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# Steady-state behaviors in nonlinear systems with an application to robust disturbance rejection☆

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**Abstract**

Despite the widespread intuitive appeal of the concept of steady-state response and its use in shaping the asymptotic behavior of control systems, this concept has only been rigorously defined for finite-dimensional, linear time invariant systems. In this paper, we investigate this concept for nonlinear systems, following some classical developments in nonlinear dynamics. As an application, we show how the concept in question plays a role of paramount importance in the design of control laws for asymptotic tracking and disturbance attenuation.


*Keywords:* Limit sets; Steady-state response; Behaviors; Internal model; Regulation; Nonlinear control

## 1. The classical notion of steady state

One of the main concerns in the analysis and design of control systems, is the ability to influence or shape the response of a given system to assigned external inputs. This can sometimes be achieved by finding the open-loop input which generates a prescribed trajectory. On the other hand, the use of closed-loop control is almost always the solution of choice in the presence of uncertainties affecting the control systems itself as well as the external inputs to which the response has to be shaped. Among various possible criteria by means of which responses can be analyzed and classified, a classical viewpoint – dating back to the origins of control theory – consists on the separation between *steady state* and *transient* responses.

There are several well-known strong arguments in support of the important role played by the idea of a steady-state response in system analysis and design. On one hand, in a large number of cases it is actually required, as a design specification, that the controlled system evolves, as time increases, toward a steady state in which one or more variables describing its behavior exactly match one or more prescribed functions of time. This is for instance the case in the classical *set-point control* problem, in which the output of the controlled system is required to asymptotically converge to a fixed (but otherwise arbitrary or undetermined) value. Another well-known instance is the case in which the output of a system is required to asymptotically track (or reject) a prescribed sinusoidally varying trajectory (or disturbance) of fixed frequency (but otherwise arbitrary in amplitude and phase). On the other hand, it is well known – at least in linear system theory – that the ability to analyze and shape the steady-state response to sinusoidally varying inputs also provides a powerful tool for the analysis and, to a some extent, for the design of the transient behavior.

Given the central importance of the notion of steady-state response, it is somewhat surprising that a rigorous investigation and delineation of this concept has never been fully developed in the system and control literature, especially for nonlinear systems. The separation between steady and transient states presumes, of course, the ability to be able to discern whether or not a given system exhibits either one of these two kinds of behavior. In this respect, a precise, but somewhat restricted, definition of steady state is the one that can be found in the classical textbook of Gardner and Barnes (1942): "A dynamical system is said to be in the *steady state* when the variables describing its behavior are either invariant with time, or are (sections of) periodic functions of time. A dynamical system is said to be in the *transient* (or unsteady) *state* when it is not in

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steady state.'' In this distinction there is no predetermined separation between inputs and outputs. Rather, the system is only analyzed in terms of how the variables describing its behavior depend on time. This viewpoint applies to general dynamical systems and not necessarily to control systems with input and output variables; it is a precursor (at least so long as the notion of steady state is concerned) of the ''behavioral'' viewpoint proposed in much more recent times by Willems (1991). Note also that, consistently with the behavioral framework, this definition speaks of steady state and not of steady-state *response*. However, if among the variables describing the behavior of the system one or more can be viewed as inputs, and if all such variables are either constant or periodically varying, then it makes sense to speak of a steady-state response whenever all other variables describing the behavior of the system also exhibit the same periodic variation.

This notion of steady state of course, is quite restrictive, as it only applies to cases in which all relevant variables which describe the behavior of a dynamical systems are *periodic* (constant in particular) functions of time. It excludes a number of situations that one would under any circumstances consider as manifestations of a steady-state behavior. For instance, it excludes the simple case in which the variables describing the behavior of the system can be expressed as linear combinations of sinusoidal functions (of time) with at least two frequencies whose ratio is not a rational number. In this case, the variables in question are not periodically varying, but by all means it would be natural to say that the system is still in steady state.

Motivated by this classical idea of a steady state (extended to cover the case of irrationally related sinusoidal functions of time) and by the fact that, in a stable linear system, any transient state asymptotically approaches a steady state, it is a common practice to regard a steady state as a kind of *limit* behavior. From this viewpoint, the steady state can be looked at as either the limit behavior which is approached when the *actual* time $t$ tends to $+\infty$ or, respectively, the limit behavior which is approached when the *initial* time $t_0$ tends to $-\infty$. The two alternatives are equivalent for a stable linear system. From this viewpoint we note, for instance, that in the classical book of James, Nichols, and Phillips (1947) it is observed that ''the *transient response* of [a linear] filter is the difference between the actual output of the filter for $t > t_0$ and the asymptotic form that it approaches'' and that ''only when a filter is stable it is possible to speak with full generality of its response to an input that starts indefinitely far in the past.'' In slightly more general terms, the book of Zadeh and Desoer (1963) defines a ''*ground state* [of the system], if it exists, [as] the limiting terminal state of [the system] when the zero input is applied, . . ., provided the limiting state $\gamma$ is the same for all initial states'' and afterward define ''the *steady-state response* [of the system] to an input $u_{(t_0,t]}$ [as] the limit, if it exists, of the ground-state response of [the system] to $u$ as $t_0 \to -\infty$.'' Furthermore it is observed that ''usually [the system] and $u$ are such that, in [the expression of the response], $\gamma$ can be replaced by an arbitrary initial state $\alpha$ without affecting the limiting value of the response as $t_0 \to -\infty$.''

## 2. Limit sets

### 2.1. The limit set of a point

The principle inspiring the study of the steady-state response of a linear system to sinusoidally varying inputs, which had so much influence in the analysis and design of linear control systems, is indeed the same principle which is behind the investigation of forced oscillations in nonlinear systems, a classical problem with its origin in celestial mechanics. In this respect it must be stressed that for a nonlinear system forced by a sinusoidally varying input, the situation is far more complex than those outlined above, with the possibility of one, or several, forced oscillations with varying stability characteristics occurring. In addition, the fundamental harmonic of these periodic responses may agree with the frequency of the forcing term (harmonic oscillations), or with integer multiples or divisors of the forcing frequency (higher harmonic, or subharmonic, oscillations) or none of the above. Despite a vast literature on nonlinear oscillations, only for second order systems is there much known about the existence and stability of forced oscillation and, in particular, which of these kinds of periodic responses might be asymptotically stable. Essentially most of methods for determining the existence and stability of forced periodic trajectories repose on H. Poincaré's classical idea of seeking the existence of fixed points for the map that associates with any (initial) condition $x(0)$, the point $x(T)$ reached after $T$ units of time. In fact, fixed points of this map are points from which a periodic trajectory of period $T$ is generated. On the other hand, while an isolated periodic motion is truly nonlinear phenomenon, oscillations are not the only nontrivial asymptotic behaviors for nonlinear systems, as the actual time tends to $+\infty$ or, respectively, as the initial time tends to $-\infty$.

This motivates the need for a fresh approach to the whole question of defining the steady state of a nonlinear system, capable of covering the largest possible number of applications. Since the idea of considering a system in steady state when the variables describing its behavior are periodically (or almost periodically) varying is too restrictive, it is reasonable to try to look at the other classical characterizations outlined above, associated to the ''limit behavior'' of the system. Fundamental contributions, in the theory of dynamical systems, toward the characterization of such behavior are the works of H. Poincaré and G.D. Birkhoff. In particular Birkhoff, in his classical 1927 essay (which was considered by J. Moser ''a continuation of Poincaré's profound and extensive work on celestial mechanics''), provided the appropriate definitions that made him possible to claim that ''with an arbitrary dynamical system . . . there is associated always a closed set of 'central motions' which do possess this property of regional recurrence, toward which all other motions of the system in general tend asymptotically'' (Birkhoff, 1927, p. 190).

The first of such definitions is the concept of $\omega$-limit (or $\alpha$-limit) set of a given point, which is consists in what follows. Consider an *autonomous* ordinary differential equation

$$\dot{x} = f(x) \tag{1}$$

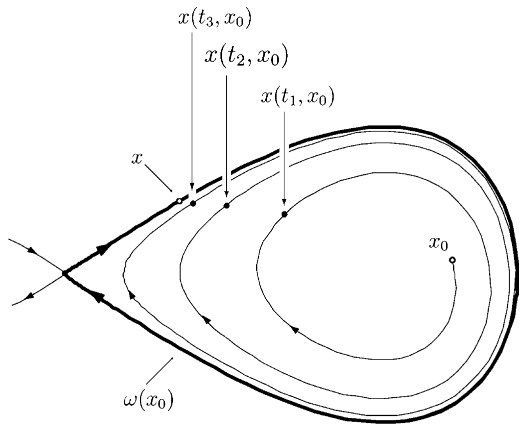


Fig. 1. The $\omega$-limit set of a point.

with $x \in \mathbb{R}^n$, $t \in \mathbb{R}$. It is well known that, if $f : \mathbb{R}^n \to \mathbb{R}^n$ is locally Lipschitz, for all $x_0 \in \mathbb{R}^n$ the solution of (1) with initial condition $x(0) = x_0$, denoted by $x(t, x_0)$, exists on some open interval of the point $t = 0$ and is unique. Assume, in particular, that $x(t, x_0)$ is defined for all $t \geq 0$. A point $x$ is said to be an $\omega$-limit *point* of the motion $x(t, x_0)$ if there exists a sequence of times $\{t_k\}$, with $\lim_{k\to\infty} t_k = \infty$, such that

$$\lim_{k\to\infty} x(t_k, x_0) = x.$$

The $\omega$-limit *set* of a point $x_0$, denoted $\omega(x_0)$, is *the union* of all $\omega$-limit points of the motion $x(t, x_0)$ (see Fig. 1).

Likewise, assume that $x(t, x_0)$ is defined for all $t \leq 0$. A point $x$ is said to be an $\alpha$-limit *point* of the motion $x(t, x_0)$ if there exists a sequence of times $\{t_k\}$, with $\lim_{k\to\infty} t_k = -\infty$, such that

$$\lim_{k\to\infty} x(t_k, x_0) = x.$$

The $\alpha$-limit *set* of a point $x_0$, denoted $\alpha(x_0)$, is the union of all $\alpha$-limit points of the motion $x(t, x_0)$.

