Data-driven control design

linear and nonlinear systems

Lecture 4

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Nonlinear stabilization via Lyapunov's linearization

In Lecture 3, we considered the problem of rendering the equilibrium $x_e = 0$ asymptotically stable for the nonlinear system

$$x^+ = f(x, u)$$

Under the assumption that f is continuously differentiable, the technical tool that we adopted was the first order Taylor's expansion of $f(x, u)$ around the equilibrium pair $(x_e, u_e) = (0, 0)$:

$$f(x, u) = \frac{\partial f}{\partial x}(0, 0)x + \frac{\partial f}{\partial u}(0, 0)u + r(x, u)$$

where $r(x, u)$ is the remainder.

As $f(x, u)$ is unknown, the matrices $\frac{\partial f}{\partial x}(0, 0)$, $\frac{\partial f}{\partial u}(0, 0)$ and the remainder $r(x, u)$ are unknown.

We collected a dataset $\mathbb{D} := \{u(k), x(k)\}_{k=0}^{T}$ and arranged it into the known matrices U_0, X_0, X_1. For the unknown matrix

$$R_0 = \begin{bmatrix} r(x(0), u(0)) & \dots & r(x(T-1), u(T-1)) \end{bmatrix}$$

we assumed the existence of a matrix Δ such that

$$R_0 R_0^\top \preceq \Delta\Delta^\top$$

We showed that if an SDP in the decision variables $S = S^\top \succ 0, Y$ and in the known matrices X_0, X_1, Δ is feasible, then the linear control law

$$u = Kx := U_0 Y S^{-1} x$$

renders $x_e = 0$ an asymptotically stable equilibrium for

$$x^+ = f(x, Kx)$$

It does so by stabilizing all the linearized plants consistent with the data. The local stability result then descends from the principle of stability by the first approximation.*

*Theorem 7.1 in J.P. La Salle, "The Stability and Control of Discrete Processes", Springer 1986.

Nonlinear systems expressed via basis functions

In this lecture, we consider a class of nonlinear systems of the form

$$x^+ = f(x) + Bu$$

The system is input affine with constant input vector field, as opposed to the general form $x^+ = f(x, u)$.

We consider constant input-vector field (B instead of $g(x)$) for the sake of simplicity.

Differently from before, we don't perform a Taylor's expansion and deal directly with the full nonlinearity $f(x)$ without approximating it.

We avoid approximating the full nonlinearity $f(x)$ by expressing it as a linear combination of basis functions.

This will allow us to design nonlinear control laws that may achieve global stabilization results extending design techniques such as (approximate) feedback linearization and backstepping to a data-driven context.

We consider systems of the form

$$x^+ = f(x) + Bu + Ed, \text{ where } f(x) = A_\star Z_\star(x),$$

- ▷ $x \in \mathbb{R}^n$ (state) and $u \in \mathbb{R}^m$ (control)
- ▷ $d \in \mathbb{R}^q$ (disturbance)
- ▷ $Z_\star : \mathbb{R}^n \to \mathbb{R}^R$ is an unknown vector of functions
- ▷ $A_\star \in \mathbb{R}^{n \times R}$, $B \in \mathbb{R}^{n \times m}$ are unknown matrices
- ▷ $E \in \mathbb{R}^{n \times q}$ known constant matrix

Any $f : \mathbb{R}^n \to \mathbb{R}^n$ can be expressed as $f(x) = A_\star Z_\star(x)$

Assumption A vector-valued function $Z : \mathbb{R}^n \to \mathbb{R}^s$ is known such that any entry of $Z_\star(x)$ is a linear combination of entries of $Z(x)$

The system can be written as

$$x^+ = AZ(x) + Bu + Ed$$

where $A \in \mathbb{R}^{n\times s}$ and A, B are unknown.

▷ $Z(x)$ includes both linear and nonlinear functions

$$Z(x) = \begin{bmatrix} x \\ Q(x) \end{bmatrix}$$

with $Q : \mathbb{R}^n \to \mathbb{R}^{s-n}$. To derive some of the results below, we will assume that $\lim_{|x|\to 0} \frac{|Q(x)|}{|x|} = 0$

$Z(x) = x$ Lecture 1

$Z(x)$ polynomial The analysis of Lecture 1 can be extended through Sum-of-Squares programs

Guo, De Persis, Tesi, "Data-driven stabilization of nonlinear polynomial systems with noisy data", IEEE Transactions on Automatic Control, 2022

Assumption A vector-valued function $Z : \mathbb{R}^n \to \mathbb{R}^s$ is known such that any entry of $Z_\star(x)$ is a linear combination of entries of $Z(x)$

Knowledge of $Z(x)$ might come from the knowledge of the physics of the system

A "dictionary" of functions might be available from some estimation technique

If the assumption is not satisfied, then we regard the discrepancy

$$A_\star Z_\star(x) - AZ(x)$$

as a neglected nonlinearity $d(x)$ and write the system as

$$x^+ = AZ(x) + Bu + Ed(x), \quad \text{where} \quad E = I_n$$

Disturbance

$$x^+ = AZ(x) + Bu + Ed$$

Assumption Disturbance d satisfies $|d| \leq \delta$ for some known δ

We have previously said that d can be state-dependent, i.e., $d = d(x)$, and used to model the discrepancy $A_\star Z_\star(x) - AZ(x)$. The analysis to be presented below in the case of a state-independent d can be repeated to this scenario if a function $\delta \colon \mathbb{R}^n \to \mathbb{R}$ and a set $\mathcal{S} \subseteq \mathbb{R}^n$ are known such that

$$|d(x)| \leq \delta(x) \quad \text{for all} \quad x \in \mathcal{S}$$

Information collection

Information about the system's dynamics is obtained from a T-long dataset of input/state samples collected during (multiple) experiment(s)

$$\mathbb{D} := \{u(k), x(k)\}_{k=0}^{T}$$

where the samples satisfy

$$x(k+1) = AZ(x(k)) + Bu(k) + Ed(k), \quad k = 0, \dots, T-1$$

Even though the values $Z(x(k))$ are not directly measured (i.e., we do not have a sensor measuring $Z(x(k))$), thanks to the knowledge of the dictionary $Z(x)$, the values $Z(x(k))$ are computable starting from $x(k)$

A control problem

Problem Based on the dataset $\mathbb{D}$ design a state feedback controller

$$u = k(x), \quad k : \mathbb{R}^n \to \mathbb{R}^m$$

that makes the origin an asymptotically stable equilibrium for

$$x^+ = AZ(x) + Bk(x) + Ed$$

when $d = 0$, with an estimate of the Region of Attraction.

- Surprisingly, not many results are available for this quintessential control problem
- Which form should $k(x)$ have?
- What is a viable strategy to design $k(x)$?
- Is this strategy "exportable" to deal with other control problems?

A control problem

Problem Based on the dataset $\mathbb{D}$ design a state feedback controller

$$u = KZ(x), \quad K \in \mathbb{R}^{m \times s}$$

that makes the origin an asymptotically stable equilibrium for

$$x^+ = (A + BK)Z(x) + Ed$$

when $d = 0$, with an estimate of the Region of Attraction.

- The state feedback controller $u = KZ(x)$ aims at directly controlling the effect of the nonlinearities in $AZ(x)$
- The choice $k(x) = KZ(x)$ allows us to express the closed-loop dynamics in terms of the dataset $\mathbb{D}$ and to reduce the design of K to a convex problem
- Setting $d = 0$ corresponds to the scenario when a disturbance is not present during the execution of the control task but it is present during the data acquisition phase

A control problem

Problem Based on the dataset $\mathbb{D}$ design a state feedback controller

$$u = KZ(x), \quad K \in \mathbb{R}^{m\times s}$$

that makes the origin an asymptotically stable equilibrium for

$$x^+ = (A + BK)Z(x) + Ed$$

when $d = 0$, with an estimate of the Region of Attraction.

- The state feedback controller $u = KZ(x)$ aims at directly controlling the effect of the nonlinearities in $AZ(x)$
- The choice $k(x) = KZ(x)$ allows us to express the closed-loop dynamics in terms of the dataset $\mathbb{D}$ and to reduce the design of K to a convex problem
- We will derive a data dependent representation in the case of perturbed data ($d \neq 0$ during the data acquisition phase). Then, for pedagogical reasons, we will design our first control policy when $d = 0$ during both the data acquisition and the control execution phase. Later, we will design a control policy when $d \neq 0$ during the data acquisition phase.

A control problem

Problem Based on the dataset $\mathbb{D}$ design a state feedback controller

$$u = KZ(x), \quad K \in \mathbb{R}^{m \times s}$$

that makes the origin an asymptotically stable equilibrium for

$$x^+ = (A + BK)Z(x) + Ed$$

if $d = 0$, with an estimate of the Region of Attraction.

