Data-driven Control Design

linear and nonlinear systems

Lecture 5

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Data-driven control of nonlinear systems so far

In this last lecture, we look again at nonlinear continuous-time systems of the form

$$\dot{x} = AZ(x) + Bu + Ed$$

We took $Z : \mathbb{R}^n \to \mathbb{R}^s$ as a $\mathcal{C}^1$ vector-valued function that includes both linear and nonlinear functions

$$Z(x) = \begin{bmatrix} x \\ Q(x) \end{bmatrix}$$

and considered for the disturbance the model $|d| \leq \delta$. Here we restrict this model:

Assumption Disturbance d is constant and unknown.

The analysis to be presented below carries over if the disturbance d is time-varying and satisfies

$$|d(t)| \leq \delta$$

with δ known. The focus on constant disturbances allows us to present a new data-dependent representation ("noise-filtered data-dependent" representation) in a more compact way.

Data-driven control of nonlinear systems so far

$$\dot{x} = AZ(x) + Bu + Ed$$

Previously, we showed that, if $d = 0$, then the solution of the SDP

$$\begin{aligned}
\text{minimize}_{P,Y_1,G_2} \quad & \|X_1 G_2\| \\
\text{subject to} \quad & Z_0 Y_1 = \begin{bmatrix} P \\ 0_{(s-n)\times n} \end{bmatrix} && \text{(1a)} \\
& X_1 Y_1 + (X_1 Y_1)^\top \prec 0 && \text{(1b)} \\
& Z_0 G_2 = \begin{bmatrix} 0_{n\times(s-n)} \\ I_{s-n} \end{bmatrix} && \text{(1c)}
\end{aligned}$$

provides a stabilising feedback $u = KZ(x)$.

- ▷ If a zero cost is attained ($\|X_1 G_2\| = 0$ – nonlinearity cancellation), then we obtain a global asymptotic result.
- ▷ For a zero cost to be attained, the system must have a special structure (e.g. nonlinearities matched by the control input, systems in strict feedback form)
- ▷ The purpose of today's lecture is to present an alternative approach to stabilization without cancelling the nonlinearity. The approach allows us to deal with tracking problems too.

Information collection

Information about the system's dynamics is obtained from a T-long dataset of input/state samples collected during (multiple) experiment(s)

$$\mathbb{D} := \{(\dot{x}_i, x_i, u_i)\}_{i=0}^{T-1}$$

where

$$\dot{x}_i := \dot{x}(t_i), x_i := x(t_i), u_i := u(t_i)$$

and

$$0 \leq t_0 < t_1 < \ldots < t_{T-1}$$

are the sampling times.

The samples satisfy

$$\dot{x}_i = AZ(x_i) + Bu_i + Ed_i, \quad i = 0, \ldots, T-1$$

Recall that we can avoid measuring the state derivatives $\dot{x}_i$ using the integral version of

$$\dot{x} = AZ(x) + Bu + Ed$$

A revised control problem for nonlinear systems

Problem Based on the dataset $\mathbb{D}$ design a state feedback controller

$$u = KZ(x)$$

that makes the closed-loop system

$$\dot{x} = (A + BK)Z(x) + Ed$$

exponentially contractive on $\mathcal{X} \subseteq \mathbb{R}^n$.

- The system is exponentially contractive on $\mathcal{X}$ if

$$\exists P \succ 0, \beta > 0 \colon ((A+BK)\frac{\partial Z}{\partial x})^\top P^{-1} + P^{-1}(A+BK)\frac{\partial Z}{\partial x} \preceq -\beta P^{-1} \; \forall x \in \mathcal{X}$$

  which is a property on the Jacobian of the dynamics.

- As before, the choice $u = KZ(x)$ allows us to express the closed-loop dynamics in terms of the dataset $\mathbb{D}$ and to reduce the design of K to a convex problem.

Why contractivity?

Global results

If $\mathcal{X} = \mathbb{R}^n$, then a unique equilibrium x_* exists, which is globally exponentially stable.

Existence of a compact forward invariant set $\mathcal{R}$ for $\dot{x} = f(x) := (A + BK)Z(x) + Ed$ Let $\overline{x}$ be any point in $\mathcal{X}$ and $V(x) = (x - \overline{x})^\top P^{-1}(x - \overline{x})$. Then

$$\begin{aligned}\dot{V}(x) := \nabla V(x)^\top f(x) = \;& 2(x-\overline{x})^\top P^{-1} \left[\frac{\partial f}{\partial y}\right]_{y \in \overline{(x,\overline{x})}} (x - \overline{x}) + 2(x - \overline{x})^\top P^{-1} f(\overline{x}) \\ \leq \;& -\beta V(x) + \|f(\overline{x})\| \, \|P^{-\frac{1}{2}}\| V(x)^{\frac{1}{2}}\end{aligned}$$

Hence, the sublevel set $\mathcal{R} := \{x \colon V(x) \leq (\beta^{-1}\|f(\overline{x})\| \, \|P^{-\frac{1}{2}}\|)^2\}$ is a forward invariant set for $\dot{x} = f(x)$.

Existence of a limit solution (Yakubovich & Demidovich) Consider $\dot{x} = f(t,x)$, where $f(t,x)$ is $\mathcal{C}^0$ in t and $\mathcal{C}^1$ in x. A compact set that is forward invariant for $\dot{x} = f(t,x)$ contains a unique (limit) solution $x_*(t)$ defined for all $t \in (-\infty, +\infty)$. Moreover, if $f(t,x)$ is independent of t, then $x_*(t) = x_*$, i.e. the limit solution is an equilibrium.

Global exponential stability We can repeat the analysis as before with $V_*(x) = (x - x_*)^\top P^{-1}(x - x_*)$, this time obtaining

$$\dot{V}(x) \leq -\beta V(x)$$

thanks to $f(x_*) = 0$.

Why contractivity?