It is obvious from this definition that an $\omega$-limit point *i* s not necessarily a limit of $x(t, x_0)$ as $t \to \infty$, because the solution in question may not admit any limit as $t \to \infty$. It happens though, that if the motion $x(t, x_0)$ is *bounded*, then $x(t, x_0)$ asymptotically approaches *the set* $\omega(x_0)$. This property is precisely characterized in the following statement (Birkhoff, 1927, p. 198).

**Lemma 1.** *Suppose there is a number $M$ such that $\|x(t, x_0)\| \leq M$ for all $t \geq 0$. Then, $\omega(x_0)$ is a nonempty compact connected set, invariant under* (1)*. Moreover, the distance of $x(t, x_0)$ from $\omega(x_0)$ tends to 0 as $t \to \infty$.*

One of the remarkable features of $\omega(x_0)$, as indicated in Lemma 1, is the fact that this set is *invariant* for (1). Invariance means that for all initial condition $\bar{x}_0 \in \omega(x_0)$ the solution $x(t, \bar{x}_0)$ of (1) exists for all $t \in (-\infty, +\infty)$ and that $x(t, \bar{x}_0) \in \omega(x_0)$ for all such $t$. In particular, it is somewhat surprising to observe that, even in case the solution $x(t, x_0)$ is not defined for all negative times, the solution $x(t, \bar{x}_0)$, i.e. the solution passing through a point $\bar{x}_0$ of the $\omega$-limit set of $x_0$, is always defined for all negative (and positive) times, and is moreover bounded, since the set $\omega(x_0)$ is compact. Put in different terms, the set $\omega(x_0)$ is filled by motions of (1) which are bounded backward and forward in time. The other remarkable feature is that $x(t, x_0)$ approaches $\omega(x_0)$ as $t \to \infty$, in the sense that the distance *of the point $x(t, x_0)$ from the set $\omega(x_0)$* tends to 0 as $t \to \infty$ (recall that the distance of a point $x$ of $\mathbb{R}^n$ from a set $S$ of $\mathbb{R}^n$, denoted $\mathrm{dist}(x, S)$, is the nonnegative number $\inf_{y\in S}\|x - y\|$). Corresponding properties hold, of course, for the $\alpha$-limit set of a motion.

We recognize in these properties some of the keywords which have already occurred in the summary of the classical notion of steady state: the existence of motions which are defined backward and forward in time (as periodic motions are) and the convergence of any actual motion to a set filled with such special motions, as observed by Birkhoff. However, as promising as this seems, there are some very limiting features about a definition of steady-state behavior based only on the concept of $\omega$-limit set of a point. Before proceeding further, we illustrate this concept with a couple of simple examples.

**Example.** Let the stable, one-dimensional, linear system

$$\dot{y} = -y + u \tag{2}$$

be forced by the input $u(t) = U \sin(\omega t + \phi)$. According to the classical definition given in the previous section, this system is in steady state if $y(t)$ is a periodic function of period $T = 2\pi/\omega$ and it is well known that this occur if $y(0)$ is chosen appropriately. To determine the value of $y(0)$ which makes this happen, i.e. to evaluate the steady-state response of the system to the given $u(t)$, regard $u(t)$ as one of the state variables of a harmonic oscillator oscillating at angular frequency $\omega$, that is, set

$$x_1(t) = u = U \sin(\omega t + \phi),$$
$$x_2(t) = \frac{1}{\omega}\dot{u}(t) = U \cos(\omega t + \phi)$$

in which case

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 & \omega \\ -\omega & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}. \tag{3}$$

Set also $y = x_3$, to represent the system, along with its forcing input, as a three-dimensional autonomous linear system

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} = \begin{pmatrix} 0 & \omega & 0 \\ -\omega & 0 & 0 \\ 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \quad x(0) = \begin{pmatrix} U \sin\phi \\ U \cos\phi \\ y(0) \end{pmatrix}. \tag{4}$$

Integration of this system is a standard exercise. Change $x_3$ into

$$z = x_3 - \Pi \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

with $\Pi$ solution of the Sylvester equation

$$\Pi\begin{pmatrix} 0 & \omega \\ -\omega & 0 \end{pmatrix} = -\Pi + (\,1 \quad 0\,).$$

This yields

$$\dot{z} = -z$$

from which it is concluded that

$$\begin{aligned} y(t) &= x_3(t) \\ &= \mathrm{e}^{-t}\left[y(0) - \Pi\begin{pmatrix} U\sin\phi \\ U\cos\phi \end{pmatrix}\right] + \Pi\begin{pmatrix} U\sin(\omega t+\phi) \\ U\cos(\omega t+\phi) \end{pmatrix}. \end{aligned}$$

It is seen from this that the value of $y(0)$ for which $x(t, x_0)$ is periodic, i.e. for which the system is in steady state according to the classical definition, is

$$y(0) = \Pi\begin{pmatrix} U\sin\phi \\ U\cos\phi \end{pmatrix}.$$

If this is the case, then

$$y(t) = \Pi\begin{pmatrix} U\sin(\omega t+\phi) \\ U\cos(\omega t+\phi) \end{pmatrix},$$

and this is precisely what is commonly considered a steady-state response of system (2) to the input $u(t) = U\sin(\omega t + \phi)$.

Viewing system (2) driven by the harmonic input $u(t)$ as a single autonomous system, such as (4), has a number of advantages. For instance, it simplifies the calculation of the steady-state response to a given *family* of inputs: those obtained for different values of $U$ and $\phi$. As a matter of fact, what we need to consider in this case is no longer a specific response to a forcing input, but rather the behavior of an autonomous system as certain initial conditions (those of $x_1$ and $x_2$ in (4)) are allowed to vary. In this respect, the previous conclusion can be rephrased by saying that system (4) is in steady state if and only if the initial condition $x(0)$ is a point of the plane

$$P = \left\{x \in \mathbb{R}^3 : x_3 = \Pi\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right\}.$$

Revisiting this elementary analysis from the viewpoint of the concept of $\omega$-limit set, it is readily observed that, for any $x_0 \in \mathbb{R}^3$, the set $\omega(x_0)$ is the ellipse defined as follows:

$$\omega(x_0) = \left\{\begin{array}{l} (x_1, x_2, x_3) \in \mathbb{R}^3 : x_1^2 + x_2^2 = x_{1,0}^2 + x_{2,0}^2, \\ x_3 = \Pi\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \end{array}\right\}.$$

**Example.** As a second, nonlinear, example, consider now the classical Van der Pol oscillator, written in state-space form as

$$\dot{x} = y, \qquad \dot{y} = -x - \varepsilon(1 - x^2)y \tag{5}$$

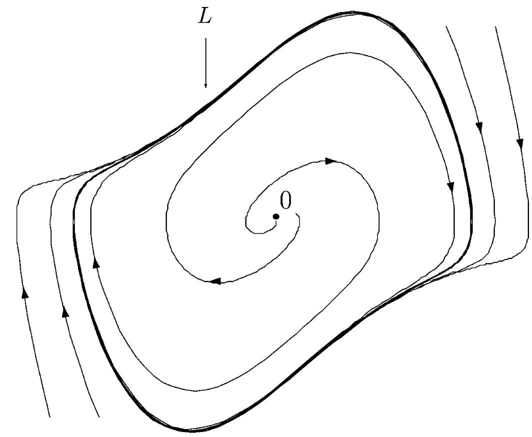


Fig. 2. The phase portrait of Van der Pol's oscillator (5).

in which, as it is well known, the damping term $-\varepsilon(1 - x^2)y$ models the effect of a nonlinear resistor (see Khalil, 2002). From the phase portrait of this system (depicted in Fig. 2 for $\varepsilon = 1$) it is seen that all motions except the trivial motion occurring for $x_0 = 0$ approach, as $t \to \infty$, a limit cycle $L$. The system is in steady state, according to the classical definition, if and only if either $x(0) = 0$ or $x(0) \in L$. Note also that the limit cycle $L$ is the $\omega$-limit set of any point $x_0 \neq 0$ while the point $x = 0$ is the $\omega$-limit set of the point $x_0 = 0$.

Since any motion $x(t, x_0)$ which is bounded in positive time asymptotically approaches the $\omega$-limit set $\omega(x_0)$ as $t \to \infty$, one may be tempted to look, for a system (1) in which all motions are bounded in positive time, at the *union* of the limit sets of all points $x_0$, that is, at the set

$$\Omega = \bigcup_{x_0 \in \mathbb{R}^n} \omega(x_0)$$

and to say that the system is in steady state if $x(0) \in \Omega$.

In this respect, note that in system (4) the set $\Omega$ is the entire plane $P$, while in system (5) the set $\Omega$ consists of the union of the equilibrium point $\{0\}$ and of the limit cycle $L$. Since all motions of systems (4) and (5) are bounded in positive time and asymptotically approach the (respective) sets $\Omega$ as $t$ increases, it may seem that the set in question is the right object to look at for a definition of steady state. There is however a remarkable difference between the two cases: the type of convergence.

In both cases, as Lemma 1 says, the distance of $x(t, x_0)$ from $\Omega$ tends to 0 as $t \to \infty$, but while in the first example the convergence is uniform in $x_0$, for all $x_0$ within a set of finite distance from $\Omega$, in the second example it is not. To examine this difference in more detail, recall that to say that the distance of $x(t, x_0)$ from a set $S$ tends to 0 as $t \to \infty$ is to say that for every $\varepsilon$ there exists $T$ such that

$$\mathrm{dist}(x(t, x_0), S) \le \varepsilon \quad \text{for all } t \ge T. \tag{6}$$

The number $T$ in this expression obviously depends on $\varepsilon$, but it also generally depends on $x_0$; as a matter of fact, the larger the distance of $x_0$ from $S$ is, the more one can expect to wait until $x(t, x_0)$ comes within an $\varepsilon$-distance from $S$. However, one might hope that if the initial distance of $x_0$ from $S$ is bounded by a fixed number, then the time required to get within an ε-distance from $S$ would only depend on ε and not on $x_0$. Formalizing this concept, let $B_d(S)$ denote the set of all points whose distance from $S$ does not exceed a given number $d$, that is

$$B_d(S) = \{x \in \mathbb{R}^n : \text{dist}(x, S) \leq d\}.$$

The distance of $x(t, x_0)$ from $S$ is said to tend to 0, as $t \to \infty$, *uniformly* in $x_0$ on $B_d(S)$, if for every ε there exists $T$, which depends on ε and $d$ but not on $x_0$, such that (6) holds for all $x_0 \in B_d(S)$.