- Nonvanishing perturbations If $d \neq 0$ during the execution of the control task, then the result determines an estimate of the Robustly Positively Invariant set
- Neglected nonlinearities $d = d(x)$ The result determines an estimate of the RoA/PI set (depending on whether or not $d(x)$ vanishes at the origin)

Consider the dataset

$$\mathbb{D} = \{u(k), x(k)\}_{k=0}^{T}, \quad x(k+1) = AZ(x(k)) + Bu(k) + Ed(k), \quad k = 0, \ldots, T-1$$

and store it into matrices U_0, X_0, X_1, Z_0 defined as

$$\begin{aligned}
U_0 &:= \begin{bmatrix} u(0) & u(1) & \cdots & u(T-1) \end{bmatrix} \\
X_0 &:= \begin{bmatrix} x(0) & x(1) & \cdots & x(T-1) \end{bmatrix} \\
X_1 &:= \begin{bmatrix} x(1) & x(2) & \cdots & x(T) \end{bmatrix} \\
Z_0 &:= \begin{bmatrix} Z(x(0)) & Z(x(1)) & \ldots & Z(x(T-1)) \end{bmatrix}
\end{aligned}$$

which satisfy the identity

$$\begin{aligned}
&\underbrace{\begin{bmatrix} x(1) & x(2) & \ldots & x(T) \end{bmatrix}}_{X_1} \\
= \;& A\underbrace{\begin{bmatrix} Z(x(0)) & Z(x(1)) & \ldots & Z(x(T-1)) \end{bmatrix}}_{Z_0} + B\underbrace{\begin{bmatrix} u(0) & u(1) & \ldots & u(T-1) \end{bmatrix}}_{U_0} \\
&+ E\underbrace{\begin{bmatrix} d(0) & d(1) & \ldots & d(T-1) \end{bmatrix}}_{D_0}
\end{aligned}$$

$$X_1 = AZ_0 + BU_0 + ED_0$$

Data-dependent representations of closed-loop nonlinear systems

In Lectures 1 and 2, the starting point was a data-dependent representation of the closed-loop system $x^+ = (A + BK)x + Ed$, derived under the key condition

$$\begin{bmatrix} K \\ I_n \end{bmatrix} = \begin{bmatrix} U_0 \\ X_0 \end{bmatrix} G$$

Analogously, here we are interested in a data-dependent representation of the closed-loop nonlinear system

$$x^+ = (A + BK)Z(x) + Ed \text{ where } Z(x) = \begin{bmatrix} x \\ Q(x) \end{bmatrix}$$

Updated key condition Consider any matrices $K \in \mathbb{R}^{m \times s}$, $G \in \mathbb{R}^{T \times s}$ such that

$$\begin{bmatrix} K \\ I_s \end{bmatrix} = \begin{bmatrix} U_0 \\ Z_0 \end{bmatrix} G$$

where

$$Z_0 = \begin{bmatrix} Z(x(0)) & \dots & Z(x(T-1)) \end{bmatrix} \quad X_1 = AZ_0 + BU_0 + ED_0$$

The matrix $A+BK$ of the closed-loop system $x^+ = (A+BK)Z(x) + Ed$ is arranged as

$$\begin{aligned} & A+BK \\ = & \begin{bmatrix} B & A \end{bmatrix} \begin{bmatrix} K \\ I_s \end{bmatrix} \\ \overset{\begin{bmatrix} K \\ I_s \end{bmatrix} = \begin{bmatrix} U_0 \\ Z_0 \end{bmatrix} G}{=} & \begin{bmatrix} B & A \end{bmatrix} \begin{bmatrix} U_0 \\ Z_0 \end{bmatrix} G \\ \overset{X_1 = \begin{bmatrix} B & A \end{bmatrix} \begin{bmatrix} U_0 \\ Z_0 \end{bmatrix} + ED_0}{=} & (X_1 - ED_0)G \end{aligned}$$

We will need to differentiate between the linear and the nonlinear part of $x^+ = (X_1 - ED_0)GZ(x) + Ed$. Partition G as

$$G = \begin{bmatrix} \underset{n}{G_1} & \underset{s-n}{G_2} \end{bmatrix}^T$$

n dimension of x, $s-n$ dimension of $Q(x)$. Then

$$A + BK = \begin{bmatrix} (X_1 - ED_0)G_1 & (X_1 - ED_0)G_2 \end{bmatrix}$$

Consider any matrices $K \in \mathbb{R}^{m\times s}$, $G \in \mathbb{R}^{T\times s}$ such that

$$\begin{bmatrix} K \\ I_s \end{bmatrix} = \begin{bmatrix} U_0 \\ Z_0 \end{bmatrix} G$$

Partition G as

$$G = \begin{bmatrix} \underset{n}{G_1} & \underset{s-n}{G_2} \end{bmatrix} T$$

n dimension of x, $s-n$ dimension of $Q(x)$

The closed-loop system $x^+ = (A + BK)Z(x) + Ed$, where $Z(x) = \begin{bmatrix} x \\ Q(x) \end{bmatrix}$, results in the data-dependent representation

$$x^+ = \quad (X_1 - ED_0)G_1 x + (X_1 - ED_0)G_2 Q(x) + Ed$$

- ▷ The use of condition $\left[\begin{smallmatrix} K \\ I_s \end{smallmatrix}\right] = \left[\begin{smallmatrix} U_0 \\ Z_0 \end{smallmatrix}\right] G$ instead of $\left[\begin{smallmatrix} K \\ I_n \end{smallmatrix}\right] = \left[\begin{smallmatrix} U_0 \\ X_0 \end{smallmatrix}\right] G$ leads to an "exact" representation without any unknown remainder.
- ▷ The representation depends on the data U_0, Z_0, X_1 and design variables G_1, G_2
- ▷ As in the case of linear (or "linearized") systems, the disturbance d affecting the dataset causes a perturbation D_0 to appear in the dynamics.

Consider any matrices $K \in \mathbb{R}^{m\times s}$, $G \in \mathbb{R}^{T\times s}$ such that

$$\begin{bmatrix} K \\ I_S \end{bmatrix} = \begin{bmatrix} U_0 \\ Z_0 \end{bmatrix} G$$

Partition G as

$$G = \left[\begin{array}{cc} G_1 & G_2 \end{array} \right] \; T$$
$$\qquad\quad n \quad s-n$$

n dimension of x, $s-n$ dimension of $Q(x)$

The closed-loop system $x^+ = (A+BK)Z(x) + Ed$, where $Z(x) = \begin{bmatrix} x \\ Q(x) \end{bmatrix}$, results in the data-dependent representation

$$x^+ = \quad (X_1 - ED_0)G_1 x + (X_1 - ED_0)G_2 Q(x) + Ed$$

- As $D_0 = \begin{bmatrix} d(0) \dots d(T-1) \end{bmatrix}$ and $|d| \leq \delta$, then

$$D_0 \in \mathcal{D} := \{D \in \mathbb{R}^{q\times T} \colon DD^\top \preceq \Delta\Delta^\top \text{ with } \Delta \text{ known}\}$$

Since $|d| \leq \delta \Leftrightarrow dd^\top \preceq \delta^2 I_q$, we have that $D_0 D_0^\top = \sum_{k=0}^{T-1} d(k)d(k)^\top \preceq \delta^2 T I_q$ $\Delta = \delta\sqrt{T} I_q$

Control design strategy

To design a controller $u = KZ(x)$, we look for $G = \begin{bmatrix} G_1 & G_2 \end{bmatrix}$ such that

$$\begin{bmatrix} K \\ I_S \end{bmatrix} = \begin{bmatrix} U_0 \\ Z_0 \end{bmatrix} G$$

and makes the origin an asymptotically stable equilibrium for the closed-loop dynamics

$$x^+ = (X_1 - ED_0)G_1 x + (X_1 - ED_0)G_2 Q(x) + Ed,$$

(1) by stabilizing the linear part via G_1
(2) by minimizing the impact of the nonlinearities via G_2

Since the nonlinear term $Q(x)$ is generic (i.e., it does not belong to a class of systems that favours control design, such as polynomial systems) and $(X_1 - ED_0)$ has little structure, the strategy above is quite natural (we will see another one in Lecture 5).