Local results

If $\mathcal{X} \subset \mathbb{R}^n$ is a convex set and there exists an equilibrium $x_* \in \text{int}(\mathcal{X})$ of $\dot{x} = f(x)$, then it is locally exponentially stable.

Alternatively, if $\mathcal{X}$ is a convex set, which is forward invariant wrt $\dot{x} = f(x)$ and $\dot{x} = f(x)$ is forward complete on $\mathcal{X}$, then there exists a unique equilibrium $x_* \in \mathcal{X}$, which is locally exponentially stable.

Regulation problem

If the equilibrium x_* is not the desired one, then contractivity allows for the design of a controller that regulates a certain function of the state $y = h(x)$ to a desired value r.

Consider the dataset

$$\mathbb{D} := \{(\dot{x}_i, x_i, u_i)\}_{i=0}^{T-1}, \quad \dot{x}_i = AZ(x_i) + Bu_i + Ed_i, \quad i = 0, \ldots, T-1$$

and store the samples into matrices U_0, X_0, X_1, Z_0 defined as

$$\begin{aligned} U_0 &:= \begin{bmatrix} u_0 & u_1 & \cdots & u_{T-1} \end{bmatrix} \\ X_0 &:= \begin{bmatrix} x_0 & x_1 & \cdots & x_{T-1} \end{bmatrix} \\ X_1 &:= \begin{bmatrix} \dot{x}_0 & \dot{x}_1 & \cdots & \dot{x}_{T-1} \end{bmatrix} \\ Z_0 &:= \begin{bmatrix} Z(x_0) & Z(x_1) & \ldots & Z(x_{T-1}) \end{bmatrix} \end{aligned}$$

which satisfy the identity

$$\begin{aligned} &\underbrace{\begin{bmatrix} \dot{x}_0 & \dot{x}_1 & \cdots & \dot{x}_{T-1} \end{bmatrix}}_{X_1} \\ = \quad &A \underbrace{\begin{bmatrix} Z(x_0) & Z(x_1) & \ldots & Z(x_{T-1}) \end{bmatrix}}_{Z_0} + B \underbrace{\begin{bmatrix} u_0 & u_1 & \cdots & u_{T-1} \end{bmatrix}}_{U_0} \\ &+ E \underbrace{\begin{bmatrix} d_0 & d_1 & \cdots & d_{T-1} \end{bmatrix}}_{D_0} \end{aligned}$$

$$X_1 = AZ_0 + BU_0 + ED_0$$

Data-dependent representations of closed-loop nonlinear systems

Closed-loop nonlinear system $\dot{x} = (A + BK)Z(x) + Ed$

Consider any matrices $K \in \mathbb{R}^{m\times s}$, $G \in \mathbb{R}^{T\times n}$ such that

$$\begin{bmatrix} K \\ I_s \end{bmatrix} = \begin{bmatrix} U_0 \\ Z_0 \end{bmatrix} G$$

where

$$\begin{aligned} U_0 &= \begin{bmatrix} u_0 & u_1 & \cdots & u_{T-1} \end{bmatrix} \\ Z_0 &= \begin{bmatrix} Z(x_0) & Z(x_1) & \ldots & Z(x_{T-1}) \end{bmatrix} \end{aligned} \qquad X_1 = AZ_0 + BU_0 + ED_0$$

The matrix $A + BK$ of the closed-loop system $\dot{x} = (A + BK)Z(x) + Ed$ is arranged as

$$\begin{aligned} & A + BK \\ = \; & \begin{bmatrix} B & A \end{bmatrix} \begin{bmatrix} K \\ I_s \end{bmatrix} \\ \overset{\begin{bmatrix} K \\ I_s \end{bmatrix} = \begin{bmatrix} U_0 \\ Z_0 \end{bmatrix} G}{=} \; & \begin{bmatrix} B & A \end{bmatrix} \begin{bmatrix} U_0 \\ Z_0 \end{bmatrix} G \\ \overset{X_1 = \begin{bmatrix} B & A \end{bmatrix} \begin{bmatrix} U_0 \\ Z_0 \end{bmatrix} + ED_0}{=} \; & (X_1 - ED_0)G \end{aligned}$$

A noise-filtered representation

Consider (again) the dataset

$$\mathbb{D} := \{(\dot{x}_i, x_i, u_i)\}_{i=0}^{T-1}, \quad \dot{x}_i = AZ(x_i) + Bu_i + Ed_i, \quad i = 0, \ldots, T-1$$

and the data matrices that satisfy

$$X_1 = AZ_0 + BU_0 + ED_0$$

We focus on D_0 and observe that

$$D_0 = \begin{bmatrix} d_0 & d_1 & \ldots & d_{T-1} \end{bmatrix} = d_0 \mathbb{1}_{1\times T} =: LM$$

Hence

$$X_1 = AZ_0 + BU_0 + ELM$$

The perturbation D_0 due to the disturbance can be split into two factors, the first one of which, L, is unknown and the second one, M, is known.

Consider any matrices $K \in \mathbb{R}^{m\times s}$, $G \in \mathbb{R}^{T\times s}$ such that

$$\begin{bmatrix} K \\ I_s \\ 0_{1\times s} \end{bmatrix} = \begin{bmatrix} U_0 \\ Z_0 \\ M \end{bmatrix} G$$

The matrix $A + BK$ of the closed-loop system $\dot{x} = (A + BK)Z(x) + Ed$ is arranged as

$$\begin{aligned} & A + BK \\ = \; & \begin{bmatrix} B & A & EL \end{bmatrix} \begin{bmatrix} K \\ I_s \\ 0 \end{bmatrix} \\ \overset{\begin{bmatrix} K \\ I_n \\ 0 \end{bmatrix} = \begin{bmatrix} U_0 \\ Z_0 \\ M \end{bmatrix} G}{=} \; & \begin{bmatrix} B & A & EL \end{bmatrix} \begin{bmatrix} U_0 \\ Z_0 \\ M \end{bmatrix} G \\ \overset{X_1 = \begin{bmatrix} B & A & EL \end{bmatrix} \begin{bmatrix} U_0 \\ Z_0 \\ M \end{bmatrix}}{=} \; & X_1 G \end{aligned}$$