Well, it is readily seen that in the case of example (4) the convergence is uniform, while in the second example it is not. To this end observe that in the first example, in which $\Omega$ coincides with the plane $P$, it is always possible to find two numbers $c_1$ and $c_2$ such that, for all $x$,

$$c_1 \,\text{dist}(x, \Omega) \leq |z| \leq c_2 \,\text{dist}(x, \Omega).$$

Thus, using the fact that $\dot{z} = -z$, one obtains the estimate

$$\text{dist}(x(t, x_0), \Omega) \leq \frac{c_2}{c_1} \text{dist}(x_0, \Omega)\, \mathrm{e}^{-t},$$

from which it immediately follows that $x(t, x_0)$ converges to the set $\Omega$ uniformly in $x_0$ on $B_d(\Omega)$, for any given $d > 0$.

Consider now the second example, in which the set $\Omega$ consists of the union of the equilibrium point $\{0\}$ and of the limit cycle $L$. Observe that all $x_0$'s inside the limit cycle $L$ are within a finite distance $d$ from $L$. All $x_0 \in B_d(\Omega)$ are such that $\text{dist}(x(t, x_0), \Omega) \to 0$ as $t \to \infty$ (as a matter of fact, if $x_0 \neq 0$, the motion $x(t, x_0)$ asymptotically approaches $L$, while, if $x_0 = 0$, the motion $x(t, x_0)$ trivially remains at 0). If the convergence were uniform in $x_0$ on $B_d(\Omega)$, it would be possible, for any choice of ε, to find a number $T$, only depending on ε and not $x_0$, such that (6) holds. This, however, is not the case. In fact, observe that, if $x_0 \neq 0$ is inside $L$, the motion $x(t, x_0)$ is bounded in negative time and remains inside $L$ for all $t \leq 0$ (as a matter of fact, it converges to 0 as $t \to -\infty$). Pick any $x_1 \neq 0$ inside $L$ such that $\text{dist}(x_1, L) > \varepsilon$ and let $T_1$ be the minimal time needed to have $\text{dist}(x(t, x_1), L) \leq \varepsilon$ for all $t \geq T_1$. Now, go backwards $T_0 > 0$ units of time, to the point $x_0 = x(-T_0, x_1)$. Then, the minimal time $T$ needed to have $\text{dist}(x(t, x_0), \Omega) \leq \varepsilon$ for all $t \geq T$ is $T = T_0 + T_1$ and, since $T_0$ can be taken arbitrarily large (while keeping $x_0$ inside the limit cycle, within distance $d$ from $\Omega$), we see that the time $T$ needed to have property (6) fulfilled cannot be made independent of $x_0$.

Now, recall that one of the main motivations for looking into the concept of steady state is the aim to shape the steady-state response of a system to a given (or to a given family of) forcing input(s). But this motivation looses much of its meaning if the time needed to get within an ε-distance from the steady state may grow unbounded as the initial state changes (even when the latter is picked within a fixed bounded set). In other words, uniform convergence to the steady state (which is automatically guaranteed in the case of linear systems) is an indispensable feature to be required in a nonlinear version of this notion. However, as shown above, the set consisting of the union of the $\omega$-limit sets of all points in the state space does not have this property of uniform convergence.

### 2.2. The limit set of a set

As it turns out, the features in question are in fact shared by an object which extends the concept of limit set from the case of a *single* initial point to the case of a *set* of initial points. This object, defined in Bhatia and Szego (1970) as *prolongational* limit set and defined in Hale, Magalhães, and Oliva (2002) and Sell and You (2002) as *omega limit set* of a given set $B$, addresses in a convenient way the issue of uniform convergence (when initial conditions are picked in a bounded set) and also lends itself to the possibility of looking at steady-state behaviors as limits taken when the initial time tends to $-\infty$. The notion in question is defined as follows.

Consider again system (1), let $B$ be a subset of $\mathbb{R}^n$ and suppose $x(t, x_0)$ is defined for all $t \geq 0$ and all $x_0 \in B$. The $\omega$-limit set of $B$, denoted $\omega(B)$, is the set of all points $x$ for which there exists a sequence of pairs $\{x_k, t_k\}$, with $x_k \in B$ and $\lim_{k \to \infty} t_k = \infty$ such that

$$\lim_{k \to \infty} x(t_k, x_k) = x.$$

It is clear from the definition that if $B$ consists of only one single point $x_0$, all $x_k$'s in the definition above are necessarily equal to $x_0$ and the definition in question reduces to the definition of $\omega$-limit set of a point, given earlier. It is also clear from this definition that, if for some $x_0 \in B$ the set $\omega(x_0)$ is nonempty, all points of $\omega(x_0)$ are points of $\omega(B)$. In fact, all such points have the property indicated in the definition, if all the $x_k$'s are taken equal to $x_0$. Thus, in particular, if all motions with $x_0 \in B$ are bounded in positive time,

$$\bigcup_{x_0 \in B} \omega(x_0) \subset \omega(B).$$

However, the converse inclusion is not true in general.

This can be immediately checked on the second of our earlier examples. In fact, consider again the van der Pol oscillator (5), and let the set $B$ be, for instance, a closed disc of sufficiently large radius, to include the limit cycle $L$ in its interior. We already know that $\{0\}$ and $L$, being $\omega$-limit sets of points of $B$, are in $\omega(B)$. But it is also easy to see that any other point inside $L$ is a point of $\omega(B)$. In fact, let $\bar{x}$ be any of such points and pick any sequence $\{t_k\}$ such that $\lim_{k \to \infty} t_k = \infty$. It is seen from the phase portrait that $x(t, \bar{x})$ remains inside $L$ (and hence in $B$) for all negative values of $t$; hence

$$x_k := x(-t_k, \bar{x})$$

is a point in $B$ for all $k$. Since the sequence $\{x_k, t_k\}$ is such that $x(t_k, x_k) = \bar{x}$, the property required for $\bar{x}$ to be in $\omega(B)$ is trivially satisfied. This shows that $\omega(B)$ includes not just $\{0\}$ and $L$, but also all points of the open region bounded by $L$. As a matter of fact, it is not difficult to prove that no other point can be a point of $\omega(B)$. This can be done either by direct arguments, or – more simply – by appealing to the result indicated in the next Lemma, which says that, in this example, all motions with initial conditions in $\omega(B)$ have to be bounded backward in time. Now, it is observed from the phase portrait that for any point $\bar{x}$ which is not on or inside $L$, the motion $x(t, \bar{x})$ is such that

$\|x(t,\bar{x})\| \to \infty$ as $t \to -\infty$. Thus, any of such $\bar{x}$ cannot be a point of $\omega(B)$.

The relevant properties of the $\omega$-limit set of a set, which extend those presented earlier in Lemma 1, can be summarized as follows (see, for instance, Bhatia & Szego, 1970; Hale, Magalhães, & Oliva, 2002; Sell & You, 2002).

**Lemma 2.** *Let $B$ be a nonempty bounded subset of $\mathbb{R}^n$ and suppose there is a number $M$ such that $\|x(t,x_0)\| \leq M$ for all $t \geq 0$ and all $x_0 \in B$. Then $\omega(B)$ is a nonempty compact set, invariant under (1). Moreover, the distance of $x(t,x_0)$ from $\omega(B)$ tends to 0 as $t \to \infty$, uniformly in $x_0 \in B$. If $B$ is connected, so is $\omega(B)$.*

Thus, as it is the case for the $\omega$-limit set of a point, we see that the $\omega$-limit set of a bounded set, being compact and invariant, is filled with motions which exist for all $t \in (-\infty, +\infty)$ and are bounded backward and forward in time (the set of all such trajectories is a *behavior*, in the sense of Willems, 1991). But, above all, we see that the set in question is *uniformly* approached by motions with initial state $x_0 \in B$, a property that the $\omega$-limit set of a point does not have. The property in question makes this notion more suitable, as explained above, for a definition of steady-state behavior of a nonlinear system.

We conclude the section with another property, that will be useful in the sequel.

**Lemma 3.** *If $B$ is a compact set invariant for (1), then $\omega(B) = B$.*

**Proof.** As shown in the previous example, for any $\bar{x} \in B$ it is trivially always possible to find a sequence $\{x_k, t_k\}$, with $x_k \in B$ and $\lim_{k\to\infty} t_k = \infty$ such that $x(t_k, x_k) = \bar{x}$, and this shows that $B \subset \omega(B)$. To show that the reverse inclusion $\omega(B) \subset B$ also holds, pick any point $\tilde{x} \in \omega(B)$, and observe that – by definition – for some sequence $\{x_k\}$ of points of $B$ there is a sequence of times $\{t_k\}$ such that $\lim_{k\to\infty} x(t_k, x_k) = \tilde{x}$. Since $B$ is by assumption invariant for (1), $x(t_k, x_k) \in B$ for all $k$. Thus $\tilde{x}$ is the limit of a sequence of points of $B$ and, since $B$ is also compact, necessarily $\tilde{x} \in B$. □

## 3. Limit sets and stability

It is well known, in the classical theory of the stability of motion, that (in a nonlinear system) an equilibrium point which attracts all motions starting from initial conditions in some (small) open neighborhood of this point is not necessarily stable in the sense of Lyapunov. Examples of systems in which convergence to an equilibrium does not imply stability can even be built in dimension two, as seen for instance from the following classical example due to Vinograd (1957) (and thoroughly analyzed in Hahn (1967, pp. 191–194)).

**Example.** Consider the nonlinear system

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} f(x,y) \\ g(x,y) \end{pmatrix}$$

in which $f(0,0) = g(0,0) = 0$ and

$$\begin{pmatrix} f(x,y) \\ g(x,y) \end{pmatrix} = \frac{1}{(x^2+y^2)(1+(x^2+y^2)^2)} \begin{pmatrix} x^2(y-x)+y^5 \\ y^2(y-2x) \end{pmatrix} \tag{7}$$

for $(x,y) \neq (0,0)$. The phase portrait of this system is the one depicted in Fig. 3. In particular, this system has only one equilibrium at $(x,y) = (0,0)$ and any initial condition $(x_0, y_0)$ in the plane produces a motion that asymptotically tends to this point. However, this equilibrium point is not stable in the sense of Lyapunov, because it is not possible to find, for every $\varepsilon > 0$, a number $\delta > 0$ such that every initial condition in a disc of radius $\delta$ produces a motion which remains for all $t \geq 0$ in a disc of radius $\varepsilon$. As a matter of fact, as is seen from the phase portrait, there are points arbitrarily close to the origin from which the motion always travels a finite fixed distance away from the origin. Thus, no matter how small $\delta$ is chosen, it is impossible to keep the motion within an $\varepsilon$-distance from the origin, if $\varepsilon$ is not large enough.

