\,(QP) + \text{trace}\,(L) \\ \text{subject to} & \begin{cases} (A+BK)P(A+BK)^\top - P + I_n \preceq 0 \\ P \succeq I_n \\ L - R^{1/2}KPK^\top R^{1/2} \succeq 0 \end{cases} \end{array}$$

are related as follows

Proposition A solution $(\overline{K}, \overline{P}, \overline{L})$ to the convex relation is such that $(\overline{K}, \overline{P})$ is the solution to the $\mathcal{H}_2$-norm minimization problem. Moreover, $\overline{K} = K_\star$, that is, $\overline{K}$ is the solution to the optimal LQR problem.

A data-dependent solution to the LQR

The previous optimization problem leads to the following data-dependent SDP for designing the LQR from data

$$\min_{G,P,L} \ \operatorname{trace}(QP) + \operatorname{trace}(L)$$

subject to

$$\begin{cases} X_1 G P G^\top X_1^\top - P + I_n \preceq 0 \\ P \succeq I_n \\ L - R^{1/2} U_0 G P G^\top U_0^\top R^{1/2} \succeq 0 \\ X_0 G = I_n \end{cases} \qquad \text{(DD-SDP-LQR)}$$

Theorem Assume that $\begin{bmatrix} U_0 \\ X_0 \end{bmatrix}$ has full row rank. Any optimal solution (G^o, P^o, L^o) to (DD-SDP-LQR) is such that $K^o := U_0 G^o$ satisfies

$$K_\star = K^o$$

and

$$\|\mathcal{T}(K^o)\|_2^2 = \operatorname{trace}(QP^o) + \operatorname{trace}(L^o)$$

A sketch of proof

Lemma 1 Consider any control gain K stabilising for

$$\begin{aligned} x(k+1) &= Ax(k) + Bu(k) + \xi(k) \\ z(k) &= \begin{bmatrix} Q^{1/2} & 0 \\ 0 & R^{1/2} \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} \end{aligned} \qquad (\star)$$

Then there exists a triple (G_K, P, L) feasible for (DD-SDP-LQR) such that

$$K = U_0 G_K \quad \text{and} \quad \|\mathcal{T}(K)\|_2^2 = \text{trace}(QP) + \text{trace}(L)$$

A sketch of proof

Lemma 1 Consider any control gain K stabilising for $(\star)$. Then there exists a triple (G_K, P, L) feasible for (DD-SDP-LQR) such that

$$K = U_0 G_K \quad \text{and} \quad \|\mathcal{T}(K)\|_2^2 = \text{trace}(QP) + \text{trace}(L)$$

For a given K, let G_K be such that

$$\begin{bmatrix} K \\ I_n \end{bmatrix} = \begin{bmatrix} U_0 \\ X_0 \end{bmatrix} G_K \quad \Longleftrightarrow \quad K = U_0 G_K,\ I_n = X_0 G_K$$

As K is stabilizing, $A + BK = X_1 G_K$ is Schur stable and there exists a unique controllability gramian P such that

$$X_1 G_K P X_1 G_K^\top - P + I = 0, \quad P \succeq I$$

Moreover, $\|\mathcal{T}(U_0 G_K)\|_2^2 = \text{trace}(QP) + \text{trace}(R^{1/2} U_0 G_K P G_K^\top U_0^\top R^{1/2})$

Set $L := R^{1/2} U_0 G_K P G_K^\top U_0^\top R^{1/2}$. Then

$$\|\mathcal{T}(U_0 G_K)\|_2^2 = \text{trace}(QP) + \text{trace}(L)$$

and (G_K, P, L) is feasible for (DD-SDP-LQR)

A sketch of proof

Lemma 1 Consider any control gain K stabilising for $(\star)$. Then there exists a triple (G_K, P, L) feasible for (DD-SDP-LQR) such that

$$K = U_0 G_K \quad \text{and} \quad \|\mathcal{T}(K)\|_2^2 = \text{trace}(QP) + \text{trace}(L)$$

The feasible solution (G_K, P, L) to

$$\min_{G,P,L} \ \text{trace}\,(QP) + \text{trace}\,(L)$$
$$\text{subject to}$$
$$\begin{cases} X_1 G P G^\top X_1^\top - P + I_n \preceq 0 \\ P \succeq I_n \\ L - R^{1/2} U_0 G P G^\top U_0^\top R^{1/2} \succeq 0 \\ X_0 G = I_n \end{cases} \qquad \text{(DD-SDP-LQR)}$$

was obtained by

- computing G_K as a solution to $\begin{bmatrix} K \\ I_n \end{bmatrix} = \begin{bmatrix} U_0 \\ X_0 \end{bmatrix} G_K$
- Setting P equal to the controllability gramian, i.e. $X_1 G_K P G_K^\top X_1^\top - P + I_n = 0$, $P \succeq I_n$
- Setting $L = R^{1/2} U_0 G_K P G_K^\top U_0^\top R^{1/2}$