Consider any matrices $K \in \mathbb{R}^{m\times s}$, $G \in \mathbb{R}^{T\times s}$ such that

$$\begin{bmatrix} K \\ I_s \\ 0 \end{bmatrix} = \begin{bmatrix} U_0 \\ Z_0 \\ M \end{bmatrix} G$$

Partition G as

$$G = \begin{bmatrix} \underbrace{G_1}_{n} & \underbrace{G_2}_{s-n} \end{bmatrix} T$$

n dimension of x, $s-n$ dimension of $Q(x)$

The closed-loop system $\dot{x} = (A+BK)Z(x) + Ed$, where $Z(x) = \begin{bmatrix} x \\ Q(x) \end{bmatrix}$, results in the data-dependent representation

$$\dot{x} = \quad X_1 G_1 x + X_1 G_2 Q(x) + Ed$$

- ▷ The representation depends on data U_0, Z_0, X_1 and design variables G_1, G_2
- ▷ The unknown disturbance d affecting the dataset is filtered out via a suitable choice of G enforced by the additional condition $0_{1\times s} = MG$.

Enforcing contractivity from data – recap so far

A dataset

$$\mathbb{D} := \{(\dot{x}_i, x_i, u_i)\}_{i=0}^{T-1}$$

is obtained from off-line experiments conducted on the system

$$\dot{x} = AZ(x) + Bu + Ed$$

and data are organized into matrices U_0, X_0, X_1, Z_0 that satisfy

$$X_1 = AZ_0 + BU_0 + ED_0$$

To design a controller $u = KZ(x)$, we look for $G = \begin{bmatrix} G_1 & G_2 \end{bmatrix}$ that satisfies

$$\begin{bmatrix} K \\ I_s \\ 0 \end{bmatrix} = \begin{bmatrix} U_0 \\ Z_0 \\ M \end{bmatrix} G \qquad \text{where} \quad M = \mathbb{1}_{1\times T}$$

and makes

$$\dot{x} = X_1 G_1 x + X_1 G_2 Q(x) + Ed$$

exponentially contractive.

D_0 is unknown but can be factored as $D_0 = LM$, where $M = \mathbb{1}_{1\times T}$ known.

A formula for enforcing contraction via data

$$\begin{bmatrix} K \\ I_s \\ 0 \end{bmatrix} = \begin{bmatrix} U_0 \\ Z_0 \\ M \end{bmatrix} G, \quad \dot{x} = X_1 G_1 x + X_1 G_2 Q(x) + Ed$$

Consider the decision variables $P \in \mathbb{S}^{n\times n}$, $Y_1 \in \mathbb{R}^{T\times n}$, $G_2 \in \mathbb{R}^{T\times n}$, $\alpha \in \mathbb{R}_{>0}$ and the SDP

$$P \succ 0 \tag{2a}$$

$$Z_0 Y_1 = \begin{bmatrix} P \\ 0_{(s-n)\times n} \end{bmatrix} \tag{2b}$$

$$\begin{bmatrix} X_1 Y_1 + (X_1 Y_1)^\top + \alpha I_n & X_1 G_2 & P R_Q \\ (X_1 G_2)^\top & -I_{s-n} & 0_{s-n\times r} \\ (P R_Q)^\top & 0_{r\times s-n} & -I_r \end{bmatrix} \preceq 0 \tag{2c}$$

$$Z_0 G_2 = \begin{bmatrix} 0_{n\times(s-n)} \\ I_{s-n} \end{bmatrix} \tag{2d}$$

$$0 = M \begin{bmatrix} Y_1 & G_2 \end{bmatrix} \tag{2e}$$

where $\frac{\partial Q}{\partial x}(x)^\top \frac{\partial Q}{\partial x}(x) \preceq R_Q R_Q^\top$ for every $x \in \mathcal{X}$. If it is feasible then the control law $u = KZ(x)$ with

$$K = U_0 \begin{bmatrix} Y_1 P^{-1} & G_2 \end{bmatrix}$$

makes the closed-loop system exponentially contractive on $\mathcal{X}$.

Constraint (2b) can be equivalently written as

$$\text{(2b) } Z_0 Y_1 = \begin{bmatrix} P \\ 0_{(s-n)\times n} \end{bmatrix} \Leftrightarrow Z_0 Y_1 P^{-1} = \begin{bmatrix} I_n \\ 0_{(s-n)\times n} \end{bmatrix}, \quad \text{(2d) } Z_0 G_2 = \begin{bmatrix} 0_{n\times(s-n)} \\ I_{s-n} \end{bmatrix}$$

Perform the change of variable $G_1 := Y_1 P^{-1}$, to obtain $Z_0 \begin{bmatrix} G_1 & G_2 \end{bmatrix} = I_s$.

By the same change of variable, the control gain $K = U_0 \begin{bmatrix} Y_1 P^{-1} & G_2 \end{bmatrix}$ can be written as $K = U_0 \begin{bmatrix} G_1 & G_2 \end{bmatrix}$

(2e) $0 = M \begin{bmatrix} Y_1 & G_2 \end{bmatrix}$ can be written as $0 = M \begin{bmatrix} G_1 & G_2 \end{bmatrix}$

Hence, $\begin{bmatrix} K \\ I_s \\ 0 \end{bmatrix} = \begin{bmatrix} U_0 \\ Z_0 \\ M \end{bmatrix} G$. This yields $A + BK = (AZ_0 + BU_0 + ELM) = X_1 G$.