Note that, in this example, the point $(x,y) = (0,0)$ is an $\omega$-limit point of every point in the plane and thus the set $\Omega$ introduced before, that is, the union of all $\omega$-limit sets of all points in the plane, is simply the point $(0,0)$. However, if $B$ is a disc of sufficiently large radius, centered at the origin, the $\omega$-limit set of $B$ is the nontrivial set consisting of the larger "figure eight" and of all points in its interior. As expected, the set $\omega(B)$ in question is filled with motions of system (7), all bounded in backward and forward time. Note, in particular, that the orbit of any of such motions is a motion in which the $\omega$-limit set and the $\alpha$-limit set coincide (a *homoclinic* orbit).

It should be stressed that while, in general, convergence to an equilibrium may not imply stability of the latter, this is no longer the case if the convergence to the equilibrium is *uniform*

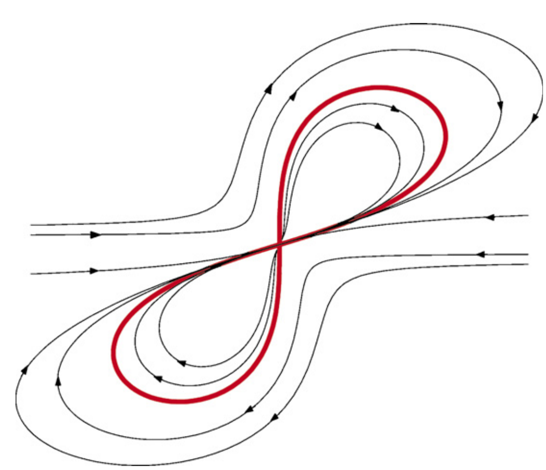

Fig. 3. The phase portrait of Vinograd's system (7).

(in the sense made precise in the previous section). As a matter of fact, uniform convergence to an equilibrium *does imply* stability in the sense of Lyapunov. This is a consequence of the fact that $x(t, x_0)$ depends continuously on $x_0$ (see for example Hahn (1967, p. 181)). Of course, in the previous example the convergence to the equilibrium is not uniform: no matter how large $T$ is taken it is always possible to pick initial states arbitrarily close to the origin (for instance on any of the homoclinic orbits) from which the motion needs a time larger than $T$ to enter a disc of small radius $\varepsilon$.

In this respect, the $\omega$-limit set of a (bounded) set $B$ of initial conditions, which carries with it the property of uniform convergence, appears to be a more satisfactory object to look at in the quest for a set (other than a simple equilibrium) having both the properties of attractivity and stability in the sense of Lyapunov. For motions converging to a closed invariant set $\mathcal{A}$, the notion of asymptotic stability, a straightforward extension of the notion of asymptotic stability of an equilibrium, is defined as follows. Let $\mathcal{A} \subset \mathbb{R}^n$ be a closed set invariant for (1). The set $\mathcal{A}$ is asymptotically stable if the following hold:

(i) for every $\varepsilon > 0$, there exists $\delta > 0$ such that,

$$\text{dist}(x_0, \mathcal{A}) \leq \delta \quad \text{implies} \quad \text{dist}(x(t, x_0), \mathcal{A}) \leq \varepsilon$$

for all $t \geq 0$.

(ii) there exists a number $d > 0$ such that

$$\text{dist}(x_0, \mathcal{A}) \leq d \quad \text{implies} \quad \lim_{t \to \infty} \text{dist}(x(t, x_0), \mathcal{A}) = 0.$$

As in the case of equilibria, for a closed invariant set $\mathcal{A}$ which is asymptotically stable for (1), the *domain of attraction* is the set of all $x_0$ for which $x(t, x_0)$ is defined for all $t \geq 0$ and $\text{dist}(x(t, x_0), \mathcal{A}) \to 0$ as $t \to \infty$.

It is not difficult to show (see for example Celani, 2003; Sontag & Wang, 1995) that if the set $\mathcal{A}$ in $\mathbb{R}^n$ is also *bounded* and hence compact, and the convergence in (ii) is *uniform* in $x_0$, then property (ii) implies property (i). Using this fact and the result of Lemma 2, this yields the following important property.

**Lemma 4.** *Let $B$ be a nonempty bounded subset of $\mathbb{R}^n$ and suppose there is a number $M$ such that $\|x(t, x_0)\| \leq M$ for all $t \geq 0$ and all $x_0 \in B$. Then $\omega(B)$ is a nonempty compact set, invariant under* (1). *Suppose also that $\omega(B)$ is contained in the interior of $B$. Then, $\omega(B)$ is asymptotically stable, with a domain of attraction that contains $B$.*

## 4. The steady-state behavior of a nonlinear system

Consider now again system (1), with initial conditions in a closed subset $X \subset \mathbb{R}^n$. Suppose the set $X$ is *positively* invariant, which means that for all initial conditions $x_0 \in X$, the solution $x(t, x_0)$ exists for all $t \geq 0$ and $x(t, x_0) \in X$ for all $t \geq 0$. The motions of this system are said to be *ultimately bounded* if there is a bounded subset $B$ with the property that, for every compact subset $X_0$ of $X$, there is a time $T > 0$ such that $x(t, x_0) \in B$ for all $t \geq T$ and all $x_0 \in X_0$. In other words, if the motions of the system are ultimately bounded, every motion eventually enters and remains in the bounded set $B$.

Note that, since by hypothesis $X$ is positively invariant, there is no loss of generality in assuming $B \subset X$ in the definition above. Hence, there exists a number $M$ such that $\|x(t, x_0)\| \leq M$ for all $t \geq 0$ and all $x_0 \in B$ and, from Lemma 2, it is concluded that the set $\omega(B)$ is nonempty (and has all the properties indicated in that Lemma). It is worth stressing that, for a system whose motions are ultimately bounded, the set $\omega(B)$ is a unique well-defined set, regardless of how $B$ is taken.

**Lemma 5.** *Let the motions of* (1) *be bounded and let $B'$ be any other bounded subset of $X$ with the property that, for every compact subset $X_0$ of $X$, there is a time $T > 0$ such that $x(t, x_0) \in B'$ for all $t \geq T$ and all $x_0 \in X_0$. Then, $\omega(B) = \omega(B')$.*

**Proof.** Let $\bar{x}$ be a point of $\omega(B')$. By hypothesis, there exists a sequence $\{\bar{x}_k, \bar{t}_k\}$, with $\bar{x}_k \in B'$ and $\lim_{k \to \infty} \bar{t}_k = \infty$ such that $x(\bar{t}_k, \bar{x}_k)$ converges to $\bar{x}$ as $k \to \infty$. As all such $\bar{x}_k$'s are in a compact subset of $X$, by definition of $B$ there exist a time $T > 0$ such that all points $x_k = x(T, \bar{x}_k)$ are points of $B$. Set $t_k = \bar{t}_k - T$ and consider the sequence $\{x_k, t_k\}$. Trivially $x(t_k, x_k)$, being equal to $x(\bar{t}_k, \bar{x}_k)$, converges to $\bar{x}$ as $k \to \infty$. Thus, $\bar{x}$ is a point of $\omega(B)$ also. We have shown in this way that $\omega(B') \subset \omega(B)$. Reversing the role of the two sets shows that $\omega(B) \subset \omega(B')$, that is, that the two sets in question are identical. □

For systems whose motions are ultimately bounded, the notion of steady state can be defined as follows.

**Definition.** Suppose the motions of system (1), with initial conditions in a closed and positively invariant set $X$, are ultimately bounded. A *steady-state motion* is any motion with initial condition in $x(0) \in \omega(B)$. The set $\omega(B)$ is the *steady-state locus* of (1) and the restriction of (1) to $\omega(B)$ is the *steady-state behavior* of (1).

The notion thus introduced recaptures the classical notion of steady state for linear systems and provides a new powerful tool to deal with similar issues in the case of nonlinear systems. To see how this notion includes the classical viewpoint, consider an $n$-dimensional, single-input, linear system

$$\dot{z} = Fz + Gu \tag{8}$$

forced by the harmonic input $u(t) = U \sin(\omega t + \phi)$. As shown in the first example, a simple method to determine the periodic motion of (8) consists in viewing the forcing input $u(t)$ as provided by an autonomous signal generator of the form (3) and in analyzing the state state behavior of the associated augmented system

$$\begin{pmatrix} \dot{w} \\ \dot{z} \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} 0 & \omega \\ -\omega & 0 \end{pmatrix} & 0 \\ G\begin{pmatrix} 1 & 0 \end{pmatrix} & F \end{pmatrix} \begin{pmatrix} w \\ z \end{pmatrix} \tag{9}$$

As a matter of fact, let $\Pi$ be the unique solution of the Sylvester equation

$$\Pi\begin{pmatrix} 0 & \omega \\ -\omega & 0 \end{pmatrix} = F\Pi + G(1 \quad 0)$$

and observe that the graph of the linear map

$$\pi: \quad \mathbb{R}^2 \to \mathbb{R}^n \\ w \mapsto \Pi w$$

is an invariant subspace for the system (9). Since all trajectories of (9) approach this subspace as $t \to \infty$, the steady-state behavior of (9) is determined by the restriction of its motion to this invariant subspace.

Revisiting this analysis from the viewpoint of the more general notion of steady state introduced above, let $W \subset \mathbb{R}^2$ be a set of the form

$$W = \{w \in \mathbb{R}^2 : \|w\| \leq c\} \tag{10}$$

in which $c$ is a fixed number, and suppose the set of initial conditions for (9) is $W \times \mathbb{R}^n$. This is in fact the case when the problem of evaluating the periodic response of (8) to harmonic inputs whose amplitude does not exceed a fixed number $c$ is addressed. The set $W$ is compact and invariant for the upper subsystem of (9). Therefore, as shown before, the $\omega$-limit set of $W$ under the motion of the upper subsystem of (9) is the subset $W$ itself.