A sketch of proof

Lemma 2 Any feasible solution (G, P, L) to (DD-SDP-LQR) is such that $K = U_0 G$ is stabilizing for $(\star)$ and

$$\|\mathcal{T}(K)\|_2^2 \leq \text{trace}(QP) + \text{trace}(L)$$

Proof – see Exercise #1

A sketch of proof

Exercise #1

(a) Show that $K = U_0 G$ is stabilising.

As $I_n = X_0 G$, setting $K = U_0 G$ yields $A + BK = X_1 G$. Since (G, P, L) is a feasible solution, $P \succeq I$ and $X_1 G_K P X_1 G_K{}^\top - P + I \preceq 0$ show that $X_1 G_K$ is Schur stable, hence $K = U_0 G$ is stabilising.

(b) Show that the inequality $X_1 G_K P X_1 G_K{}^\top - P + I \preceq 0$ implies the existence of a matrix Θ such that P is the controllability Gramian of the system

$$\begin{aligned} x(k+1) &= X_1 G_K x(k) + \begin{bmatrix} I & \Theta \end{bmatrix} \xi(k) \\ z(k) &= \begin{bmatrix} Q^{1/2} \\ R^{1/2} K \end{bmatrix} x(k) \end{aligned}$$

Since $X_1 G_K$ is Schur stable, P is the controllability gramian for the system if and only if

$$X_1 G_K P X_1 G_K{}^\top - P + I + \begin{bmatrix} I & \Theta \end{bmatrix} \begin{bmatrix} I & \Theta \end{bmatrix}^\top = 0$$

Hence, one needs to prove the existence of a matrix Θ such that the equation above holds. Since $X_1 G_K P G_K{}^\top X_1^\top - P + I \preceq 0$, then there exists Θ such that

$$X_1 G_K P X_1 G_K{}^\top - P + I + \Theta\Theta^\top = 0$$

In fact, set $\Xi := -(X_1 G_K P X_1 G_K{}^\top - P + I)$. Then $X_1 G_K P X_1 G_K{}^\top - P + I + \Xi = 0$. Since $\Xi \succeq 0$, by Cholesky factorization, we have $\Xi = \Theta\Theta^\top$.

A sketch of proof

(c) Show that $\|\mathcal{T}_e(K)\|_2^2 = \text{trace}(QP) + \text{trace}(R^{1/2}KPK^\top R^{1/2})$, where $\mathcal{T}_e(K)$ is the transfer function of

$$\begin{aligned} x(k+1) &= X_1 G_K x(k) + \begin{bmatrix} I & \Theta \end{bmatrix} \xi(k) \\ z(k) &= \begin{bmatrix} Q^{1/2} \\ R^{1/2}K \end{bmatrix} x(k) \end{aligned}$$

Since P is the controllability gramian for the system, then

$$\|\mathcal{T}_e(K)\|_2^2 = \text{trace}\Big(\begin{bmatrix} Q^{1/2} \\ R^{1/2}K \end{bmatrix} P \begin{bmatrix} Q^{1/2} \\ R^{1/2}K \end{bmatrix}^\top \Big)$$

and the claim follows immediately by the definition of trace.

(d) Conclude $\|\mathcal{T}(K)\|_2^2 \leq \|\mathcal{T}_e(K)\|_2^2 \leq \text{trace}(QP) + \text{trace}(L)$

By Parseval's theorem the total energy of the output impulsive responses equals the $\mathcal{H}_2$-norm squared of the system

$$\|\mathcal{T}_e(K)\|_2^2 = \text{trace}\Big(\sum_{k=0}^{\infty} C_c \cdot (X_1 G_K)^k \begin{bmatrix} I & \Theta \end{bmatrix} \begin{bmatrix} I \\ \Theta^\top \end{bmatrix} (G_K^\top X_1^\top)^k C_c^\top \Big)$$