Applying the Schur complement, constraint (2c) $\begin{bmatrix} X_1 Y_1 + (X_1 Y_1)^\top + \alpha I_n & X_1 G_2 & PR_Q \\ (X_1 G_2)^\top & -I_{s-n} & 0_{s-n\times r} \\ (PR_Q)^\top & 0_{r\times s-n} & -I_r \end{bmatrix} \preceq 0$ is equivalently written as

$$\begin{bmatrix} X_1 Y_1 + (X_1 Y_1)^\top + \alpha I_n & X_1 G_2 \\ (X_1 G_2)^\top & -I_{s-n} \end{bmatrix} + \begin{bmatrix} PR_Q \\ 0_{s-n\times r} \end{bmatrix} \begin{bmatrix} (PR_Q)^\top & 0_{r\times s-n} \end{bmatrix} \preceq 0$$

and, applying the Schur complement once more, as

$$X_1 Y_1 + (X_1 Y_1)^\top + X_1 G_2 (X_1 G_2)^\top + PR_Q (PR_Q)^\top + \alpha I_n \preceq 0$$

Thus, we have written the constraint (2c) as

$$X_1Y_1 + (X_1Y_1)^\top + X_1G_2(X_1G_2)^\top + PR_Q(PR_Q)^\top + \alpha I_n \preceq 0$$

Recall the change of variable $G_1 = Y_1P^{-1}$ and note that

$$\frac{\partial Q}{\partial x}(x)^\top \frac{\partial Q}{\partial x}(x) \preceq R_Q R_Q^\top \quad \forall x \in \mathcal{X} \Longrightarrow P\frac{\partial Q}{\partial x}(x)^\top \frac{\partial Q}{\partial x}(x)P \preceq PR_Q R_Q^\top P \quad \forall x \in \mathcal{X}$$

Hence,

$$X_1G_1P + (X_1G_1P)^\top + X_1G_2G_2^\top X_1^\top + P\frac{\partial Q}{\partial x}(x)^\top \frac{\partial Q}{\partial x}(x)P + \alpha I_n \preceq 0 \;\; \forall x \in \mathcal{X}$$

By a completion of the squares argument

$$0 \preceq (X_1G_2 - P\frac{\partial Q}{\partial x}^\top)(G_2^\top X_1^\top - \frac{\partial Q}{\partial x}P) = X_1G_2G_2^\top X_1^\top - X_1G_2\frac{\partial Q}{\partial x}P - P\frac{\partial Q}{\partial x}^\top G_2^\top X_1^\top + P\frac{\partial Q}{\partial x}^\top \frac{\partial Q}{\partial x}P$$

we obtain

$$X_1G_1P + (X_1G_1P)^\top + P\frac{\partial Q}{\partial x}^\top G_2^\top X_1^\top + X_1G_2\frac{\partial Q}{\partial x}P + \alpha I_n \preceq 0 \;\; \forall x \in \mathcal{X}$$

i.e.

$$X_1 \begin{bmatrix} G_1 & G_2 \end{bmatrix} \begin{bmatrix} I_n \\ \frac{\partial Q}{\partial x} \end{bmatrix} P + P \begin{bmatrix} I_n \\ \frac{\partial Q}{\partial x} \end{bmatrix}^\top \begin{bmatrix} G_1 & G_2 \end{bmatrix}^\top X_1^\top + \alpha I_n \preceq 0 \;\; \forall x \in \mathcal{X}$$

or

$$P^{-1}(A + BK)\frac{\partial Z}{\partial x} + \frac{\partial Z}{\partial x}^\top (A + BK)^\top P^{-1} \preceq -\beta P^{-1} \;\; \forall x \in \mathcal{X}, \text{with } \beta := \frac{\alpha}{\lambda_{\max}(P)}$$

Discussion

Growth condition To derive the result, we introduce a growth condition on the nonlinearities of $Z(x)$

The nonlinearities $Q(x)$ satisfy

$$\frac{\partial Q}{\partial x}(x)^\top \frac{\partial Q}{\partial x}(x) \preceq R_Q R_Q^\top \text{ for any } x \in \mathcal{X}$$

for some known matrix R_Q.

Since $Q(x)$ is continuous, R_Q exists when $\mathcal{X}$ is a compact set. Otherwise, the existence of R_Q must be considered as an assumption.

Aim of the controller Constraint (2c) $\begin{bmatrix} X_1 Y_1 + (X_1 Y_1)^\top + \alpha I_n & X_1 G_2 & P R_Q \\ (X_1 G_2)^\top & -I_{s-n} & 0_{s-n\times r} \\ (P R_Q)^\top & 0_{r\times s-n} & -I_r \end{bmatrix} \preceq 0$ implies

$$P^{-1} X_1 G_1 + (P^{-1} X_1 G_1)^\top + P^{-1} X_1 G_2 (X_1 G_2)^\top P^{-1} + R_Q R_Q^\top + \alpha P^{-2} \preceq 0$$

This reveals that the linear part of the controller aims at dominating the nonlinear part $X_1 G_2 \frac{\partial Q}{\partial x}$ of the dynamics.

Asymptotic behavior

Global results If $\mathcal{X} = \mathbb{R}^n$, then the designed feedback controller induces the equilibrium x_* in the closed-loop dynamics

$$\dot{x} = (A + BK)Z(x) + Ed$$

and makes it globally exponentially stable.

Local result If $\mathcal{X} \subset \mathbb{R}^n$ and the designed feedback controller

$$u = KZ(x), \quad K = U_0 \begin{bmatrix} Y_1 P^{-1} & G_2 \end{bmatrix}$$

induces an equilibrium x_* in $\mathrm{int}(\mathcal{X})$ for the closed-loop dynamics, i.e.,

$$(A + BK)Z(x_*) + Ed = 0 \quad \text{for some } x_* \in \mathrm{int}(\mathcal{X})$$

then x_* is exponentially stable.

The case $d = 0$ The result holds under the same SDP (the controller must still dominate the nonlinearities) but in this case we can analytically determine whether or not there exists x_* such that $(A + BK)Z(x_*) = 0$ because $A + BK = X_1 G$.