The set $W \times \mathbb{R}^n$ is closed and positively invariant for the full system (9) and, moreover, since the lower subsystem of (9) is a linear asymptotically stable system driven by a bounded input, it is immediate to check that the motions of system (9), with initial conditions taken in $W \times \mathbb{R}^n$, are ultimately bounded. As a matter of fact, any bounded set $B$ of the form

$$B = \{(w, z) \in \mathbb{R}^2 \times \mathbb{R}^n : w \in W, \|z - \Pi w\| \leq d\}$$

in which $d$ is any positive number, has the property indicated in the definition of ultimate boundedness. Note also that any such $B$ satisfies $B \subset W \times \mathbb{R}^n$. It is easy to check that

$$\omega(B) = \{(w, z) \in \mathbb{R}^2 \times \mathbb{R}^n : w \in W, z = \Pi w\},$$

that is, $\omega(B)$ is the graph of the restriction of the map $\pi$ to the set $W$. Note that $\omega(B)$ is independent of the choice of $B$ (so long as $B$ is a set having the properties indicated in the definition of ultimate boundedness). The restriction of (9) to the invariant set $\omega(B)$ characterizes the steady-state behavior of (8) under the family of all harmonic inputs of fixed angular frequency $\omega$, and amplitude not exceeding $c$.

A similar result, that is, the fact that the steady-state locus is the graph of a map, can be reached if the signal generator is any nonlinear system, with initial conditions chosen in a compact invariant set $W$. More precisely, consider an augmented system of the form (we retain, throughout, the assumption that both $s(w)$ and $q(w)$ are locally Lipschitz functions)

$$\dot{w} = s(w), \qquad \dot{z} = Fz + Gq(w), \tag{11}$$

in which $w \in W \subset \mathbb{R}^r$, $z \in \mathbb{R}^n$, and assume that: (i) the eigenvalues of $F$ have negative real part, (ii) the set $W$ is a compact set, invariant for the upper subsystem of (11). As in the previous example, the $\omega$-limit set of $W$ under the motion of the upper subsystem of (11) is the subset $W$ itself. Moreover, since the lower subsystem of (11) is a linear asymptotically stable system driven by the bounded input $u(t) = q(w(t))$, it is easy to check that the motions of system (11), with initial conditions taken in $W \times \mathbb{R}^n$, are ultimately bounded. As a matter of fact, so long as $w(0) \in W$, the input $q(w(t))$ to the lower subsystem of (11) is bounded by some fixed number $U$ and standard arguments can be invoked to show that

$$\|z(t)\| \leq K\,\mathrm{e}^{-\lambda t}\|z(0)\| + LU$$

for all $t \geq 0$, in which $K, \lambda$ and $L$ are appropriate positive numbers. Thus, any bounded set $B$ of the form

$$B = \{(w, z) \in \mathbb{R}^r \times \mathbb{R}^n : w \in W, \|z\| \leq 2LU\}$$

has the property indicated in the definition of ultimate boundedness.

Moreover, it is possible to show that, regardless of how $B$ is taken, $\omega(B)$ is the graph of the map

$$\pi: \quad W \to \mathbb{R}^n \\ w \mapsto \pi(w),$$

defined by

$$\pi(w) = \int_{\infty}^{0} \mathrm{e}^{-F\tau} Gq(w(\tau, w))\,\mathrm{d}\tau. \tag{12}$$

The explanation of this fact reposes on the following arguments. First of all, observe that – since $q(w(t, w))$ is by hypothesis a bounded function of $t$ and all eigenvalues of $F$ have negative real part – the improper integral on the right-hand side of (12) exists. Then, a simple calculation shows that the graph of the map $\pi$ is invariant for (11). To see why this is the case, pick any initial condition $(w_0, z_0)$ on the graph of $\pi$, i.e. with $z_0 = \pi(w_0)$ and compute the solution $z(t)$ of the lower equation of (11) by means of the classical variation of constants formula, to obtain

$$z(t) = \mathrm{e}^{Ft}\int_{-\infty}^{0} e^{-F\tau} Gq(w(\tau, w_0))\,\mathrm{d}\tau \\ + \int_0^t \mathrm{e}^{F(t-\tau)} Gq(w(\tau, w_0))\,\mathrm{d}\tau$$

From this, an easy manipulation yields $z(t) = \pi(w(t, w_0))$, proving the invariance of the graph of $\pi$ for (11). Then, it is immediately concluded that any point of the graph of $\pi$ is necessarily a point of $\omega(B)$. To complete the proof of the claim it remains to show that no other point of $W \times \mathbb{R}^n$ can be a point of $\omega(B)$. But this is a direct consequence of the fact that $F$ has eigenvalues with negative real part. In fact, this assumption implies that all motions of (11) whose initial condition is not on the graph of $\pi$ are unbounded in backward time and therefore cannot be contained in $\omega(B)$, which we know is a bounded set.

There are various ways in which this result can be generalized. For instance, it can be extended to describe the steady-state response of a nonlinear system

$$\dot{z} = f(z,u) \tag{13}$$

in the neighborhood of a locally exponentially stable equilibrium point. To this end, suppose that $f(0,0) = 0$ and that the matrix

$$F = \left[\frac{\partial f}{\partial z}\right](0,0)$$

has all eigenvalues with negative real part. Then, it is well known (see for example Hahn (1967, p. 275)) that it is always possible to find a compact subset $Z \subset \mathbb{R}^n$, which contains $z = 0$ in its interior and a number $\sigma > 0$ such that, if $z_0 \in Z$ and $\|u(t)\| \leq \sigma$ for all $t \geq 0$, the solution of (13) with initial condition $z(0) = z_0$ satisfies $z(t) \in Z$ for all $t \geq 0$. Suppose that the input $u$ to (13) is produced, as before, by a signal generator of the form

$$\dot{w} = s(w), \qquad u = q(w) \tag{14}$$

with initial conditions chosen in a compact invariant set $W$ and, moreover, suppose that, $\|q(w)\| \leq \sigma$ for all $w \in W$. If this is the case, the set $Z \times W$ is positively invariant for

$$\dot{w} = s(w), \qquad \dot{z} = f(z, q(w)), \tag{15}$$

and the motions of the latter are ultimately bounded, with $B = Z \times W$. The set $\omega(B)$ may have a complicated structure but it is possible to show, using the Center Manifold theorem, that if $s(0) = 0$ and the matrix

$$S = \left[\frac{\partial s}{\partial w}\right](0)$$

has all eigenvalues on the imaginary axis and if $Z$ and $B$ are small enough, the set in question can still be expressed as the graph of a map $z = \pi(w)$. Specifically, the graph in question is precisely the center manifold of (15) at $(0,0)$.

Of course, the possibility of expressing the steady-state locus of a system of the form (15) as the graph of a map $z = \pi(w)$ is not necessarily tied to the assumption that the equilibrium point $(w,z) = (0,0)$ of (13) be locally exponentially stable. This is shown for instance in the following simple example. It should be stressed, though, that if the equilibrium $(w,z) = (0,0)$ of (13) is not locally exponentially stable, the map $\pi$ may fail to be differentiable at the point $w = 0$.

**Example.** Consider the system

$$\dot{z} = -z^3 + u \tag{16}$$

forced by an input $u = w_1$ provided by the autonomous signal generator (3), in which we assume, for simplicity $\omega = 1$ and $W$ as in (10). The set $W \times \mathbb{R}$ is positively invariant and, by means of simple arguments, it is easy to see that the motions with initial conditions in $W \times \mathbb{R}$ are ultimately bounded. Now, by means of the classical method of Poincaré for the study of periodic solutions of a nonlinear differential equation (whose details cannot be included here for reasons of space, but which can be found for instance in Byrnes, Gilliam, Isidori, and Ramsey (2003)), it is possible to show that, for each $w(0) \in W$, there is one and only one value $z(0) \in \mathbb{R}$ from which the motion of

$$\dot{w}_1 = w_2, \qquad \dot{w}_2 = -w_1, \qquad \dot{z} = -z^3 + w_1$$

is a periodic motion. The set of all such pairs identifies a map $\pi : W \to \mathbb{R}$, whose graph coincides with the steady-state locus $\omega(B)$ of the system. The map in question, depicted in Fig. 4, is continuously differentiable at any nonzero $w$ but only continuous at $w = 0$, as shown in Byrnes et al. (2003).

Note that the motions of the autonomous system generator (14) that drives system (13) are not supposed to be periodic motions. For instance, the system in question could be a stable Van der Pol oscillator, with $W$ defined as the set of all points inside and on the boundary of the limit cycle. In this case, our approach makes it possible to define the steady-state response of (13) not just to the (single) periodic motion generated by (14) when the initial condition is taken on the boundary of $W$, but also to all (non-periodic) motions generated by (14) when the initial condition is taken in the interior of $W$. We consider this as an advantage of the proposed approach.

A common feature of the examples discussed above is the fact that the set $\omega(B)$ can be expressed as the graph of a map $z = \pi(w)$. This means that, so long as this is the case, a system of the form (13) has a *unique* well defined steady-state response to the input $u(t) = q(w(t))$. As a matter of fact, the response in question is precisely $z(t) = \pi(w(t))$. Of course, in general, this may not be the case and the global structure of the steady-state locus can be very complicated. In particular, the set $\omega(B)$ may fail to be the graph of a map $z = \pi(w)$ and *multiple* steady-state responses to a given input may occur. This is the counterpart – in the context of forced motions – of the fact that, in general, a nonlinear system may possess multiple equilibria. In these cases, the steady-state response is determined not only by the forcing input, but also by the initial state of the system to which the input is applied.

Even though, in general, uniqueness of the steady-state response of system (13) to inputs generated by a system of the form (14) cannot be guaranteed, it is useful to stress that, if the set $W$ is compact and invariant (as assumed above), for each $w \in W$ there is always at least one $z \in Z$ such that the pair $(w,z)$ produces a steady-state response.

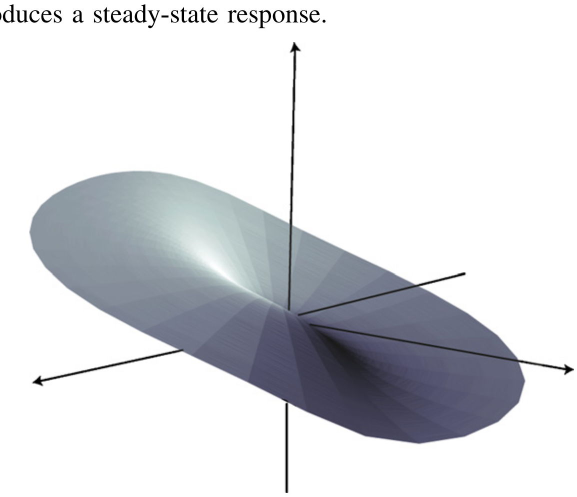

Fig. 4. The steady-state locus of Example (16).