We first examine the noiseless case ($D_0 = 0$, $d = 0$)

Control design from noise-free data – recap

A dataset

$$\mathbb{D} = \{u(k), x(k)\}_{k=0}^{T}$$

is obtained from off-line experiments conducted on the system

$$x^+ = AZ(x) + Bu$$

and data are organized into matrices U_0, X_0, X_1, Z_0 that satisfy

$$X_1 = AZ_0 + BU_0$$

To design a controller $u = KZ(x)$, we look for $G = \begin{bmatrix} G_1 & G_2 \end{bmatrix}$ such that

$$\begin{bmatrix} K \\ I_S \end{bmatrix} = \begin{bmatrix} U_0 \\ Z_0 \end{bmatrix} G$$

and makes the origin an asymptotically stable equilibrium for

$$x^+ = X_1 G_1 x + X_1 G_2 Q(x)$$

A formula for data-driven nonlinear control design

Theorem Consider the decision variables $P \in \mathbb{R}^{n\times n}$, $Y_1 \in \mathbb{R}^{T\times n}$, $G_2 \in \mathbb{R}^{T\times n}$ and the following SDP

$$\begin{aligned} \text{minimize}_{P,Y_1,G_2} \quad & \|X_1 G_2\| \\ \text{subject to} \quad & Z_0 Y_1 = \begin{bmatrix} P \\ 0_{(s-n)\times n} \end{bmatrix} & \text{(1a)} \\ & \begin{bmatrix} -P & Y_1^\top X_1^\top \\ X_1 Y_1 & -P \end{bmatrix} \prec 0 & \text{(1b)} \\ & Z_0 G_2 = \begin{bmatrix} 0_{n\times(s-n)} \\ I_{s-n} \end{bmatrix} & \text{(1c)} \end{aligned}$$

(i) If the SDP is feasible and achieves zero cost ($\|X_1 G_2\| = 0$), then $u = KZ(x)$ with

$$K = U_0 \begin{bmatrix} Y_1 P^{-1} & G_2 \end{bmatrix}$$

linearizes the closed-loop system and renders the origin a globally asymptotically stable equilibrium.

(ii) Assume that $\lim_{|x|\to 0} \frac{|Q(x)|}{|x|} = 0$. If the SDP is feasible, then $u = KZ(x)$ renders the origin a locally asymptotically stable equilibrium.

Constraint (1a) can be equivalently written as

$$\text{(1a) } Z_0 Y_1 = \begin{bmatrix} P \\ 0_{(s-n)\times n} \end{bmatrix} \Leftrightarrow Z_0 Y_1 P^{-1} = \begin{bmatrix} I_n \\ 0_{(s-n)\times n} \end{bmatrix}, \quad \text{(1c) } Z_0 G_2 = \begin{bmatrix} 0_{n\times(s-n)} \\ I_{s-n} \end{bmatrix}$$

Perform the change of variable $G_1 := Y_1 P^{-1}$, to obtain $Z_0 \begin{bmatrix} G_1 & G_2 \end{bmatrix} = I_s$.

By the same change of variable, the control gain $K = U_0 \begin{bmatrix} Y_1 P^{-1} & G_2 \end{bmatrix}$ can be written as $K = U_0 \begin{bmatrix} G_1 & G_2 \end{bmatrix}$

Hence, $\begin{bmatrix} K \\ I_s \end{bmatrix} = \begin{bmatrix} U_0 \\ Z_0 \end{bmatrix} G$. This returns the data-dependent representation of the closed-loop system, $x^+ = X_1 G_1 x + X_1 G_2 Q(x)$

Constraint (1b) $\begin{bmatrix} -P & Y_1^\top X_1^\top \\ X_1 Y_1 & -P \end{bmatrix} \prec 0$ implies Schur stability of $X_1 G_1$ with Lyapunov function $V(x) := x^\top P^{-1} x$, i.e. $P \succ 0, \quad (X_1 G_1)^\top P^{-1} X_1 G_1 - P^{-1} \prec 0$

If (i) holds, hence $X_1 G_2 = 0$, then $x^+ = X_1 G_1 x$ (exact linearization) and global asymptotic stability descends from $X_1 G_1$ being Schur.

If (ii) holds, then $x^+ = X_1 G_1 x + X_1 G_2 Q(x)$, where $X_1 G_1$ is Schur and $\lim_{|x|\to 0} |Q(x)|/|x| = 0$. Then local asymptotic stability descends from the stability principle by the first approximation.

Example 1

Consider the system $x^+ = AZ(x) + Bu$, with $x, u \in \mathbb{R}$, $Z(x) = \left[\begin{smallmatrix} x \\ x^2 \end{smallmatrix}\right]$, and the dataset $\mathbb{D} = \{u(0), x(0), x(1)\} \cup \{u(1), x(1), x(2)\}$ $(T = 2)$, obtained in 2 experiments, where

$$x(0) = 1, u(0) = -1, x(1) = 0 \text{ and } x(1) = -1, u(1) = -1, x(2) = 0$$

In this case,

$$Z_0 = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}, \quad X_1 = \begin{bmatrix} 0 & 0 \end{bmatrix}, \quad U_0 = \begin{bmatrix} -1 & -1 \end{bmatrix}$$

The SDP can be written as

$$\min_{P, Y_1, G_2} \| \begin{bmatrix} 0 & 0 \end{bmatrix} G_2 \| \quad \text{subject to} \quad \begin{bmatrix} Y_1 & G_2 \end{bmatrix} = Z_0^{-1} \begin{bmatrix} P & 0 \\ 0 & 1 \end{bmatrix}, \quad P > 0 \left(\Leftrightarrow \begin{bmatrix} -P & Y_1^\top X_1^\top \\ X_1 Y_1 & -P \end{bmatrix} \prec 0 \right)$$

The constraints return the feasible solutions

$$\begin{bmatrix} Y_1 & G_2 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} P & 0 \\ 0 & 1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} P & 1 \\ -P & 1 \end{bmatrix}, \quad P > 0$$

and all of them attain $X_1 G_2 = 0$. Since $K = U_0 \left[\begin{smallmatrix} Y_1 P^{-1} & G_2 \end{smallmatrix}\right] = \left[\begin{smallmatrix} -1 & -1 \end{smallmatrix}\right] \frac{1}{2} \left[\begin{smallmatrix} 1 & 1 \\ -1 & 1 \end{smallmatrix}\right] = \left[\begin{smallmatrix} 0 & -1 \end{smallmatrix}\right]$, the resulting control law is $u = KZ(x) = -x^2$ and is guaranteed to globally asymptotically stabilize the origin.

By construction, $K = \begin{bmatrix} 0 & -1 \end{bmatrix}$ makes the origin a globally asymptotically stable equilibrium for the closed-loop system $x^+ = (A + BK)Z(x)$, for all A, B that satisfy $X_1 = AZ_0 + BU_0$

The constraint $X_1 = AZ_0 + BU_0$ written explicitly as

$$Z_0 = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 \end{bmatrix} = A \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} + B \begin{bmatrix} -1 & -1 \end{bmatrix}$$

returns

$$A = -B \begin{bmatrix} -1 & -1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}^{-1} = -B \begin{bmatrix} -1 & -1 \end{bmatrix} \frac{1}{2} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} = -B \begin{bmatrix} 0 & -1 \end{bmatrix}$$

The set of all systems consistent with the dataset is

$$x^+ = B \begin{bmatrix} 0 & 1 \end{bmatrix} Z(x) + Bu$$

By replacing $u = KZ(x) = \begin{bmatrix} 0 & -1 \end{bmatrix} Z(x)$ in $x^+ = B \begin{bmatrix} 0 & 1 \end{bmatrix} Z(x) + Bu$, we obtain $x^+ = 0$, for which $x = 0$ is indeed globally asymptotically stable.

Example 2

Inverted pendulum Euler discretization

$$\begin{cases} x_1^+ = & x_1 + T_s x_2 \\ x_2^+ = & \dfrac{T_s g}{\ell} \sin x_1 + \left(1 - \dfrac{T_s \mu}{m\ell^2}\right) x_2 + \dfrac{T_s}{m\ell^2} u \end{cases}$$

Here $Z(x) = \begin{bmatrix} x_1 & x_2 & \sin x_1 \end{bmatrix}^\top$, $T_s = 0.1, m = 1, \ell = 1, \mu = 0.01$

Experiment

$x_0 \in [-0.5, 0.5]^2$

$u \in [-0.5, 0.5]$

$T = 10$

Measured data are collected in X_1, Z_0, U_0 and used to solve the SDP, which returns the solution

$$K = \begin{bmatrix} -23.5644 & -10.3904 & -9.8 \end{bmatrix}$$

$$P = \begin{bmatrix} 5.5115 & -11.3604 \\ -11.3604 & 75.8512 \end{bmatrix}$$

As prescribed by the SDP, the control law learns from the data how to cancel out the nonlinear term $\frac{T_s g}{\ell} \sin x_1$