An example - A flexible robot arm

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -\frac{K_c}{J_2}x_1 - \frac{F_2}{J_2}x_2 + \frac{K_c}{J_2 N_c}x_3 - \frac{mgd}{J_2}\cos x_1$$
$$\dot{x}_3 = x_4$$
$$\dot{x}_4 = -\frac{K_c}{J_1 N_c}x_1 + \frac{K_c}{J_1 N_c^2}x_3 - \frac{F_1}{J_1}x_4 + \frac{1}{J_1}u$$

Experiment

$x_0 \in [-0.1, 0.1]^4$

$u \in [-0.1, 0.1]$

$T = 10$

$$Q(x) = \cos x_1, \quad R_Q = R_Q^\top = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \mathcal{X} = \mathbb{R}^4$$

Results The SDP is feasible and returns the controller gain

$$K = \begin{bmatrix} -3.1639 & -4.6751 & -3.9299 & -0.7614 & 0.0106 \end{bmatrix}$$

and the closed-loop system $\dot{x} = \begin{bmatrix} 0.0000 & 1.0000 & 0.0000 & 0.0000 & 0.0000 \\ -2.0000 & -0.7500 & 1.0000 & 0.0000 & -1.9600 \\ 0.0000 & 0.0000 & 0.0000 & 1.0000 & 0.0000 \\ -22.4261 & -31.1671 & -25.5328 & -5.7430 & 0.0708 \end{bmatrix} Z(x).$

Since the latter is contractive, the equilibrium x_* exists and computable via $X_1 G Z(x_*) = 0$, which gives $x_* = \begin{bmatrix} -0.5718 & 0 & 0.5046 & 0 \end{bmatrix}^\top$.

Example (continued)

Let us include additional nonlinearities in $Q(x)$

$$Q(x) = \begin{bmatrix} \cos x_1 \\ x_1^2 \\ \sin x_2 \end{bmatrix}, \quad \frac{\partial Q}{\partial x}(x)^\top \frac{\partial Q}{\partial x}(x) = \begin{bmatrix} \sin(x_1)^2 + 4x_1^2 & 0 & 0 & 0 \\ 0 & \cos(x_2)^2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

If we choose $\mathcal{X} = [-w, w] \times \mathbb{R}^3$, where $w \in \mathbb{R}_{>0}$, then the growth condition

$$\frac{\partial Q}{\partial x}(x)^\top \frac{\partial Q}{\partial x}(x) \preceq R_Q R_Q^\top$$

is satisfied on $\mathcal{X}$ with

$$R_Q = R_Q^\top = \begin{bmatrix} \sqrt{4w^2+1} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Here, we set $w = 1$ and solve the SDP with the new value of R_Q and find the controller gain

$$K = \begin{bmatrix} -280.8884 & -257.1662 & -91.3493 & -8.2326 & 0.1591 & -0.0032 & -0.0030 \end{bmatrix}$$

The controller returns the closed-loop system

$$\dot{x} = \begin{bmatrix} 0.0000 & 1.0000 & 0.0000 & 0.0000 & 0.0000 & -0.0000 & -0.0000 \\ -2.0000 & -0.7500 & 1.0000 & 0.0000 & -1.9600 & 0.0000 & 0.0000 \\ 0.0000 & 0.0000 & 0.0000 & 1.0000 & -0.0000 & 0.0000 & 0.0000 \\ -1873.9225 & -1714.4416 & -608.3286 & -55.5510 & 1.0603 & -0.0213 & -0.0201 \end{bmatrix} Z(x)$$

which is known to the designer since it coincides with $\dot{x} = X_1 G Z(x)$. We can solve for x_* in $X_1 G Z(x_*) = 0$ and find out that the equation has a solution

$$x_* = \begin{bmatrix} -0.3607 & 0 & 1.1126 & 0 \end{bmatrix}^\top,$$

which belongs to $\mathrm{int}(\mathcal{X})$. Hence, it is exponentially stable.

Comment 1 Solutions initialized outside the largest sublevel set of $x^\top P^{-1} x$ contained $\mathcal{X}$ still converge to x_*, suggesting a much larger RoA than the certifiable one.

Comment 2 The SDP remains feasible for values of w up to 100, to which it corresponds higher controller gains, consistent with the observation that the proposed control design dominate the growth of the nonlinearities.

Steering towards a desired equilibrium

Making the closed-loop system contractive allows the designer to guarantee asymptotic properties of an equilibrium without knowing the equilibrium. This is useful in the data-driven control design context where the model is uncertain. A drawback, though, is that the equilibrium might differ from the desired one. We deal now with this problem formulating an output regulation problem.

As a first step, we endow the system with an output of interest

$$\begin{aligned} \dot{x} &= AZ(x) + Bu + Ed \\ y &= CZ(x) \end{aligned}$$

where C is an unknown matrix. We would like to design a controller that steers the output $y \in \mathbb{R}^p$ to a prescribed constant reference signal r.

To this purpose, we assume to measure the regulation error

$$e = r - y$$

Given the state x and the regulated error e, we would like to design the controller

$$\begin{aligned} \dot{\xi} &= e \\ u &= \underbrace{\begin{bmatrix} K_x & K_Q \end{bmatrix}}_{K} Z(x) + K_\xi \xi \end{aligned}$$

that, for the closed-loop system

$$\begin{aligned} \dot{x} &= AZ(x) + Bu + Ed \\ e &= r - CZ(x) \\ \dot{\xi} &= e \\ u &= KZ(x) + K_\xi \xi, \end{aligned}$$

guarantees
(i) boundedness of the solution $(x(t), \xi(t))$;
(ii) $\lim_{t\to+\infty} e(t) = 0$.