**Lemma 6.** *Consider a system of the form (15) with $(w,z)\in W\times Z$. Suppose its motions are ultimately bounded. If W is a compact set invariant for $\dot{w}=s(w)$, the steady-state locus of (15) is the graph of a (possibly set-valued) map defined on W.*

**Proof.** Recall that, as shown above, the limit set of $W$ under the flow of $\dot{w}=s(w)$ coincides with $W$ itself, that is $\omega(W)=W$. As a consequence, for all $\bar{w}\in W$ there is a sequence $\{w_k, t_k\}$ with $w_k$ in $W$ for all $k$ such that $\bar{w}=\lim_{k\to\infty} w(t_k, w_k)$. Set $p=\mathrm{col}(w,z)$ and let $\phi(t, p_0)$ denote the integral curve of (15) passing through $p_0$ at time $t=0$. Pick *any* point $z_0\in Z$ and let $p_k=\mathrm{col}(w_k, z_0)$. If the motions of (15) are ultimately bounded, there is a bounded set $B$ and a time $T>0$ such that $\phi(t, p_k)\in B$ for all $t\geq T$ and all $k>0$. Pick any integer $h$ such that $t_h\geq T$, set $\bar{p}_k=\phi(t_h, p_k)$ and $\bar{t}_k=t_k-t_h$, for $k\geq h$, and observe that, by construction, $\phi(t_k, p_k)=\phi(\bar{t}_k, \bar{p}_k)$. The sequence $\{\phi(\bar{t}_k, \bar{p}_k)\}$ is bounded. Hence, there exists a subsequence $\{\phi(\hat{t}_k, \hat{p}_k)\}$ converging to a point $\hat{p}=\mathrm{col}(\hat{w},\hat{x})$, which is a point of $\omega(B)$ because all $\bar{p}_k$'s are in $B$. Since system (15) is upper triangular, necessarily $\hat{w}=\bar{w}$. This shows that, for any point $\bar{w}\in W$, there is a point $\hat{z}\in Z$ such that $(\bar{w},\hat{Z})\in\omega(B)$, as claimed. □

## 5. Asymptotic tracking and disturbance rejection

### *5.1. Background*

One of the main motivations for the importance of an appropriate notion of steady state is the need to address control problems in which the output of a system is required to asymptotically track prescribed trajectories and/or to asymptotically reject prescribed disturbances. Problems of this kind are commonly known as *generalized tracking* problems, as *generalized servomechanism* problems, or – more often – *output regulation* problems. In any realistic scenario, this control goal has to be achieved in spite of a good number of phenomena which would cause a system to behave differently than expected. These phenomena could be endogenous, for instance parameter variations, or exogenous, such as additional undesired inputs affecting the behavior of the plant.

In most cases of practical interest, the trajectories to be tracked (or the disturbance to be rejected) are not available for measurement. Rather, it is only known that these trajectories are simply (undefined) members of a set of functions, for instance the set of all possible solutions of an ordinary differential equation. Theses cases include the classical problem of the set-point control, the problem of active suppression of harmonic disturbances of unknown amplitude, phase and even frequency, the synchronization of nonlinear oscillations, and similar others.

For linear multivariable systems, the generalized servomechanism problem has been successfully addressed by various authors (Davison, 1976; Francis, 1977; Francis & Wonham). In particular Francis and Wonham (1976) and Francis (1977) provide a very elegant analysis, that was later taken as a paradigm for the study of the nonlinear version of the problem. One of the contributions of Francis and Wonham (1976) was a precise characterization of what the author called (and since then has become known as) the *internal model principle*. In a suitable framework, they proved that the property of perfect tracking is insensitive to plant parameter variations ''only if the controller utilizes feedback of the regulated variable, and incorporates in the feedback path a suitably reduplicated model of the dynamic structure of the exogenous signals which the regulator is required to process''. The converse of this property, also shown in Francis (1977), is that if the controller embeds an internal model of the exogenous signals, stable perfect tracking can be achieved, regardless of plant parameter variations (so long as the stability of the closed loop is preserved).

A nonlinear enhancement of this theory, which uses a combination of geometry and nonlinear dynamical systems theory, was initiated by pioneering works of Isidori and Byrnes (1990), Huang and Rugh (1990) and Huang and Lin (1994) who showed how to design a controller that provides a local solution near an equilibrium point, in the presence of exogenous signals which were produced by a Poisson stable system. In particular, Isidori and Byrnes (1990) showed how the use center manifold theory determines – also in the case of nonlinear systems – the necessity of the existence of an internal model and Khalil (1994) showed how issues of global convergence to the required steady state could be addressed. Since these early contributions, the theory has experienced a tremendous growth, culminating in the recent development of design methods able to handle the case of parametric uncertainties affecting the autonomous (linear) system which generates the exogenous signals (such as in Delli Priscoli, Marconi, & Isidori, 2006; Serrani, Isidori, & Marconi, 2001), the case of nonlinear exogenous systems (such as in Byrnes & Isidori, 2004), or a combination thereof (as in Marconi, Praly, & Isidori, 2006).

### *5.2. The generalized tracking problem*

The generalized tracking problem is cast in the following terms. The controlled plant is a finite-dimensional, time-invariant, nonlinear system modelled by equations of the form

$$\dot{x}=f(w,x,u),\qquad e=h(w,x),\qquad y=k(w,x),\tag{17}$$

in which $x\in\mathbb{R}^n$ is a vector of state variables, $u\in\mathbb{R}^m$ is a vector of inputs used for *control* purposes, $w\in\mathbb{R}^s$ is a vector of inputs which cannot be controlled and include *exogenous* commands, exogenous disturbances and model uncertainties, $e\in\mathbb{R}^p$ is a vector of *regulated* outputs which include tracking errors and any other variable that needs to be steered to 0, $y\in\mathbb{R}^q$ is a vector of outputs that are available for *measurement* and hence used to feed the device that supplies the control action. The problem is to design a controller, which receives $y(t)$ as input and produces $u(t)$ as output, to the purpose that, in the resulting closed-loop system, $x(t)$ remains bounded and

$$\lim_{t\to\infty} e(t)=0,\tag{18}$$

regardless of what the exogenous input $w(t)$ actually is.

The specific characteristic of the problem at issue is that the exogenous input $w(t)$ is assumed to be a (undefined) member of a fixed family of functions of time, the family of all solutions obtained when the initial condition $w(0)$ of a fixed ordinary differential equation of the form

$$\dot{w} = s(w) \tag{19}$$

is allowed to vary on a prescribed set $W$. This system is usually referred to as the *exosystem*. This approach can be viewed as intermediate between one extreme case in which $w(t)$ is totally unknown, and the opposite extreme case in which $w(t)$ is available for measurement. As observed earlier, there is abundance of design problems in which parameter uncertainties, reference commands and/or exogenous disturbances can be modelled in this way: the case of set point control, in which the exosystem has a trivial dynamics, namely $\dot{w} = 0$, the case in which $w(t)$ is any combination sinusoidal signals of fixed frequency but unspecified amplitude and phase, in which case the exosystem is characterized by a bench of harmonic oscillators

$$\begin{pmatrix} \dot{w}_1 \\ \dot{w}_2 \end{pmatrix} = \begin{pmatrix} 0 & \omega \\ -\omega & 0 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix},$$

but even the case in which the frequencies of such sinusoidal signals are undetermined, in which case the exosystem is characterized by a bench of (nonlinear) equations of the form

$$\begin{pmatrix} \dot{w}_1 \\ \dot{w}_2 \\ \dot{w}_3 \end{pmatrix} = \begin{pmatrix} 0 & w_3 & 0 \\ -w_3 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}.$$

The control law for (17) is to be provided by a system modelled by equations of the form

$$\begin{aligned} \dot{\xi} &= \varphi(\xi, y) \\ u &= \gamma(\xi, y) \end{aligned} \tag{20}$$

with state $\xi \in \mathbb{R}^\nu$. The initial conditions $x(0)$ of the *plant* (17), $w(0)$ of the *exosystem* (19) and $\xi(0)$ of the *controller* (20) are allowed to range over a fixed *compact* sets $X \subset \mathbb{R}^n$, $W \subset \mathbb{R}^s$ and, respectively $\Xi \subset \mathbb{R}^\nu$. All maps characterizing the model of the controlled plant, of the exosystem and of the controller are assumed to be sufficiently differentiable.

The problem to be addressed, is to design a feedback controller of the form (20) so as to obtain a closed-loop system in which all trajectories are bounded and the regulated output $e(t)$ asymptotically decays to 0 as $t \to \infty$. More precisely, it is required that

$$\begin{aligned} &\dot{w} = s(w), \qquad \dot{x} = f(w, x, \gamma(\xi, k(w, x))), \\ &\dot{\xi} = \varphi(\xi, k(w, x)), \end{aligned} \tag{21}$$

viewed as an autonomous system with output

$$e = h(w, x),$$

be such that:

(i) the positive orbit of $W \times X \times \Xi$ is bounded, i.e. there exists a bounded subset $S$ of $\mathbb{R}^s \times \mathbb{R}^n \times \mathbb{R}^\nu$ such that, for any $(w_0, x_0, \xi_0) \in W \times X \times \Xi$, the integral curve $(w(t), x(t), \xi(t))$ of (21) passing through $(w_0, x_0, \xi_0)$ at time $t = 0$ remains in $S$ for all $t \geq 0$.
(ii) $\lim_{t \to \infty} e(t) = 0$, *uniformly* in the initial condition, i.e. for every $\varepsilon > 0$ there exists a time $\bar{t}$, depending only on $\varepsilon$ and *not on* $(w_0, x_0, \xi_0)$ such that the integral curve $(w(t), x(t), \xi(t))$ of (21) passing through $(w_0, x_0, \xi_0)$ at time $t = 0$ yields $\|e(t)\| \leq \varepsilon$ for all $t \geq \bar{t}$.

Condition (i) replaces, and actually extends to a nonlinear setting, the classical requirement – of linear system theory – that the un-driven closed-loop system be an asymptotically stable system. Condition (ii) expresses the property of asymptotic regulation (or tracking). The property that convergence of the regulated variable $e(t)$ to zero be uniform, which is granted in the case of a linear system, needs now to be explicitly requested, since in the case of nonlinear systems (as shown earlier) this may no longer be the case even if the initial conditions are taken in a compact set. Form a practical viewpoint, in fact, the only meaningful case is the one in which there is a guaranteed ‘‘rate of decay’’ of the regulated variable to zero.