A sketch of proof

Hence

$$
\begin{aligned}
&\|\mathcal{T}_e(K)\|_2^2 = \text{trace}\Big(\sum_{k=0}^{\infty} C_c \cdot (X_1 G_K)^k \begin{bmatrix} I & \Theta \end{bmatrix} \begin{bmatrix} I \\ \Theta^\top \end{bmatrix} (G_K^\top X_1^\top)^k C_c^\top \Big) = \\
&\text{trace}\Big(\sum_{k=0}^{\infty} C_c (X_1 G_K)^k \, I \, (G_K^\top X_1^\top)^k C_c^\top \Big) + \text{trace}\Big(\sum_{k=0}^{\infty} C_c (X_1 G_K)^k \Theta\Theta^\top (G_K^\top X_1^\top)^k C_c^\top \Big) \geq \\
&\text{trace}\Big(\sum_{k=0}^{\infty} C_c (X_1 G_K)^k \, I \, (G_K^\top X_1^\top)^k C_c^\top \Big) = \|\mathcal{T}(K)\|_2^2
\end{aligned}
$$

The claim follows since

$$
\begin{aligned}
\|\mathcal{T}_e(K)\|_2^2 &\overset{(c)}{=} \text{trace}(QP) + \text{trace}(R^{1/2} K P K^\top R^{1/2}) \\
&\overset{R^{1/2} K P K^\top R^{1/2} \preceq L}{\leq} \text{trace}(QP) + \text{trace}(L)
\end{aligned}
$$

A sketch of proof – final argument

An optimal solution (G^o, P^o, L^o) to (DD-SDP-LQR) satisfies (Lemma 2)

$$\|\mathcal{T}(K^o)\|_2^2 \leq \text{trace}(QP^o) + \text{trace}(L^o) \quad \text{with} \quad K^o := U_0 G^o$$

On the other hand, since $K_\star$ is stabilizing, there exists a feasible $(G_{K_\star}, P_\star, L_\star)$ for (DD-SDP-LQR) such that (Lemma 1)

$$K_\star = U_0 G_{K_\star} \quad \text{and} \quad \|\mathcal{T}(K_\star)\|_2^2 = \text{trace}(QP_\star) + \text{trace}(L_\star)$$

As (G^o, P^o, L^o) is an optimal solution to (DD-SDP-LQR), it is true that

$$\text{trace}(QP^o) + \text{trace}(L^o) \leq \text{trace}(QP_\star) + \text{trace}(L_\star)$$

which implies

$$\|\mathcal{T}(K^o)\|_2^2 \leq \text{trace}(QP^o) + \text{trace}(L^o) \leq \text{trace}(QP_\star) + \text{trace}(L_\star) = \|\mathcal{T}(K_\star)\|_2^2$$

As $K_\star$ is the optimal solution to the $\mathcal{H}_2$-norm minimization problem, $\|\mathcal{T}(K_\star)\|_2^2 \leq \|\mathcal{T}(K^o)\|_2^2$, that is $\|\mathcal{T}(K_\star)\|_2^2 = \|\mathcal{T}(K^o)\|_2^2$ and by uniqueness of the optimal gain, $K^o = K_\star$

A data-dependent solution to the LQR

Recap We have shown the correctness of the following data-dependent SDP for designing the LQR from data

$$\min_{G,P,L}\ \operatorname{trace}(QP)+\operatorname{trace}(L)$$
subject to
$$\begin{cases} X_1GPG^\top X_1^\top - P + I_n \preceq 0 \\ P \succeq I_n \\ L - R^{1/2}U_0GPG^\top U_0^\top R^{1/2} \succeq 0 \\ X_0G = I_n \end{cases} \qquad \text{(DD-SDP-LQR)}$$

Theorem Assume that $\begin{bmatrix} U_0 \\ X_0 \end{bmatrix}$ has full row rank. Any optimal solution (G^o, P^o, L^o) to (DD-SDP-LQR) is such that $K^o := U_0G^o$ satisfies

$$K_\star = K^o$$

and

$$\|\mathcal{T}(K^o)\|_2^2 = \operatorname{trace}(QP^o) + \operatorname{trace}(L^o)$$

A data-dependent SDP for solving the LQR

The change of variables $Y = GP$ and an application of Schur complement lead to the semidefinite program

$$\min_{Y,P,L} \ \operatorname{trace}(QP) + \operatorname{trace}(L)$$

subject to

$$\begin{cases} \begin{bmatrix} P - I_n & X_1 Y \\ Y^\top X_1^\top & P \end{bmatrix} \succeq 0 \\ \begin{bmatrix} L & R^{1/2} U_0 Y \\ Y^\top U_0^\top R^{1/2} & P \end{bmatrix} \succeq 0 \\ \\ P = X_0 Y \end{cases}$$

with the optimal gain matrix given by

$$K_\star = U_0 Y P^{-1}$$

C. De Persis, P. Tesi. "Formulas for data-driven control: stabilization, optimality, robustness". IEEE Transactions on Automatic Control, 65(3), 909-924, 2020.