Code

```
clear all
close all
rng(1);

%% GENERATING THE DATASET

n = 2;  % system dimensions
m = 1;

T = 10; % number of samples

Ts = 0.1; mas=1; ell=1; g=9.8; mu=0.01; % system parameters

mag = 0.5;
U0  = (2*mag).*rand(m,T)-mag;      % experiment input
x   = (2*mag).*rand(n,1)-mag;      % experiment initial state
```

```
X0     = [];  % initialization data matrices
X1     = [];
Q0     = [];

X  = []; % auxiliary data matrices
Q  = [];

X   = x;
Q   = sin(x(1)); % Q(x)=sin(x1)

for i=1:T % system input/state response
    upd_1 = x(1) + Ts*x(2);
    upd_2 = x(2) + Ts*(g/ell)*sin(x(1)) - Ts*mu/(mas*ell^2)*x(2) +...
                    Ts/(mas*ell^2)*U0(:,i);
    x = [upd_1; upd_2];
    X   = [X x];
    Q   = [Q sin(x(1))];
end
```

```
X0   = X(:,1:end-1);
X1   = X(:,2:end);
Q0   = Q(:,1:end-1);
Z0 = [ X0; Q0];
[s col] = size(Z0); % s number of functions in Z(x)

%% Controller design
cvx_begin sdp
    variable P(n,n) symmetric
    variable Y1(T,n)
    variable G2(T,s-n)
    Z0*Y1 == [P; zeros(s-n,n)];
    Z0*G2 == [zeros(n,s-n); eye(s-n)];
    [P-eye(n), X1*Y1; Y1'*X1', P] >= 0 % nonstrict LMI implementation
    minimize ( norm(X1*G2) )
cvx_end

G1 = Y1/P;  G = [G1 G2]; K = U0*G;
```

Estimate of the region of attraction (RoA)

In the case $\|X_1 G_2\| \neq 0$, the result gives a controller that renders the origin a locally asymptotically stable equilibrium for the closed-loop system.

It also allows us to provide estimates of the (RoA) of the closed-loop system

$$x^+ = (A + BK)Z(x) = X_1 G_1 x + X_1 G_2$$

Lyapunov difference along the solutions of the closed-loop system Recall that $V(x) := x^\top P^{-1} x$. Then

$$\begin{aligned} dV(x) &:= V(x^+) - V(x) = (X_1 G_1 x + X_1 G_2 Q(x))^\top P^{-1} (X_1 G_1 x + X_1 G_2 Q(x)) - x^\top P^{-1} x \\ &= x^\top (G_1^\top X_1^\top P^{-1} X_1 G_1 - P^{-1}) x + 2 Q(x)^\top G_2^\top X_1^\top P^{-1} X_1 G_1 x + Q(x)^\top G_2^\top X_1^\top P^{-1} X_1 G_2 Q(x) \end{aligned}$$

The set

$$\mathcal{L} := \{x : dV(x) < 0\} \neq \emptyset$$

and

any Lyapunov sub-level set $\mathcal{R}_\gamma := \{x : V(x) \leq \gamma\}$ of V contained in $\mathcal{L} \cup \{0\}$ is an estimate of the RoA of the closed-loop system

Both $\mathcal{L}$ and $\mathcal{R}_\gamma$ are known from data and the solution of the SDP, hence the RoA is computable.

Example 3

Consider the nonlinear system

$$\begin{aligned} x_1^+ &= x_2 + x_1^3 + u \\ x_2^+ &= 0.5x_1 + 0.2x_2^2 \end{aligned}$$

Choose

$$Z(x) = \begin{bmatrix} x^\top & x_1^2 & x_2^2 & x_1x_2 & x_1^3 & x_2^3 & x_1x_2^2 & x_1^2x_2 \end{bmatrix}^\top$$

The SDP is feasible and returns the control law $u = KZ(x)$, where

$$K = \begin{bmatrix} \underbrace{-0.0113}_{x_1} & \underbrace{-1.0862}_{x_2} & \underbrace{0.0005}_{x_1^2} & \underbrace{0}_{x_2^2} & \underbrace{0.0039}_{x_1x_2} & \underbrace{-1.0010}_{x_1^3} & \underbrace{-0.0130}_{x_2^3} & \underbrace{0.0119}_{x_1x_2^2} & \underbrace{-0.0010}_{x_1^2x_2} \end{bmatrix}.$$

Then the closed-loop system is $x^+ = (A + BK)Z(x) = X_1G_1x + X_1G_2Q(x)$, where

$$X_1G_1 = \begin{bmatrix} -0.0113 & -0.0862 \\ 0.5000 & 0 \end{bmatrix}, \quad X_1G_2 = \begin{bmatrix} 0.0005 & 0 & 0.0039 & -0.0010 & -0.0130 & 0.0119 & -0.0010 \\ 0 & 0.2000 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The minimum cost attained by the SDP is $\|X_1G_2\| = 0.2$, achieved thanks to $u = KZ(x)$ that almost cancels the term x_1^3. This value cannot be further reduced due to the unmatched nonlinearity $0.2x_2^2$ in the second equation (exact nonlinearity cancellation is not possible).

As $\|X_1 G_2\| \neq 0$, the stability result is local, and we estimate the RoA.

The SDP returns the matrix P that allows us to determine the Lyapunov function

$$V(x) = x^\top P^{-1} x.$$

We can numerically compute (grey area)

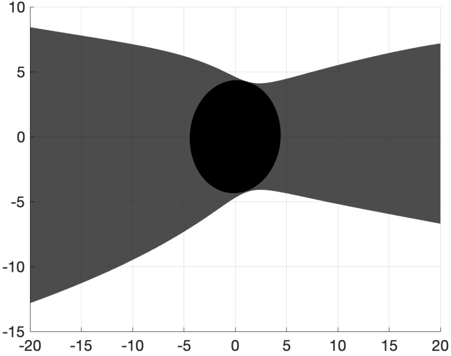
$$\mathcal{L} := \{x : dV(x) < 0\},$$

where

$$dV(x) = (X_1 G_1 x + X_1 G_2 Q(x))^\top P^{-1} \cdot \\ \cdot (X_1 G_1 x + X_1 G_2 Q(x)) - x^\top P^{-1} x.$$

Then we determine the largest value $\gamma > 0$ such that the Lyapunov sub-level set $\mathcal{R}_\gamma = \{x : V(x) \leq \gamma\}$ (black area) satisfies

$$\mathcal{R}_\gamma \subseteq (\mathcal{L} \cup \{0\})$$



On the condition $\lim_{|x|\to 0} \dfrac{|Q(x)|}{|x|} = 0$

In the case $\|X_1 G_2\| \neq 0$, the local stabilization result under the assumption that

$$\lim_{|x|\to 0} \frac{|Q(x)|}{|x|} = 0$$

Suppose that the domain knowledge about $f(x)$ suggests to adopt a $Q(x)$ that does not satisfy such an assumption. We can reduce this case to the previous one provided that $\underline{Q(0) = 0 \text{ and } Q(x) \text{ is continuously differentiable}}$.

By Taylor's theorem

$$Q(x) = \frac{\partial Q}{\partial x}(0)x + r_Q(x), \text{ where } \lim_{|x|\to 0} \frac{|r_Q(x)|}{|x|} = 0.$$

Then

$$AZ(x) = \overline{A}x + \widehat{A}Q(x) = \left(\overline{A} + \widehat{A}\frac{\partial Q}{\partial x}(0)\right)x + \widehat{A}r_Q(x)$$

Hence, if $Q(x)$ does not satisfy $\lim_{|x|\to 0} \frac{|Q(x)|}{|x|} = 0$, it is sufficient to replace it with $r_Q(x) = Q(x) - \dfrac{\partial Q}{\partial x}(0)x$

A remark on the case of continuous-time systems

Continuous-time systems As for the case of linear systems, the result carries over to continuous-time nonlinear systems $\dot{x} = AZ(x) + Bu$.

Input and state sampled trajectories Given a sequence of sampling times $0 \le t_0 < t_1 < \ldots < t_{T-1}$, let

$$\begin{aligned} U_0 &= \begin{bmatrix} u_d(t_0) & u_d(t_1) & \ldots & u_d(t_{T-1}) \end{bmatrix} \\ Z_0 &= \begin{bmatrix} Z(x_d(t_0)) & Z(x_d(t_1)) & \ldots & Z(x_d(t_{T-1})) \end{bmatrix} \\ X_1 &= \begin{bmatrix} \dot{x}_d(t_0) & \dot{x}_d(t_1) & \ldots & \dot{x}_d(t_{T-1}) \end{bmatrix} \end{aligned}$$

For continuous-time systems the SDP becomes

$$\begin{aligned} \text{minimize}_{P,Y_1,G_2} \quad & \|X_1 G_2\| \\ \text{subject to} \quad & Z_0 Y_1 = \begin{bmatrix} P \\ 0_{(s-n)\times n} \end{bmatrix} \\ & X_1 Y_1 + Y_1^\top X_1^\top \prec 0 \\ & Z_0 G_2 = \begin{bmatrix} 0_{n\times(s-n)} \\ I_{s-n} \end{bmatrix} \end{aligned}$$

The case of continuous-time systems

If the SDP is feasible and attains zero cost ($\|X_1 G_2\| = 0$), then $u = KZ(x)$ with $K = U_0 \left[\, Y_1 P^{-1} \;\; G_2 \,\right]$ linearizes the systems and renders the origin a globally asymptotically equilibrium.