### 5.3. Steady-state analysis

The notion of steady state introduced earlier is instrumental to prove the following, elementary – but fundamental – result, which is a nonlinear enhancement of a Lemma of Francis (1977) on which all the theory of output regulation for linear systems is based (see Byrnes and Isidori (2003) for a proof).

**Lemma 7.** *Suppose the positive orbit of $W \times X \times \Xi$ is bounded. Then*

$$\lim_{t \to \infty} e(t) = 0$$

*if and only if*

$$\omega(W \times X \times \Xi) \subset \{(w, x, \xi) : h(w, x) = 0\}. \tag{22}$$

This result is simply a characterization of the generalized tracking problem in geometric terms. The problem in question, in fact, can be cast as a problem of *shaping the steady-state locus* of the closed-loop system, in such a way that the latter becomes a subset of the set of all points at which the regulated variable is 0.

To proceed with the analysis in a more concrete fashion, we restrict our discussion to a very special case, on which nevertheless most of the relevant features of the theory can be illustrated. This is the case of a controlled plant modelled by equations of the form

$$\begin{aligned} &\dot{z} = f_0(w, z) + f_1(w, z, e)e, \\ &\dot{e} = q_0(w, z) + q_1(w, z, e)e + u, \qquad y = e. \end{aligned} \tag{23}$$

This system is very special because the relative degree (Isidori, 1995, p. 137) between the control input $u$ and the regulated output $e$ is equal to one, the coefficient of $u$ in the second equation (known, with an abuse of terminology, as ‘‘high-frequency gain’’) is unitary, and, moreover, the regulated variable $e$ is assumed to coincide with the measured variable $y$. It must be observed, though, that there is not much loss of generality in considering a system having this simple structure because, as shown for instance in Delli Priscoli et al. (2006) and Marconi et al. (2006), the case of a more general system (having arbitrary relative degree, but with the dynamics of $z$ still driven only by $e$, and having non-unitary high-frequency gain) can be handled in a very similar manner, after suitable preliminary manipulations. The initial conditions of (23) are assumed to range on a set $Z \times E$, in which $Z$ is a fixed *compact* subset of $\mathbb{R}^{n-1}$ and $E = \{e \in \mathbb{R} : |e| \leq c\}$, with $c$ a fixed number.

Since the problem in question is a problem concerning how the closed-loop system behaves in steady state, there is no special interest in considering exosystems that are not ‘‘in steady state’’. Thus – without loss of generality – we assume that the set $W$ is invariant for (19), and hence – by Lemma 3– that $W = \omega(W)$.

Suppose that a controller of the form (20) solves the problem of output regulation. Then Lemma 7 applies and, the following conclusions immediately come true:

- The steady-state locus $\omega(W \times Z \times E \times \Xi)$ of the closed-loop system is a subset of the set $W \times \mathbb{R}^{n-1} \times \{0\} \times \mathbb{R}^{\nu}$.
- The restriction of the closed-loop system to its steady-state locus $\omega(W \times Z \times E \times \Xi)$ reduces to
$$\dot{w} = s(w), \qquad \dot{z} = f_0(w, z), \qquad \dot{\xi} = \varphi(\xi, 0). \tag{24}$$
- For each $(w, z, 0, \xi) \in \omega(W \times Z \times E \times \Xi)$
$$0 = q_0(w, z) + \gamma(\xi, 0). \tag{25}$$

With this in mind we observe that, by Lemma 6, if the positive orbit of $W \times Z \times E \times \Xi$ under the flow of (21) is bounded, then $\omega(W \times Z \times E \times \Xi)$ is the graph of a (possibly set-valued) map defined on the whole of $W$. As a consequence, the set

$$\mathcal{A}_{ss} = \{(w, z) : (w, z, 0, \xi) \in \omega(W \times Z \times E \times \Xi) \text{ for some } \xi \in \mathbb{R}^{\nu}\}$$

is the graph of a (possibly set-valued) map defined on the whole of $W$, and is invariant for the dynamics of

$$\dot{w} = s(w), \qquad \dot{z} = f_0(w, z). \tag{26}$$

Define the map

$$u_{ss}: \quad \mathcal{A}_{ss} \to \mathbb{R}, \quad (w, z) \mapsto -q_0(w, z).$$

The conclusions reached above can be rephrased in the following terms. Suppose that a controller of the form (20) solves the problem of output regulation for (23) with exosystem (19). Then, there exists a (possibly set-valued) map, defined on the whole of $W$, whose graph $\mathcal{A}_{ss}$ is invariant for the autonomous system (26). Moreover, for each $(w_0, z_0) \in \mathcal{A}_{ss}$ there is a point $\xi_0 \in \mathbb{R}^{\nu}$ such that the integral curve of (26) issued from $(w_0, z_0)$ and the integral curve of

$$\dot{\xi} = \varphi(\xi, 0)$$

issued from $\xi_0$ satisfy

$$u_{ss}(w(t), z(t)) = \gamma(\xi(t), 0) \quad \forall t \in \mathbb{R}.$$

This is a nonlinear version of the celebrated *internal model principle* of Francis and Wonham (1976).

## 6. Regulator design

### 6.1. Controller structure

The steady-state analysis presented above has identified certain features that any controller must have to be able to solve the problem at issue. As a matter of fact, this controller must include a subsystem that behaves as a ‘‘generator’’ of all inputs of the form $u_{ss}(w(t), z(t))$ in which $(w(t), z(t))$ is a trajectory of (26), issued at any point $(w_0, z_0)$ of the compact invariant set $\mathcal{A}_{ss}$. As such, the characterization seems to be independent of the controller, because the system (26) and the map $u_{ss}(w, z)$ only depend on the plant. However, this is not strictly speaking true, because the invariant set $\mathcal{A}_{ss}$ may depend on how the controller is chosen. In view of the property thus established, it appears that a first, fundamental, step in the design of a controller that solves the problem of output regulation is to find a suitable candidate for the invariant set $\mathcal{A}_{ss}$ and to build a ‘‘device’’ able to generate – as outputs – all inputs of the form $u_{ss}(w(t), z(t))$. Driving the controlled plant by means of a device of this kind yields a system possessing a (compact) invariant set on which the regulated variable is identically zero. This is a necessary prerequisite for the solution of the problem in question, but clearly not sufficient yet, because the convergence to this invariant set still has to be secured. To this purpose, an additional assumption (which in general is not necessary, though) is useful.

**Assumption (A).** There exists a bounded subset $B \subset W \times \mathbb{R}^{n-1}$ which contains the positive orbit of the set $W \times Z$ under the flow of (26) and the resulting omega-limit set $\mathcal{A} := \omega(W \times Z)$ is locally exponentially stable.

While in the analysis of the necessity we have only identified the existence of a compact set (actually, the graph of a map defined on $W$) which is invariant for (26), Assumption (A) implies, in its first part, the existence of a compact set $\mathcal{A}$ (still the graph of a map defined on $W$) which is not only invariant but also uniformly attractive of all trajectories of (26) issued from points of $W \times Z$. The second part of the Assumption, in turn, strengthens this property by also requiring the set $\mathcal{A}$ to be locally exponentially stable.

For convenience, rewrite the "augmented" system (19)–(23) as

$$\dot{\hat{z}} = \hat{f}_0(\hat{z}) + \hat{f}_1(\hat{z}, e)e \qquad \dot{e} = \hat{q}_0(\hat{z}) + \hat{q}_1(\hat{z}, e)e + u \tag{27}$$

having set $\hat{z} = (w, z)$. Consistently let $\hat{Z} := W \times Z$ denote the compact set where the initial condition $\hat{z}(0)$ is supposed to range, and set $\hat{u}_{ss}(\hat{z}) = -\hat{q}_0(\hat{z})$. In these notations, Assumption (A) expresses the property that, in the autonomous system

$$\dot{\hat{z}} = \hat{f}_0(\hat{z}), \tag{28}$$

the compact invariant set $\mathcal{A}$ is asymptotically and locally exponentially stable, with a domain of attraction that contains the set $\hat{Z}$.

With a view to the internal model principle, we choose the following candidate controller

$$\dot{\xi} = \varphi(\xi) + Gv, \qquad u = \gamma(\xi) + v, \qquad v = -ke. \tag{29}$$

In fact, when the regulated variable $e$ is identically zero (as it should occur in steady state), this controller reduces to the autonomous system with output

$$\dot{\xi} = \varphi(\xi) \qquad u = \gamma(\xi), \tag{30}$$

which is supposed to be a "generator" capable – as the internal model principle dictates – to produce inputs of the form $\hat{u}_{ss}(\hat{z}(t))$, with $\hat{z}(t)$ a trajectory of (28) issued at any point $\hat{z}_0$ of $\mathcal{A}$.

Controlling system (27) by means of (29) yields a closed-loop system

$$\dot{\hat{z}} = \hat{f}_0(\hat{z}) + \hat{f}_1(\hat{z}, e)e, \qquad \dot{e} = \hat{q}_0(\hat{z}) + \hat{q}_1(\hat{z}, e)e + \gamma(\xi) + v,$$
$$\dot{\xi} = \varphi(\xi) + Gv \tag{31}$$

which, regarded as a system with input $v$ and output $e$, has relative degree 1. It can be put in normal form by changing variables as

$$x = \xi - Ge$$

which yields

$$\dot{\hat{z}} = \hat{f}_0(\hat{z}) + \hat{f}_1(\hat{z}, e)e,$$
$$\dot{x} = \varphi(x + Ge) - G\gamma(x + Ge) - G\hat{q}_0(\hat{z}) - G\hat{q}_1(\hat{z}, e)e, \tag{32}$$
$$\dot{e} = \hat{q}_0(\hat{z}) + \hat{q}_1(\hat{z}, e)e + \gamma(x + Ge) + v.$$

Setting $p = (\hat{z}, x)$, this system can be further rewritten in the form

$$\dot{p} = F(p) + P(p, e)e \qquad \dot{e} = Q(p) + R(p, e)e + v \tag{33}$$

in which

$$F(p) = \begin{pmatrix} \hat{f}_0(\hat{z}) \\ \varphi(x) - G[\gamma(x) + \hat{q}_0(\hat{z})] \end{pmatrix}$$
$$Q(p) = \gamma(x) + \hat{q}_0(\hat{z})$$

and $P(p, e)$, $R(p, e)$ are suitable continuous functions.