Discussion

- The data-based problem is solvable via efficient numerical algorithms (cvx)

```
cvx_begin sdp
    variable Y(T,n)
    variable L(m,m) symmetric
    variable P(n,n) symmetric
    minimize ( trace(Q*P) +trace(L) )
    [L, sqrtm(R)*U0*Y; Y'*U0'*sqrtm(R)', P] >= 0
    [P-eye(n), X1*Y; Y'*X1', P] >= 0
    P=X0*Y
cvx_end
K = U0*Y*inv(P);
```

- It only requires data collected in low sample-complexity experiments
- Solution is exactly computed via a single SDP and not approximated via sequential iterations as in, e.g., LQR via policy iteration

Policy iteration and LQR

Algorithm 1 Policy iteration applied to the LQR problem

```
Guess initial stabilizing gain K_0
Set initial time k = 0
for i = 0 to ∞ do
    for j = 1 to N do
        Apply u(k) = K_i x(k) + e(k), e(k) PE "exploration signal"
        Estimate K_i(j) using RLS and I/O measurements
        k = k + 1
    end for
    Set K_{i+1} = K_i(N)
end for
```

There exists an estimation interval N such that the algorithm generates a sequence $\{K_i \,:\, i = 0, 1, 2, \ldots\}$ such that $\lim_{i \to \infty} \|K_i - K_\star\| = 0$

S.J. Bradtke, B.E. Ydstie and A.G. Barto. Adaptive linear quadratic control using policy iteration. Proceedings of the 1994 American Control Conference, 3475–3479, 1994.

The data-dependent solution to LQR with noisy data

$$\min_{G,P,L} \ \text{trace}(QP) + \text{trace}(L)$$
subject to
$$\begin{cases} X_1 G P G^\top X_1^\top - P + I_n \preceq 0 \\ P \succeq I_n \\ L - R^{1/2} U_0 G P G^\top U_0^\top R^{1/2} \succeq 0 \\ X_0 G = I_n \end{cases}$$

As any other result in this Lecture 1, this program is derived from noise-free data

In the presence of noise, brought in by the unknown matrix D_0 (Lecture 2), the data-dependent representation leads to the SDP $\Rightarrow$

$$\min_{G,P,L} \ \text{trace}(QP) + \text{trace}(L)$$
subject to
$$\begin{cases} (X_1 - D_0) G P G^\top (X_1 - D_0)^\top - P + I_n \preceq 0 \\ P \succeq I_n \\ L - R^{1/2} U_0 G P G^\top R^{1/2} \succeq 0 \\ X_0 G = I_n \end{cases}$$

The resulting optimal gain matrix is $K^o = U_0 Y P^{-1}$, which coincides with $K_\star$

The data-dependent solution to LQR with noisy data

$$\min_{G,P,L} \ \text{trace}\,(QP) + \text{trace}\,(L)$$

subject to

$$\begin{cases} X_1 G P G^\top X_1^\top - P + I_n \preceq 0 \\ P \succeq I_n \\ L - R^{1/2} U_0 G P G^\top U_0^\top R^{1/2} \succeq 0 \\ X_0 G = I_n \end{cases}$$

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$$\min_{G,P,L} \ \text{trace}\,(QP) + \text{trace}\,(L)$$

subject to

$$\begin{cases} (X_1 - D_0) G P G^\top (X_1 - D_0)^\top - P + I_n \preceq 0 \\ P \succeq I_n \\ L - R^{1/2} U_0 G P G^\top R^{1/2} \succeq 0 \\ X_0 G = I_n \end{cases}$$

The resulting optimal gain matrix is $K^o = U_0 Y P^{-1}$, which coincides with $K_\star$

Data-dependent solution to LQR - Soft constraint

- Since D_0 is unknown, one option is to neglect D_0 and require the term $M = GPG^\top$ to be small via the hard constraint $2M \preceq \epsilon I$
- The hard constraint, however, favours too much robustness to the detriment of performance

We instead look for a solution that trades off robustness for performance via a soft constraint

$$
\begin{aligned}
&\min_{Y,P,L,V} \ \operatorname{trace}(QP) + \operatorname{trace}(L) + \alpha \operatorname{trace}(V) \\
&\text{subject to} \\
&\begin{cases}
X_1 GPG^\top X_1^\top - P + I_n \preceq 0 \\
P \succeq I_n \\
L - R^{1/2} U_0 GPG^\top U_0^\top R^{1/2} \succeq 0 \\
V - GPG^\top \succeq 0 \\
X_0 G = I_n
\end{cases}
\end{aligned}
$$

where

$\alpha \gg 1$ favours robustness

$\alpha \ll 1$ favours performance

C. De Persis, P. Tesi. "Low-complexity learning of Linear Quadratic Regulators from noisy data". Automatica 128, 109548, 2021