If $\lim_{|x|\to 0} \frac{|Q(x)|}{|x|} = 0$ and the SDP is feasible, then $u = KZ(x)$ renders the origin a locally asymptotically stable equilibrium.

Derivatives As for the case of linear systems, the use of state derivatives can be avoided considering the integral version of $\dot{x} = AZ(x) + Bu$

$$\overbrace{x(t_{k+1}) - x(t_k)}^{\xi(k)} = A \overbrace{\int_{t_k}^{t_{k+1}} Z(x(t))dt}^{r(k)} + B \overbrace{\int_{t_k}^{t_{k+1}} u(t)dt}^{v(k)}$$

and working with the data matrices identity

$$\overbrace{\left[\xi(0) \ldots \xi(T-1)\right]}^{X_1} = A \overbrace{\left[r(0) \ldots r(T-1)\right]}^{Z_0} + B \overbrace{\left[v(0) \ldots v(T-1)\right]}^{U_0}$$

Feedback linearization

“Cancelling nonlinearities” in

$$x^+ = f(x,u)$$

is classically enabled by <u>normal forms</u> revealed by coordinate transformations.

Let f be continuously differentiable ($\mathcal{C}^1$) and let $x^0 = f(x^0, u^0)$. Here, $(x^0, u^0) = (0, 0)$.

<u>Exact feedback linearization</u> The system is (locally) feedback linearizable at (x^0, u^0) if there exist
(i) a $\mathcal{C}^1$ change of coordinates $w = \Phi(x)$ such that $0 = \Phi(x^0)$ and $\text{rank}\frac{\partial \Phi}{\partial x}(x^0) = n$;
(ii) a $\mathcal{C}^1$ feedback $u = \gamma(x, v)$ such that $\gamma(x^0, 0) = u^0$ and $\text{rank}\frac{\partial \gamma}{\partial v}(x^0, 0) = m$
for which the closed-loop system in the coordinates w takes the form

$$w^+ = \Phi(f(x, \gamma(x, v)))|_{x=\Phi^{-1}(w)} = Aw + Bv$$

with (A, B) a controllable pair.

If $\Phi(x)$, $\gamma(x, v)$ are found such that

$$w^+ = \Phi(f(x, \gamma(x, v)))|_{x=\Phi^{-1}(w)} = Aw + Bv$$

then the powerful design tools available for linear control systems can be used to design nonlinear control laws.

Suppose the system

$$x^+ = f(x, u)$$

is feedback linearizable at (x^0, u^0) and we want to design a feedback law that renders x^0 an asymptotically stable equilibrium for the closed-loop system.

The feedback law

$$u = \gamma(x, K\Phi(x)),$$

where K is a matrix that renders $A + BK$ a Schur stable matrix, makes x^0 an asymptotically stable equilibrium of

$$x^+ = f(x, \gamma(x, K\Phi(x)))$$

This is because, in the coordinates $w = \Phi(x)$, the system above is $w^+ = (A + BK)w$ whose solution is $w(k) = (A + BK)^k w(0)$. Hence, $x(k) = \Phi^{-1}(w(k)) = \Phi^{-1}((A + BK)^k \Phi(x(0)))$, from which stability and attractivity can be shown.

For continuous-time input affine systems $\dot{x} = f(x) + g(x)u$, the feedback linearization problem is solvable if and only if there exists an "output" function $y = h(x)$ with respect to which the system has relative degree n at x^0.

For discrete-time systems $x^+ = f(x, u)$, we take a similar approach and we assume that an "output" function $h(x)$, $h \in \mathcal{C}^1$, $h(0) = 0$, is available with respect to which the system has a relative degree n.

We focus on single input systems and scalar output functions $h\colon \mathbb{R}^n \to \mathbb{R}$.

The system

$$x^+ = f(x, u), \quad y = h(x)$$

has relative degree n if

$$\frac{\partial h \circ f}{\partial u}(x, u) = \frac{\partial h \circ f_0 \circ f}{\partial u}(x, u) = \ldots = \frac{\partial h \circ f_0^{n-2} \circ f}{\partial u}(x, u) = 0,$$
$$\frac{\partial h \circ f_0^{n-1} \circ f}{\partial u}(x, u) \neq 0, \quad \forall (x, u) \in \mathbb{R}^{n+1}$$

where $f_0(x) = f(x, 0)$, $f_0^d = \underbrace{f_0 \circ f_0 \circ \ldots \circ f_0}_{d \text{ times}}$.

S. Monaco, D. Normand-Cyrot. "Minimum-phase nonlinear discrete-time systems and feedback stabilization." 28th IEEE CDC, 979–986, 1987.

The function

$$\Phi(x) := \begin{bmatrix} h(x) \\ h \circ f_0(x) \\ \vdots \\ h \circ f_0^{n-1}(x) \end{bmatrix}$$

is a global $\mathcal{C}^1$ change of coordinates, that is, $0 = \Phi(x^0)$, $\text{rank}\frac{\partial \Phi}{\partial x} = n$ for all x and $\lim_{\|x\| \to \infty} \|\Phi(x)\| = \infty$;

The function $\beta(x, u) := h \circ f_0^{n-1} \circ f(x, u)$ is globally invertible wrt u, i.e., there exists $\gamma(x, v)$ such that $\beta(x, \gamma(x, v)) = v$ for all $(x, v) \in \mathbb{R}^n \times \mathbb{R}^m$.

To avoid keeping track of the domain where the feedback linearization holds, we consider the global feedback linearization problem.

If we set

$$w(k) := \begin{bmatrix} y(k) \\ y(k+1) \\ \vdots \\ y(k+n-1) \end{bmatrix}$$

we obtain that

$$w(k) = \Phi(x(k)) = \begin{bmatrix} h(x(k)) \\ h \circ f_0(x(k)) \\ \vdots \\ h \circ f_0^{n-1}(x(k)) \end{bmatrix}$$

By Taylor's theorem

$$\begin{aligned} y(k+1) = &\quad h(x(k+1)) = h(f(x(k), u(k)) \\ = &\quad h(f(x(k), 0)) + \left[\tfrac{\partial h(f(x(k), v))}{\partial v}\right]_{v=\alpha u(k), \alpha \in (0,1)} u(k) = h(f(x(k), 0)) = h \circ f_0(x(k)) \end{aligned}$$

$$\begin{aligned} y(k+2) = &\quad h(x(k+2)) = h(f(x(k+1), u(k+1)) = h \circ f_0(x(k+1)) = h \circ f_0(f(x(k), u(k))) \\ = &\quad h \circ f_0(f(x(k), 0)) + \left[\tfrac{\partial h \circ f_0(f(x(k), 0))}{\partial v}\right]_{v=\alpha u(k), \alpha \in (0,1)} u(k) \\ = &\quad h \circ f_0(f(x(k), 0)) = h \circ f_0^2(x(k)) \\ &\vdots \end{aligned}$$

In the new coordinates $w(k) = \begin{bmatrix} y(k) \\ y(k+1) \\ \vdots \\ y(k+n-1) \end{bmatrix} = \begin{bmatrix} h(x(k)) \\ h \circ f_0(x(k)) \\ \vdots \\ h \circ f_0^{n-1}(x(k)) \end{bmatrix}$, the system is written as

$$w(k+1) = \begin{bmatrix} y(k+1) \\ y(k+2) \\ \vdots \\ y(k+n-1) \\ y(k+n) \end{bmatrix} = \begin{bmatrix} w_2(k) \\ w_3(k) \\ \vdots \\ w_n(k) \\ h \circ f_0^{n-1} \circ f(x(k), u(k)) \end{bmatrix}, \quad y(k) = w_1(k)$$

In matrix form

$$w(k+1) = \underbrace{\begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix}}_{A_c} w(k) + \underbrace{\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}}_{B_c} \underbrace{h \circ f_0^{n-1} \circ f(x(k), u(k))}_{\beta(x(k),u(k))}, \quad y(k) = w_1(k)$$

In the new coordinates, the system $x(k+1) = f(x(k), u(k))$ becomes

$$w(k+1) = \underbrace{\begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix}}_{A_c} w(k) + \underbrace{\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}}_{B_c} \underbrace{h \circ f_0^{n-1} \circ f(x(k), u(k))}_{\beta(x(k), u(k))}$$

<u>If the model were known</u>, then the feedback law

$$u(k) = \gamma(x(k), v(k))$$

would return

$$w_n(k+1) = \beta(x(k), \gamma(x(k), v(k))) = v(k),$$

i.e., in the coordinates $w = \Phi(x)$, the closed-loop system

$$x(k+1) = f(x(k), \gamma(x(k), v(k)))$$

becomes

$$w(k+1) = A_c w(k) + B_c v(k),$$

where the pair (A_c, B_c) is reachable. In other words, the feedback $u = \gamma(x, v)$ "cancels the nonlinearity" in the new coordinates w (feedback linearization).