The feedback law $u = KZ(x) + K_\xi \xi$ has the task of making the augmented dynamics

$$\begin{bmatrix} \dot{x} \\ \dot{\xi} \end{bmatrix} = \begin{bmatrix} A & 0 \\ C & 0 \end{bmatrix} \begin{bmatrix} Z(x) \\ \xi \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} u + \begin{bmatrix} E & 0 \\ 0 & -I \end{bmatrix} \begin{bmatrix} d \\ r \end{bmatrix}$$

contractive.
The internal model controller $\dot{\xi} = e$ induces an equilibrium for which $e = 0$.

The goal is to design the feedback law

$$u = KZ(x) + K_\xi \xi$$

that makes the augmented dynamics

$$\begin{bmatrix} \dot{x} \\ \dot{\xi} \end{bmatrix} = \begin{bmatrix} A & 0 \\ C & 0 \end{bmatrix} \begin{bmatrix} Z(x) \\ \xi \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} u + \begin{bmatrix} E & 0 \\ 0 & -I \end{bmatrix} \begin{bmatrix} d \\ r \end{bmatrix}$$

contractive.

It is easier to to realize that we are in the same setup considered previously if we arrange the quantities $Z(x)$, ξ in a suitable order, namely as in the vector

$$\begin{bmatrix} x \\ \xi \\ Q(x) \end{bmatrix}$$

and rewrite the augmented dynamics above accordingly, i.e.

$$\begin{bmatrix} \dot{x} \\ \dot{\xi} \end{bmatrix} = \underbrace{\begin{bmatrix} \overline{A} & 0 & \widehat{A} \\ \overline{C} & 0 & \widehat{C} \end{bmatrix}}_{\mathcal{A}} \begin{bmatrix} x \\ \xi \\ Q(x) \end{bmatrix} + \underbrace{\begin{bmatrix} B \\ 0 \end{bmatrix}}_{\mathcal{B}} u + \underbrace{\begin{bmatrix} E & 0 \\ 0 & -I \end{bmatrix}}_{\mathcal{E}} \begin{bmatrix} d \\ r \end{bmatrix}$$

where we used the partition

$$AZ(x) = \begin{bmatrix} \overline{A} & \widehat{A} \end{bmatrix} \begin{bmatrix} x \\ Q(x) \end{bmatrix}, \quad CZ(x) = \begin{bmatrix} \overline{C} & \widehat{C} \end{bmatrix} \begin{bmatrix} x \\ Q(x) \end{bmatrix}$$

The goal is to design the feedback law

$$u = KZ(x) + K_\xi \xi$$

that makes the augmented dynamics

$$\begin{bmatrix} \dot{x} \\ \dot{\xi} \end{bmatrix} = \underbrace{\begin{bmatrix} \overline{A} & 0 & \widehat{A} \\ \overline{C} & 0 & \widehat{C} \end{bmatrix}}_{\mathcal{A}} \begin{bmatrix} x \\ \xi \\ Q(x) \end{bmatrix} + \underbrace{\begin{bmatrix} B \\ 0 \end{bmatrix}}_{\mathcal{B}} u + \underbrace{\begin{bmatrix} E & 0 \\ 0 & -I \end{bmatrix}}_{\mathcal{E}} \begin{bmatrix} d \\ r \end{bmatrix}$$

contractive.

We collect the dataset $\mathbb{D} := \{x_i, \xi_i, u_i, \dot{x}_i, e_i\}_{i=0}^{T-1}$ from the augmented system and arrange it into the data matrices $U_0, \mathcal{Z}_0, \mathcal{Z}_1, M$ that satisfy

$$\mathcal{Z}_1 = \mathcal{A}\mathcal{Z}_0 + \mathcal{B}U_0 + \mathcal{E}\mathcal{L}M$$

where

$$\begin{aligned}
U_0 &:= \begin{bmatrix} u_0 & u_1 & \cdots & u_{T-1} \end{bmatrix} \\
\mathcal{Z}_0 &:= \begin{bmatrix} x_0 & x_1 & \cdots & x_{T-1} \\ \xi_0 & \xi_1 & \cdots & \xi_{T-1} \\ Q(x_0) & Q(x_1) & \cdots & Q(x_{T-1}) \end{bmatrix} \quad (\xi_i := \xi(t_i)) \\
\mathcal{Z}_1 &:= \begin{bmatrix} \dot{x}_0 & \dot{x}_1 & \cdots & \dot{x}_{T-1} \\ e_0 & e_1 & \cdots & e_{T-1} \end{bmatrix} \quad (e_i := e(t_i)) \\
\mathcal{L} &:= \begin{bmatrix} d_0 \\ r_0 \end{bmatrix}, \quad M = \mathbb{1}_{1\times T}
\end{aligned}$$

Data-dependent representation of the augmented dynamics

Consider any matrices $\mathcal{K} = \begin{bmatrix} K_x & K_\xi & K_Q \end{bmatrix} \in \mathbb{R}^{m\times(s+p)}$, $\mathcal{G} = \begin{bmatrix} \mathcal{G}_1 & \mathcal{G}_2 \end{bmatrix} \in \mathbb{R}^{T\times(s+p)}$ such that

$$\begin{bmatrix} \mathcal{K} \\ I_{s+p} \\ 0_{1\times(s+p)} \end{bmatrix} = \begin{bmatrix} U_0 \\ \mathcal{Z}_0 \\ M \end{bmatrix} \begin{bmatrix} \mathcal{G}_1 & \mathcal{G}_2 \end{bmatrix}$$

Then the closed-loop system

$$\begin{aligned} \begin{bmatrix} \dot{x} \\ \dot{\xi} \end{bmatrix} &= \mathcal{A} \begin{bmatrix} x \\ \xi \\ Q(x) \end{bmatrix} + \mathcal{B}u + \mathcal{E} \begin{bmatrix} d \\ r \end{bmatrix} \\ u &= \mathcal{K} \begin{bmatrix} x \\ \xi \\ Q(x) \end{bmatrix} \end{aligned}$$

results in

$$\begin{bmatrix} \dot{x} \\ \dot{\xi} \end{bmatrix} = \mathcal{Z}_1 \mathcal{G} \begin{bmatrix} x \\ \xi \\ Q(x) \end{bmatrix} + \mathcal{E} \begin{bmatrix} d \\ r \end{bmatrix}$$

We can then design $\mathcal{K}$ (through $\mathcal{G}$) that makes the closed-loop system contractive and – as a byproduct – returns the integral controller that regulates e to 0.