Suppose now that the control $v$ is chosen as $v = -ke$, as indicated earlier. This yields a closed-loop system that can be regarded as pure feedback interconnection of

$$\dot{p} = F(p) + P(p, e)e \tag{34}$$

viewed as a system with input $e$ and state $p$, and

$$\dot{e} = Q(p) + R(p, e)e - ke \tag{35}$$

viewed as a system with input $p$ and state $e$. To obtain bounded trajectories, and to steer $e(t)$ to zero, one might invoke the so-called *small gain theorem*, actually an enhanced version of it (see e.g. Teel & Praly, 1995). As a matter of fact, suppose that the following properties hold:

(P1) the dynamics

$$\dot{p} = F(p) \tag{36}$$

possesses a compact invariant set $S$ which is asymptotically (and locally exponentially) stable, with a domain of attraction that contains the set $P := \hat{Z} \times X$ of all admissible initial conditions, and

(P2) the function $Q(p)$ vanishes on this invariant set.

Then, a version of the small-gain theorem (enhanced to allow the case in which one of the two pieces of the interconnection, in this case system (34), possesses an asymptotically stable compact invariant set) can be invoked to conclude that, if $k$ is large enough (the lower limit of $k$ being dependent on the actual choice of the set $P \times E$ of admissible initial conditions), all trajectories of the interconnected system (33) are bounded and do converge, as $t \to \infty$, to the invariant set $S \times \{0\}$. Thus, in particular, $e(t) \to 0$ as $t \to \infty$ and the problem of output regulation is solved.

Motivated by this observation, in what follows we will seek properties such as (P1) and (P2).

### 6.2. The internal model property

Properties (P1) and (P2) are only determined by the properties of the autonomous system (28) and of the function

$$\hat{u} = \hat{q}_0(\hat{z}) \tag{37}$$

which, in the composite system (36), can be viewed as an output of (28) driving a system of the form

$$\dot{x} = \varphi(x) - G[\gamma(x) + \hat{u}]. \tag{38}$$

For convenience, we will say that triplet $\{\varphi(x), G, \gamma(x)\}$ *is an asymptotic internal model of the pair* (28)–(37) if properties (P1) and (P2) are satisfied. In this terminology, we can summarize the conclusion of the previous subsection as follows.

**Proposition 1.** *Pick compact sets $\hat{Z}$, $E$ and $\Xi$ for the initial conditions of the closed-loop system* (19), (23), (29). *Suppose that Assumption A holds and that the triplet $\{\varphi(x), G, \gamma(x)\}$ is an asymptotic internal model for* (28)–(37). *Then there exists*

*$k^\star > 0$ such that for all $k \geq k^\star$ the controller (29) with $v = -ke$ solves the generalized tracking problem.*

The notion of steady state provides a useful interpretation of the properties in question. In fact, recall that, by Assumption A, all trajectories of system (28) with initial conditions in $\hat{Z}$ asymptotically converge to the compact invariant set $\mathcal{A}$, and the latter is also locally exponentially stable. If property (P1) holds, all trajectories of the composite system

$$\begin{aligned}\dot{\hat{z}} &= \hat{f}_0(\hat{z})\\ \dot{x} &= \varphi(x) - G[\gamma(x) + \hat{q}_0(\hat{z})]\end{aligned}$$

with initial conditions in $\hat{Z} \times X$ asymptotically converge to the limit set $\omega(\hat{Z} \times X)$. Since (36) is a triangular system, its is readily seen (see also Lemma 6), that the set $\omega(\hat{Z} \times X)$ is the graph of a set-valued map defined on $\mathcal{A}$, i.e. that there exists a map

$$\tau : \hat{z} \in \mathcal{A} \mapsto \tau(\hat{z}) \subset \mathbb{R}^\nu,$$

such that

$$\omega(\hat{Z} \times X) = \{(\hat{z}, x) : \hat{z} \in \mathcal{A}, x \in \tau(\hat{z})\} := \mathrm{gr}(\tau).$$

The set $\mathrm{gr}(\tau)$ is the steady-state locus of (36) and the restriction of the latter to this invariant set characterizes its steady-state behavior. Property (P2), on the other hand, expresses the property that at each point of $(\hat{z}, x) \in \mathrm{gr}(\tau)$

$$\hat{q}_0(\hat{z}) = -\gamma(x). \tag{39}$$

Thus, looking again at system (36), it is realized that $\mathrm{gr}(\tau)$ is in fact invariant for the simpler system

$$\dot{\hat{z}} = \hat{f}_0(\hat{z}) \qquad \dot{x} = \varphi(x). \tag{40}$$

Note that, if the map $\tau(\hat{z})$ is single-valued and $C^1$, its invariance for (40) is expressed by the property that

$$\frac{\partial \tau(\hat{z})}{\partial \hat{z}} \hat{f}_0(\hat{z}) = \varphi(\tau(\hat{z})) \quad \forall\, \hat{z} \in \mathcal{A}, \tag{41}$$

while the fact that (39) holds at each point of $(\hat{z}, x) \in \mathrm{gr}(\tau)$ is expressed by the property that

$$\hat{q}_0(\hat{z}) = -\gamma(\tau(\hat{z})) \quad \forall\, \hat{z} \in \mathcal{A}. \tag{42}$$

Properties (41) and (42) have been usually referred to, in the literature (see, e.g. Isidori, 1995), as properties of *immersion* of system

$$\begin{aligned}\dot{\hat{z}} &= \hat{f}_0(\hat{z})\\ \hat{u} &= \hat{q}_0(\hat{z})\end{aligned}$$

into system

$$\begin{aligned}\dot{x} &= \varphi(x)\\ \hat{u} &= -\gamma(x).\end{aligned}$$

### 6.3. Nonlinear observers as internal models

In this section, we discuss how the properties (P1) and (P2) can be enforced. To simplify matters, we refer to the case in which the map $\tau(\hat{z})$ that characterizes the steady-state locus of (36) is singled-valued and $C^1$. To this end, we stress that the properties in question are quite similar to properties that are usually sought in the design of *state observers*. As a matter of fact it is seen from (41)–(42) that, for each $\hat{z}_0 \in \mathcal{A}$, the function of time

$$\hat{x}(t) = \tau(\hat{z}(t, \hat{z}_0))$$

which is defined (and bounded) for all $t \in \mathbb{R}$ satisfies

$$\frac{\mathrm{d}\hat{x}(t)}{\mathrm{d}t} = \varphi(\hat{x}(t)) \tag{43}$$

and, moreover,

$$\gamma(\hat{x}(t)) = -\hat{q}_0(\hat{z}(t, \hat{z}_0)).$$

In view of the latter, system (38) can be rewritten in the form

$$\dot{x} = \varphi(x) + G[\gamma(\hat{x}) - \gamma(x)] \tag{44}$$

and interpreted as *a copy of the dynamics (43) of $\hat{x}$ corrected by an "innovation term" $[\gamma(\hat{x}) - \gamma(x)]$ weighted by an "output injection gain" $G$*. This is the classical structure on an *observer* and the requirement in (P1) precisely expresses the property that the difference $x(t) - \hat{x}(t)$ (the "observation error", in our interpretation) should asymptotically decay to zero (with ultimate exponential decay).

This interpretation is at the basis of a number of major recent advances in the design of regulators. In fact, in a number of recent papers, this interpretation has been pursued and, taking into consideration various approaches to the design of nonlinear observers, has lead to effective design methods. In Byrnes and Isidori (2004), the approach of Bornard–Gauthier–Hammouri–Kupca to the design of high-gain observers (as described, e.g. by Gauthier and Kupka (2001)) has been followed. This design requires the extra assumption that the set of all functions of the form $u(t) := \hat{u}_{ss}(\hat{z}(t))$ obtained as $\hat{z}(0)$ ranges over $\mathcal{A}$ be a subset of the set of solutions of a (nonlinear) differential equation

$$u^{(d)} + \phi(u^{(d-1)}, \ldots, u^{(1)}, u) = 0. \tag{45}$$

The controller obtained in this way cannot handle cases in which the exosystem contains uncertain constant parameters (as it is in the case of harmonic oscillator of unknown frequency). To handle this case, an adaptive nonlinear observer is needed, as that of Bastin and Gevers (1988), which in turn – though – requires stronger hypotheses, such as the possibility of expressing the state-space version of (45) in a form which can be made linear by means of output injection and diffeomorphisms. The design of a regulator on the basis of the theory of adaptive observers was pursued by Serrani et al. (2001) and by Delli Priscoli et al. (2006). Finally, in Marconi et al. (2006), the more recent advances in the theory of nonlinear observers obtained by Andrieu and Praly (2006) have been exploited, to show that a triplet $\{\varphi(\xi), G, \gamma(\xi)\}$ having internal model property controller does in fact exist *always*, and no assumption like the existence of an equation of the form (45) is actually required. In the triplet in question, $\varphi(\xi)$ has the following form

$$\varphi(\xi) = F\xi - G\gamma(\xi)$$

in which $(F, G)$ is a controllable pair, with $F$ a $d \times d$ Hurwitz matrix, and sufficiently large $d$. However, no closed form is immediately available for $\gamma(\xi)$ and this function is only guaranteed to be continuous.

## 7. Conclusions

Classically, engineers have developed two intuitive approaches to formalize the concept of steady-state response. The first is that such a motion should be the response of the system as the initial time tends to $-\infty$, independent of the initial condition. The second is based on the idea that every motion can be decomposed into the superposition of a transient response and a steady-state response. Any rigorous interpretation of the first approach would require the trajectories to be bounded backward in time, while any effective use of the second interpretation for analysis and design would require some form of uniformity: that is, for any given tolerance and any given (compact) set of initial conditions there should exist a time after which the magnitude of the transient response is less than this given tolerance regardless of the choice of initial condition.

In this paper, we outlined concepts and tools necessary to motivate, for general nonlinear systems, a simple definition of steady-state behavior which captures both of the classical intuitive approaches: boundendess of steady-state trajectories backward and forward in time and uniform attractivity of the steady-state behavior. A rigorous technical treatment is provided as well as an indication of the recent applications of these ideas and tools to the generalized tracking problem for nonlinear control systems.

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