If the dynamical model $x(k+1) = f(x(k), u(k))$ is unknown, then in the system

$$w(k+1) = \underbrace{\begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix}}_{A_c} w(k) + \underbrace{\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}}_{B_c} \underbrace{h \circ f_0^{n-1} \circ f(x(k), u(k))}_{\beta(x(k), u(k))}, \quad y(k) = w_1(k)$$

$w_2(k) = h \circ f_0(x(k)), w_3(k) = h \circ f_0^2(x(k)), w_n(k) = h \circ f_0^{n-1}(x(k))$ as well as $h \circ f_0^{n-1} \circ f(x(k), u(k))$ are unknown.

In the spirit of this lecture, we assume that

$$h \circ f_0^{n-1} \circ f(x, u) = a^\top Z(x) + bu$$

for some unknown $a \in \mathbb{R}^s, b \in \mathbb{R} \setminus \{0\}$ and $Z(x) = \begin{bmatrix} h(x) \\ x \\ Q(x) \end{bmatrix}$.

We choose $Z(x) = \begin{bmatrix} h(x) \\ x \\ Q(x) \end{bmatrix}$ instead of $Z(x) = \begin{bmatrix} x \\ Q(x) \end{bmatrix}$ because, as before, we will be using $u = KZ(x)$ and the linearizing feedback must depend on $h(x)$. In what follows, to avoid notational confusion with the previous analysis, we will use $Z(x)$ for the vector of functions $\begin{bmatrix} h(x) \\ x \\ Q(x) \end{bmatrix}$.

Dataset

We collect $\mathbb{D} := \{(x(k), u(k))\}_{k=0}^{n+T-1}$. As $y = h(x)$ is known, we compute

$$w(k) = \begin{bmatrix} y(k) \\ \vdots \\ y(k+n-1) \end{bmatrix} = \begin{bmatrix} h(x(k)) \\ \vdots \\ h(x(k+n-1)) \end{bmatrix}, \quad k = 0, 1, \dots T$$

In addition to the matrices of data U_0, Z_0 defined as before, we introduce

$$\begin{aligned} W_0 &:= \begin{bmatrix} w(0) & w(1) & \cdots & w(T-1) \end{bmatrix} \in \mathbb{R}^{n \times T}, \\ W_1 &:= \begin{bmatrix} w(1) & w(2) & \cdots & w(T) \end{bmatrix} \in \mathbb{R}^{n \times T}, \end{aligned}$$

Since

$$w(k+1) = A_c w(k) + B_c(a^\top Z(x(k)) + bu(k)),$$

these matrices satisfy the identity

$$W_1 = A_c W_0 + B_c(a^\top Z_0 + bU_0)$$

A formula for data-driven feedback linearization

Theorem Consider the decision variables

$$G_1 \in \mathbb{R}^{T\times 1}, \quad G_2 \in \mathbb{R}^{T\times (s-1)}, \quad k_1 \in \mathbb{R}$$

and the following SDP

$$\begin{aligned}
\text{minimize}_{G_1,G_2,k_1} \quad & \|(W_1 - A_c W_0)G_2\| && \text{(2a)}\\
\text{subject to} \quad & Z_0 G_1 = \begin{bmatrix} 1 \\ 0_{(s-1)\times 1} \end{bmatrix}, && \text{(2b)}\\
& (W_1 - A_c W_0)G_1 = B_c k_1 \,, && \text{(2c)}\\
& k_1 \in (-1, 1)\,, && \text{(2d)}\\
& Z_0 G_2 = \begin{bmatrix} 0_{1\times(s-1)} \\ I_{s-1} \end{bmatrix}. && \text{(2e)}
\end{aligned}$$

If the SDP is feasible and achieves zero cost, i.e. $\|(W_1 - A_c W_0)G_2\| = 0$, then $u = KZ(x)$, with

$$K = U_0 G,$$

linearizes the closed-loop system $x^+ = f(x, KZ(x))$, with $Z(x) = \begin{bmatrix} h(x) \\ x \\ Q(x) \end{bmatrix}$, and renders the origin a globally asymptotically stable equilibrium.

The feasibility of the SDP and the zero cost guarantee

$$
\begin{array}{l}
\|(W_1 - A_c W_0) G_2\| = 0 \\
Z_0 G = I_s \\
(W_1 - A_c W_0) G_1 = B_c k_1 \,, \\
k_1 \in (-1, 1)
\end{array}
$$

The closed-loop system can be manipulated as

$$
\begin{aligned}
w^+ &= A_c w + B_c(a^\top Z(x) + bKZ(x)) \\
&\overset{K=U_0G}{=} A_c w + B_c a^\top Z(x) + B_c b U_0 G Z(x) \\
&\overset{W_1=A_cW_0+B_c(a^\top Z_0+bU_0)}{=} A_c w + B_c a^\top Z(x) + (W_1 - A_c W_0 - B_c a^T Z_0) G Z(x) \\
&\overset{(W_1-A_cW_0)G_2=0}{=} A_c w + B_c a^\top Z(x) + (W_1 - A_c W_0) G_1 h(x) - B_c a^T Z_0 G Z(x) \\
&\overset{Z_0G=I_s}{=} A_c w + (W_1 - A_c W_0) G_1 e_1^\top w \\
&\overset{(W_1-A_cW_0)G_1=B_ck_1}{=} (A_c + B_c k_1 e_1^T) w
\end{aligned}
$$

where $e_1 = \begin{bmatrix} 1 & 0 & \dots & 0 \end{bmatrix}^\top \in \mathbb{R}^n$.

Global asymptotic stability follows since all the eigenvalues of $A_c + B_c k_1 e_1^T$ have magnitude $\sqrt[n]{|k_1|} < 1$

Comments

The controller exactly cancels the nonlinearity even when the nonlinearity is "not matched" by the control input.

It requires to know an output function $h(x)$ such that

$$x^+ = f(x,u), \quad y = h(x)$$

has relative degree n and $h \circ f_0^{n-1} \circ f(x,u) = a^\top Z(x) + bu$.

Because the stability analysis is carried out in the w-coordinates and only $w_1 = h(x)$ is available, to stabilize the linear part via $u = KZ(x)$, we must include $h(x)$ in $Z(x)$. This constrains to assign the eigenvalues of the closed-loop system all in $\sqrt[n]{|k_1|} < 1$.

The SDP includes the strict inequality $k_1 \in (-1, 1)$, which in CVX we approximate with a weak one.

How to relax the priors under which the result is derived and how to solve the exact feedback stabilization problem when using noisy data are open problems. A different "indirect" approach consisting of estimating the change of coordinates and the linearizing feedback law has been recently considered.

DP, Gadginmath, Pasqualetti, Tesi. "Data-driven feedback linearization with complete dictionaries." 62nd IEEE CDC, 3037–3042, 2023.
DP, Gadginmath, Pasqualetti, Tesi. "Feedback linearization through the lens of data." arXiv:2308.11229v2, 2024.

Example 3 (continued)

Consider the nonlinear system

$$\begin{aligned} x_1^+ &= x_2 + x_1^3 + u \\ x_2^+ &= 0.5x_1 + 0.2x_2^2 \end{aligned}$$

Due to the unmatched nonlinearity $0.2x_2^2$ in the second equation, exact nonlinearity cancellation is impossible with the previous approach.

Suppose we are given the output function

$$y = h(x) = x_2$$

Then

$$\begin{aligned} \frac{\partial h \circ f}{\partial u}(x,u) &= \frac{\partial(0.5x_1 + 0.2x_2^2)}{\partial u} = 0 \\ \frac{\partial h \circ f_0 \circ f}{\partial u}(x,u) &= \frac{\partial(0.5(x_2 + x_1^3 + u) + 0.2(0.5x_1 + 0.2x_2^2)^2)}{\partial u} = 0.5 \end{aligned}$$

Hence, the system has relative degree $n = 2$. Moreover,

$$h \circ f_0^{n-1} \circ f(x,u) = 0.5(x_2 + x_1^3 + u) + 0.2(0.5x_1 + 0.2x_2^2)^2)$$

If we choose

$$Z(x) = \begin{bmatrix} h(x) & x_1^2 & x_2^2 & x_1x_2 & x_1^3 & x_2^3 & x_1x_2^2 & x_1^2x_2 & x_1^4 & x_2^4 & x_1x_2^3 & x_1^2x_2^2 & x_1^3x_2 \end{bmatrix}^\top.$$

it holds that $h \circ f_0^{n-1} \circ f(x,u) = a^\top Z(x) + bu$.