Design of nonlinear regulators (PI controllers)

Consider the decision variables $\mathcal{P} \in \mathbb{S}^{n+p \times n+p}$, $\mathcal{Y}_1 \in \mathbb{R}^{T \times n+p}$, $\mathcal{G}_2 \in \mathbb{R}^{T \times s-n}$, $\alpha \in \mathbb{R}_{>0}$ and the SDP

$$
\begin{aligned}
&\mathcal{P} \succ 0 \\
&\mathcal{Z}_0 \mathcal{Y}_1 = \begin{bmatrix} \mathcal{P} \\ 0_{(s-n) \times n+p} \end{bmatrix} \\
&\begin{bmatrix} \mathcal{Z}_1 \mathcal{Y}_1 + (\mathcal{Z}_1 \mathcal{Y}_1)^\top + \alpha I_n & \mathcal{Z}_1 \mathcal{G}_2 & \mathcal{P} \begin{bmatrix} R_Q \\ 0_{p \times r} \end{bmatrix} \\ (\mathcal{Z}_1 \mathcal{G}_2)^\top & -I_{s-n} & 0_{s-n \times r} \\ (\mathcal{P} \begin{bmatrix} R_Q \\ 0_{p \times r} \end{bmatrix})^\top & 0_{r \times s-n} & -I_r \end{bmatrix} \preceq 0 \\
&\mathcal{Z}_0 \mathcal{G}_2 = \begin{bmatrix} 0_{n+p \times (s-n)} \\ I_{s-n} \end{bmatrix} \\
&0 = M \begin{bmatrix} Y_1 & G_2 \end{bmatrix}
\end{aligned}
$$

where $\frac{\partial Q}{\partial x}(x)^\top \frac{\partial Q}{\partial x}(x) \preceq R_Q R_Q^\top$ for every $x \in \mathcal{X}$. If it is feasible then the control law $u = \mathcal{K} \begin{bmatrix} x \\ \xi \\ Q(x) \end{bmatrix}$ with $\mathcal{K} = U_0 \begin{bmatrix} \mathcal{Y}_1 \mathcal{P}^{-1} & \mathcal{G}_2 \end{bmatrix}$ makes the closed-loop system exponentially contractive on $\mathcal{X} \times \mathbb{R}^p$.

If $\mathcal{X} = \mathbb{R}^n$, then the designed nonlinear regulator

$$\begin{aligned} \dot{\xi} = & \; e \\ u = & \; \mathcal{K} \begin{bmatrix} x \\ \xi \\ Q(x) \end{bmatrix} \end{aligned} \qquad \text{where} \quad \mathcal{K} = U_0 \begin{bmatrix} \mathcal{Y}_1 \mathcal{P}^{-1} & \mathcal{G}_2 \end{bmatrix}$$

induces an equilibrium (x_*, ξ_*) in the closed-loop system, i.e.

$$0 = \underbrace{\begin{bmatrix} \overline{A} + BK_x & BK_\xi & \widehat{A} + BK_Q \\ \overline{C} & 0 & \widehat{C} \end{bmatrix}}_{\mathcal{A}+\mathcal{B}\mathcal{K}=\mathcal{Z}_1\mathcal{G}} \begin{bmatrix} Z(x_*) \\ \xi_* \end{bmatrix} + \underbrace{\begin{bmatrix} E & 0 \\ 0 & -I \end{bmatrix}}_{\mathcal{L}} \begin{bmatrix} d \\ r \end{bmatrix},$$

such that

(x_*, ξ_*) is globally exponentially stable (hence, boundedness of solutions)

$0 = CZ(x_\star) - r = e_\star$ (regulation).

The designed regulator $\begin{cases} \dot{\xi} = & e \\ u = & \mathcal{K} \begin{bmatrix} x \\ \xi \\ Q(x) \end{bmatrix} \end{cases}$ is a nonlinear PI controller. The SDP gives a method to tune offline the parameters of a PI controller for nonlinear systems.

Example - A flexible robot arm (continued)

$$\dot{x}_1 = x_2 + d_1, \quad \dot{x}_2 = -\frac{K_c}{J_2}x_1 - \frac{F_2}{J_2}x_2 + \frac{K_c}{J_2 N_c}x_3 - \frac{mgd}{J_2}\cos x_1 + d_2$$

$$\dot{x}_3 = x_4 + d_3, \quad \dot{x}_4 = -\frac{K_c}{J_1 N_c}x_1 + \frac{K_c}{J_1 N_c^2}x_3 - \frac{F_1}{J_1}x_4 + \frac{1}{J_1}u + d_4$$

Experiment

$x_0 \in [-0.1, 0.1]^4$

$u \in [-0.1, 0.1]$

$T = 10$

$$Q(x) = \cos x_1 \qquad R_Q = R_Q^\top = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \mathcal{X} = \mathbb{R}^4$$

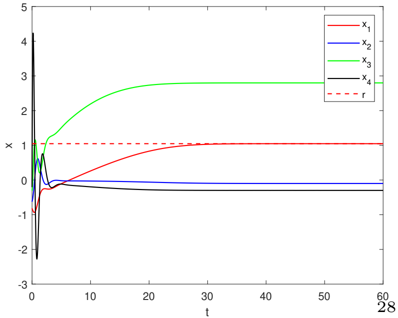
$$y = x_1,\ r = \frac{\pi}{3},\ d = \begin{bmatrix} 0.1 & 0.2 & 0.3 & 0.4 \end{bmatrix}^\top$$

Results The solution of the SDP returns

$$\mathcal{K} = \begin{bmatrix} -3.6314 & -5.9014 & -5.1133 & -1.0990 & -0.9704 & -0.0001 \end{bmatrix}$$

and allows the designer to tune the PI

$$\text{controller} \begin{cases} \dot{\xi} = e = r - y \\ u = \mathcal{K} \begin{bmatrix} x \\ \xi \\ \cos(x_1) \end{bmatrix} \end{cases}$$



Practical considerations on the collection of data

To design our PI controller we collected the dataset $\mathbb{D} := \{x_i, \xi_i, u_i, \dot{x}_i, e_i\}_{i=0}^{T-1}$ from the augmented system

$$\begin{bmatrix} \dot{x} \\ \dot{\xi} \end{bmatrix} = \underbrace{\begin{bmatrix} \overline{A} & 0 & \widehat{A} \\ \overline{C} & 0 & \widehat{C} \end{bmatrix}}_{\mathcal{A}} \begin{bmatrix} x \\ \xi \\ Q(x) \end{bmatrix} + \underbrace{\begin{bmatrix} B \\ 0 \end{bmatrix}}_{\mathcal{B}} u + \underbrace{\begin{bmatrix} E & 0 \\ 0 & -I \end{bmatrix}}_{\mathcal{E}} \begin{bmatrix} d \\ r \end{bmatrix}$$

and arranged it into the data matrices $U_0, \mathcal{Z}_0, \mathcal{Z}_1, M$ that satisfy

$$\mathcal{Z}_1 = \mathcal{A}\mathcal{Z}_0 + \mathcal{B}U_0 + \mathcal{E}\mathcal{L}M$$

where

$$U_0 := \begin{bmatrix} u_0 & u_1 & \cdots & u_{T-1} \end{bmatrix}$$

$$\mathcal{Z}_0 := \begin{bmatrix} x_0 & x_1 & \cdots & x_{T-1} \\ \xi_0 & \xi_1 & \cdots & \xi_{T-1} \\ Q(x_0) & Q(x_1) & \cdots & Q(x_{T-1}) \end{bmatrix} \quad (\xi_i := \xi(t_i))$$

$$\mathcal{Z}_1 := \begin{bmatrix} \dot{x}_0 & \dot{x}_1 & \cdots & \dot{x}_{T-1} \\ e_0 & e_1 & \cdots & e_{T-1} \end{bmatrix} \quad (e_i := e(t_i))$$

$$\mathcal{L} := \begin{bmatrix} d_0 \\ r_0 \end{bmatrix}, \quad M = \mathbb{1}_{1\times T}$$

As $\mathcal{Z}_1$ depends on the samples $e_i := r - y_i$, if the reference signal r changes, then we need a new dataset to design the PI controller, which is not desirable.

This issue can be overcome if we collect the dataset $\overline{\mathbb{D}} := \{x_i, \xi_i, u_i, \dot{x}_i, y_i\}_{i=0}^{T-1}$, instead of $\mathbb{D} := \{x_i, \xi_i, u_i, \dot{x}_i, e_i\}_{i=0}^{T-1}$ and define

$$\overline{\mathcal{Z}}_1 = \begin{bmatrix} \dot{x}_0 & \dot{x}_1 & \cdots & \dot{x}_{T-1} \\ y_0 & y_1 & \cdots & y_{T-1} \end{bmatrix}$$

instead of $\mathcal{Z}_1$. Consider the system's dynamics with output

$$\begin{bmatrix} \dot{x} \\ y \end{bmatrix} = \underbrace{\begin{bmatrix} \overline{A} & 0 & \widehat{A} \\ \overline{C} & 0 & \widehat{C} \end{bmatrix}}_{\mathcal{A}} \begin{bmatrix} x \\ \xi \\ Q(x) \end{bmatrix} + \underbrace{\begin{bmatrix} B \\ 0 \end{bmatrix}}_{\mathcal{B}} u + \underbrace{\begin{bmatrix} E & 0 \\ 0 & 0 \end{bmatrix}}_{\overline{\mathcal{E}}} \begin{bmatrix} d \\ r \end{bmatrix}$$

With the data matrices $U_0, \mathcal{Z}_0, M$ defined as before, we have the identity

$$\overline{\mathcal{Z}}_1 = \mathcal{A}\mathcal{Z}_0 + \mathcal{B}U_0 + \overline{\mathcal{E}}\mathcal{L}M$$

From here we obtain the result that if the SDP with $\mathcal{Z}_1$ replaced by $\overline{\mathcal{Z}}_1$ is feasible, then a PI controller that solves the output regulation problem exists. This time, if the reference signal r takes a different constant value, the same PI controller continues to regulate the output y to r.[1]

[1] In fact, due to the structure of $\mathcal{A}$, the measurement of ξ_i can be omitted from $\overline{\mathbb{D}}$.

Final comment

- In the case of data perturbed by constant exogenous signals ($\dot{d} = 0$), we found out that the condition

$$\begin{bmatrix} K \\ I_s \\ 0_{1\times s} \end{bmatrix} = \begin{bmatrix} U_0 \\ Z_0 \\ M \end{bmatrix} G, \quad M = \mathbb{1}_{1\times T}$$

  allows us to obtain a data-dependent representation of the closed-loop system in which the effect of d on $A + BK$ is filtered out, i.e.,

$$\dot{x} = (A + BK)Z(x) + Ed = X_1 G Z(x) + Ed$$

- Can this property be generalized to the case of time-varying disturbances $d(t)$?
- Can this generalization be used to solve an output regulation problem in the presence of time-varying exogenous signals?

Summary Lecture 5

- Contractivity We enforced contractivity as a method to design nonlinear controllers from data alternative to cancelling the nonlinearities.
- The design aims at dominating the growth of the nonlinearities.
- Tracking The method allows for a data-driven design of regulators for tracking.
- Noise filtering technique Unmeasured disturbances on data that are generated by known systems are filtered out in the data-based representation.

Designing controllers for nonlinear systems from data remains a challenging problem.

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