We conduct T independent n-long experiments, with $T \geq s$ (Z_0 having full-row rank is necessary for the SDP to be feasible) and s the number of functions in $Z(x)$. As $s = 13$, we let $T = 15$.

For each experiment i, with $i = 1, 2, \ldots, T$, we randomly select $x^{(i)}(0)$ and the input sequence $\{u^{(i)}(0), u^{(i)}(1)\}$, and conduct the experiment for $n = 2$ time steps, so as to obtain

$$W_0^{(i)} = \begin{bmatrix} y^{(i)}(0) \\ y^{(i)}(1) \end{bmatrix}, \quad W_1^{(i)} = \begin{bmatrix} y^{(i)}(1) \\ y^{(i)}(2) \end{bmatrix}, \quad Z_0^{(i)} = Z(x^{(i)}(0)), \quad U_0^{(i)} = u^{(i)}(0)$$

Then we form the matrices

$$\begin{aligned} W_0 &= \begin{bmatrix} W_0^{(1)} & \ldots & W_0^{(T)} \end{bmatrix}, \quad W_1 = \begin{bmatrix} W_1^{(1)} & \ldots & W_1^{(T)} \end{bmatrix}, \\ Z_0 &= \begin{bmatrix} Z_0^{(1)} & \ldots & Z_0^{(T)} \end{bmatrix}, \qquad U_0 = \begin{bmatrix} U_0^{(1)} & \ldots & U_0^{(T)} \end{bmatrix} \end{aligned}$$

and launch the SDP solver.

The SDP is feasible and returns the controller

$$K = \Big[\underbrace{-1}_{x_2}\ \underbrace{-0.1}_{x_1^2}\ \underbrace{0}_{x_2^2}\ \underbrace{0}_{x_1x_2}\ \underbrace{-1}_{x_1^3}\ \underbrace{0}_{x_2^3}\ \underbrace{-0.08}_{x_1x_2^2}\ \underbrace{0}_{x_1^2x_2}\ \underbrace{0}_{x_1^4}\ \underbrace{-0.016}_{x_2^4}\ \underbrace{0}_{x_1x_2^3}\ \underbrace{0}_{x_1^2x_2^2}\ \underbrace{0}_{x_1x_2^3}\Big]$$

To understand the rationale of the controller synthesized by the SDP, let us consider the system in the new coordinates

$$w^+ = A_c w + B_c\Big[0.5(x_2 + x_1^3 + u) + 0.2(0.5x_1 + 0.2x_2^2)^2\Big]$$

where $A_c = \left[\begin{smallmatrix}0 & 1\\ 0 & 0\end{smallmatrix}\right]$, $B_c = \left[\begin{smallmatrix}0\\ 1\end{smallmatrix}\right]$. Replacing $u = KZ(x)$, we have

$$\begin{array}{rl} & 0.5(x_2 + x_1^3 + u)|_{u=KZ(x)} + 0.2(0.5x_1 + 0.2x_2^2)^2 \\ = & 0.5(-0.1x_1^2 - 0.08x_1x_2^2 - 0.016x_2^4) + 0.2(0.5x_1 + 0.2x_2^2)^2 = 0 \end{array}$$

that is

$$w^+ = A_c w.$$

The feasibility of the SDP for feedback linearization

Corollary If $\begin{bmatrix} U_0 \\ Z_0 \end{bmatrix}$ has full row rank, then the SDP

$$\begin{aligned} \text{minimize}_{G_1,G_2,k_1} \quad & \|(W_1 - A_c W_0)G_2\| \\ \text{subject to} \quad & Z_0 G_1 = \begin{bmatrix} 1 \\ 0_{(s-1)\times 1} \end{bmatrix}, \\ & (W_1 - A_c W_0)G_1 = B_c k_1 \,, \\ & k_1 \in (-1,1)\,, \\ & Z_0 G_2 = \begin{bmatrix} 0_{1\times(s-1)} \\ I_{s-1} \end{bmatrix} \end{aligned}$$

is feasible and achieves zero cost ($\|(W_1 - A_c W_0)G_2\| = 0$).

The closed-loop system $w^+ = A_c w + B_c(a^\top + bK)Z(x)$ is feedback linearizable with $K = b^{-1}(-a^\top + \begin{bmatrix} k_1 & 0_{1\times s-1}\end{bmatrix})$, which returns $w^+ = (A_c + B_c \begin{bmatrix} k_1 & 0_{1\times n-1}\end{bmatrix})w$.

If $\left[\begin{smallmatrix} U_0 \\ Z_0 \end{smallmatrix}\right]$ has full row rank, then for such a K there exist G such that $\left[\begin{smallmatrix} K \\ I_s \end{smallmatrix}\right] = \left[\begin{smallmatrix} U_0 \\ Z_0 \end{smallmatrix}\right] G$, that is the constraint $Z_0 G = I_s$ must hold.

It must also hold $K = U_0 G$. Hence, the closed-loop system becomes

$$\begin{aligned} w^+ = \;& A_c w + B_c a^\top Z(x) + bU_0 G Z(x) = A_c w + B_c a^\top Z(x) + (W_1 - A_c W_0 - B_c a^T Z_0) G Z(x) \\ = \;& A_c w + (W_1 - A_c W_0) G Z(x) = A_c w + (W_1 - A_c W_0) G_1 e_1^\top w + (W_1 - A_c W_0) G_2 \left[\begin{smallmatrix} x \\ Q(x) \end{smallmatrix}\right] \end{aligned}$$

which returns the data-dependent expression for the closed-loop system

$$w^+ = (A_c + (W_1 - A_c W_0) G_1 e_1^\top) w + (W_1 - A_c W_0) G_2 \left[\begin{smallmatrix} x \\ Q(x) \end{smallmatrix}\right]$$

Comparing it with

$$w^+ = (A_c + B_c \begin{bmatrix} k_1 & 0_{1\times n-1}\end{bmatrix}) w = (A_c + B_c k_1 e_1^\top) w$$

and after some tedious arguments, we conclude that

$$(W_1 - A_c W_0) G_1 = B_c k_1, \quad (W_1 - A_c W_0) G_2 = 0$$

Approximate nonlinearity cancellation
The case of noisy data

Control design from noisy data – recap so far

A dataset

$$\mathbb{D} = \{u(k), x(k)\}_{k=0}^{T}$$

is obtained from off-line experiments conducted on the system

$$x^+ = AZ(x) + Bu + Ed$$

and data are organized into matrices U_0, X_0, X_1, Z_0 that satisfy

$$X_1 = AZ_0 + BU_0 + ED_0$$

To design a controller $u = KZ(x)$, we look for $G = \begin{bmatrix} G_1 & G_2 \end{bmatrix}$ that satisfies

$$\begin{bmatrix} K \\ I_S \end{bmatrix} = \begin{bmatrix} U_0 \\ Z_0 \end{bmatrix} G$$

and makes the origin an asymptotically stable equilibrium for

$$x^+ = (X_1 - ED_0)G_1 x + (X_1 - ED_0)G_2 Q(x) + Ed$$

D_0 is unknown but satisfies

$$D_0 \in \mathcal{D} = \left\{ D \in \mathbb{R}^{n \times T} : DD^\top \preceq \Delta\Delta^\top \text{ with } \Delta \text{ known} \right\}$$

Control design strategy

To design a controller $u = KZ(x)$, we look for $G = \begin{bmatrix} G_1 & G_2 \end{bmatrix}$ that satisfies

$$\begin{bmatrix} K \\ I_S \end{bmatrix} = \begin{bmatrix} U_0 \\ Z_0 \end{bmatrix} G$$

and makes the origin an asymptotically stable equilibrium for the closed-loop dynamics

$$x^+ = (X_1 - ED_0)G_1 x + (X_1 - ED_0)G_2 Q(x) + Ed,$$

(1) by stabilizing the linear part via G_1
(2) by minimizing the impact of the nonlinearities via G_2

Since D_0 is unknown but $D_0 \in \mathcal{D}$, stability of the linear part will be guaranteed for all $D \in \mathcal{D}$

Similar to the case of noiseless data, since the nonlinear term $Q(x)$ is generic and $(X_1 - ED_0)$ has little structure, G_2 is designed to make $(X_1 - ED_0)G_2$ small

A nonlinear feedback stabilizer with noisy data

$\begin{bmatrix} K \\ I_S \end{bmatrix} = \begin{bmatrix} U_0 \\ Z_0 \end{bmatrix} G$, $x^+ = (X_1 - ED_0)G_1 x + (X_1 - ED_0)G_2 Q(x) + Ed$, $D_0 \in \mathcal{D} := \{D \colon DD^\top \preceq \Delta\Delta^\top\}$

For a given $\Omega \succ 0$, consider the decision variables

$$P \in \mathbb{R}^{n\times n},\ Y_1 \in \mathbb{R}^{T\times n},\ G_2 \in \mathbb{R}^{T\times n},\ \varepsilon \in \mathbb{R}$$

and the following SDP

$$\begin{aligned}
\text{minimize}_{P,Y_1,G_2,\varepsilon} \quad & \|X_1 G_2\| \\
\text{subject to} \quad & Z_0 Y_1 = \begin{bmatrix} P \\ 0_{(s-n)\times n} \end{bmatrix} && \text{(3a)} \\
& \begin{bmatrix} -P+\Omega & Y_1^\top X_1^\top & Y_1^\top \\ X_1 Y_1 & -P+\varepsilon E\Delta\Delta^\top E^\top & 0_{n\times T} \\ Y_1 & 0_{T\times n} & -\varepsilon I_T \end{bmatrix} \prec 0 && \text{(3b)} \\
& Z_0 G_2 = \begin{bmatrix} 0_{n\times(s-n)} \\ I_{s-n} \end{bmatrix} && \text{(3c)}
\end{aligned}$$

If it is feasible then the control law $u = KZ(x)$ with

$$K = U_0 \begin{bmatrix} Y_1 P^{-1} & G_2 \end{bmatrix}$$

renders the origin a locally asymptotically stable equilibrium for the closed-loop system

Discussion

The SDP in the case of noisy data The difference with respect to the SDP used in the result for noiseless data is the LMI

$$\text{(3b)}\quad \begin{bmatrix} -P+\Omega & Y_1^\top X_1^\top & Y_1^\top \\ X_1 Y_1 & -P+\varepsilon E\Delta\Delta^\top E^\top & 0_{n\times T} \\ Y_1 & 0_{T\times n} & -\varepsilon I_T \end{bmatrix} \prec 0 \text{ that replaces } \begin{bmatrix} -P & Y_1^\top X_1^\top \\ X_1 Y_1 & -P \end{bmatrix} \prec 0$$

The proof goes similarly to the case of noiseless data. The constraints (3a), (3c) and the designed K guarantee that the closed-loop system is

$$x^+ = (X_1 - ED_0)G_1 x + (X_1 - ED_0)G_2 Q(x) + Ed$$

(3b) implies Schur stability of

$$(X_1 - ED)G_1 \text{ for all } D \in \mathcal{D}$$

with Lyapunov function $x^\top P^{-1} x$, i.e.

$$P \succ 0, \quad ((X_1 - ED)G_1)^\top P^{-1}(X_1 - ED)G_1 - P^{-1} \prec -P^{-1}\Omega P^{-1} \text{ for all } D \in \mathcal{D}$$

Asymptotic stability descends from (i) Schur stability of $(X_1 - ED_0)G_1 x$ and (ii) $(X_1 - ED_0)G_2 Q(x) \to 0$ with $\lim_{|x|\to 0} |Q(x)|/|x| = 0$.

Nonlinearity attenuation The cost function

$$\text{minimize}_{P,Y_1,G_2,\varepsilon} \quad \|X_1 G_2\|$$

aims at designing a controller that minimizes the magnitude of the nonlinearity and is a substitute of

$$\text{minimize}_{P,Y_1,G_2,\varepsilon} \quad \|(X_1 - ED_0)G_2\|$$

As an alternative, one can minimize an upper bound β on

$$\|(X_1 - ED_0)G_2\|^2$$

at the cost of adding one constraint and two decision variables

$$\begin{aligned} \text{minimize}_{P,Y_1,G_2,\varepsilon,\beta,\delta} \quad & \beta \\ \text{subject to} \quad & \text{same constraints (3) as before} \\ & \begin{bmatrix} -\beta\, I_{S-n} & G_2^\top X_1^\top & G_2^\top \\ X_1 G_2 & -I_n + \delta E \Delta \Delta^\top E^\top & 0_{n\times T} \\ G_2 & 0_{T\times n} & -\delta I_T \end{bmatrix} \prec 0 \end{aligned}$$

Estimate of the region of attraction (RoA)

The previous result allows us to provide estimates of the (RoA) of the closed-loop system ($d = 0$)

$$x^+ = (A + BK)Z(x) = \underbrace{(X_1 - ED_0)G_1}_{\Psi} x + \underbrace{(X_1 - ED_0)G_2}_{\Xi} Q(x)$$

<u>Lyapunov difference along the solutions of the closed-loop system</u>

$$\begin{aligned} V(x^+) - V(x) = & \quad (\Psi x + \Xi Q(x))^\top P^{-1}(\Psi x + \Xi Q(x)) - x^\top P^{-1} x \\ \overset{\|D_0\| \leq \|\Delta\|}{\leq} & \quad -x^\top P^{-1} \Omega P^{-1} x + \ell(x, \Delta) \end{aligned}$$

where $\ell(x, \Delta)$ is a function dependent on data only, which upper bounds $(2\Psi x + \Xi Q(x))^\top P^{-1} \Xi Q(x)$

Set

$$\mathcal{L} := \{x : -x^\top P^{-1} \Omega P^{-1} x + \ell(x, \Delta) < 0\} \neq \emptyset$$

Any Lyapunov sub-level set $\mathcal{R}_\gamma := \{x : V(x) \leq \gamma\}$ of V contained in $\mathcal{L} \cup \{0\}$ is an estimate of the RoA of the closed-loop system

Example 2 (continued)

Inverted pendulum

$$x_1^+ = x_1 + T_s x_2, \quad x_2^+ = \frac{T_s g}{\ell} \sin x_1 + \left(1 - \frac{T_s \mu}{m\ell^2}\right) x_2 + \frac{T_s}{m\ell^2} u + d$$

Here $Z(x) = [x_1 \quad x_2 \quad \sin x_1 - x_1]^\top$ $E = [0 \quad 1]^\top$ $T_s = 0.1, m = 1, \ell = 1, \mu = 0.01$

Experiment

$x_0 \in [-0.5, 0.5]^2$

$u \in [-0.5, 0.5]$

$T = 30$

$d \in [-\delta, \delta]$

$\delta = 0.01$

$\Delta = \delta\sqrt{T}$

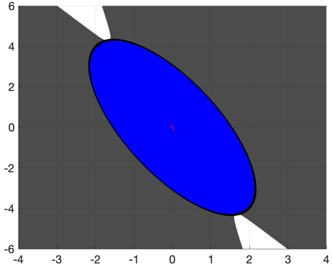
Measured data collected in X_1, Z_0, U_0 and used to solve the SDP

Results The solution to the SDP returns

$$K = \begin{bmatrix} -23.9436 & -11.4581 & -9.8564 \end{bmatrix}$$

in $u = KZ(x)$ and P in $V(x) = x^\top P^{-1} x$

Gray $= \mathcal{L}$ – Blue $= \mathcal{R}$



Flexibility of the approach

Controllers achieving different specifications can be obtained by modifying the cost function and/or the constraints.

$$\text{minimize}_{P,Y_1,G_2,\varepsilon} \quad \|X_1G_2\| + \lambda_1\|P\| + \lambda_2\|G_2\|$$

where $\lambda_1, \lambda_2 \geq 0$ are weights, might lead to a larger estimate of the RoA

$$\text{minimize}_{P,Y_1,G_2,\varepsilon,X,V} \quad \text{trace}(X) + \text{trace}(V)$$

$$\begin{bmatrix} X & X_1G_2 \\ (X_1G_2)^\top & V \end{bmatrix} \succeq 0$$

might lead to a sparse (low-complexity) control law X_1G_2 by minimizing the convex envelope of its rank

Summary Lecture 4

- ▷ Data-driven control of nonlinear systems expressible via “basis”functions
- ▷ Based on (approximate) nonlinearity cancellation
- ▷ As in the case of linear systems, simple end-to-end criterion (SDP⇒controller)
- ▷ Feedback linearization

Additional results

- ▷ Deterministic and stochastic perturbations on data
- ▷ Nonvanishing perturbations & neglected nonlinearities
- ▷ Recursive nonlinear cancellation
- ▷ Choice of “basis” functions via estimation methods

Outlook

- ▷ Alternatives to nonlinearity cancellation (Lecture 5)
- ▷ Measurement noise, I/O data, complex systems

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