System Regulation and Design, Geometric and Algebraic Methods

A. Isidori [†], L. Marconi [‡]

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[†]Department of Informatica e Sistemistica, University of Rome "La Sapienza" (Italy).
[‡]Center for research on Complex Automated SYstems (CASY), University of Bologna (Italy).

Article Outline

Glossary

Exosystem: A dynamical system modelling the set of all exogenous inputs (command/disturbances) affecting a controlled plant.

Internal Model: A model of the exogenous inputs (command/disturbances) affecting a controlled plant, embedded in the interior of the controller.

Generalized Tracking Problem: The problem of designing a controller able to asymptotically track/reject any exogenous command/disturbance in a fixed set of functions.

Observer: A device designed to asymptotically track the state of a dynamical system on the basis of measured observations.

Steady state: A family of behaviors, in a dynamical system, that are asymptotically approached, as actual time tends to infinity or as initial time tends to minus infinity.

1 Definition

A central problem in control theory is the design of feedback controllers so as to have certain outputs of a given plant *to track* prescribed reference trajectories. In any realistic scenario, this control goal has to be achieved in spite of a good number of phenomena which would cause the system to behave differently than expected. These phenomena could be endogenous, for instance parameter variations, or exogenous, such as additional undesired inputs affecting the behavior of the plant. In numerous design problems, exogenous inputs are not available for measurement, nor are known ahead of time, but rather can only seen as unspecified members of a given family of functions. Embedding a suitable "internal model" of such a family in the controller is a design strategy that has proven to be quite successful in handling uncertainties in the controlled plant as well as in the exogenous inputs.

2 Introduction

The problem of controlling the output of a system so as to achieve asymptotic tracking of prescribed trajectories and/or asymptotic rejection of disturbances is a central problem in control theory. There are essentially three different possibilities to approach the problem: tracking by dynamic inversion, adaptive tracking, tracking via internal models. Tracking by *dynamic inversion* consists in computing a precise initial state and a precise control input (or equivalently a reference trajectory of the state), such that, if the system is accordingly initialized and driven, its output exactly reproduces the reference signal. The computation of such control input, however, requires "perfect knowledge" of the entire trajectory which is to be tracked as well as "perfect knowledge" of the model of the plant to be controlled. Thus, this type of approach is not suitable in the presence of large uncertainties on plant parameters as well as on the reference signal. *Adaptive* tracking consists in tuning the parameters of a control input computed via dynamic inversion in such a way as to guarantee asymptotic convergence to zero of a tracking error. This method can successfully handle parameter uncertainties, but still presupposes the knowledge of the entire trajectory which is to be tracked (to be used in the design of the adaptation algorithm) and therefore an approach of this kind is not suited to the problem of tracking unknown trajectories. Of course, one might consider the problem of tracking a slowly varying reference trajectory as a stabilization problem in the presence of a slowly varying unknown parameter, but this would, in most cases, yield a very conservative solution.

In most cases of practical interest, the trajectory to be tracked (or the disturbance to be rejected) is not available for measurement. Rather, it is only known that this trajectory is simply an (undefined) member in a set of functions, for instance the set of all possible solutions of an ordinary differential equation. Theses cases include the classical problem of the set point control, the problem of active suppression of harmonic disturbances of unknown amplitude, phase and even frequency, the synchronization of nonlinear oscillations, and similar others. It is in these cases that *tracking via internal models* proves particularly efficient, in its ability to handle simultaneously uncertainties in plant parameters as well as in the trajectory which is to be tracked.

For linear multivariable systems, tracking problems of this kind (those in which the exogenous commands and/or disturbances are only known to be members in the set of solutions of a given ordinary differential equations) have been addressed in very elegant geometric terms by Davison, Francis, Wonham [8, 11, 10] and others. In particular, one of the most relevant contributions of [11] was a clear delineation of what is known as *internal model principle*, i.e. the fact that the property of perfect tracking is insensitive to plant parameter variations "only if the controller utilizes feedback of the regulated variable, and incorporates in the feedback path a suitably reduplicated model of the dynamic structure of the exogenous signals which the regulator is required to process". Conversely, in a stable-closed loop system, if the controller utilizes feedback of the regulated variable and incorporates an internal model of the exogenous signals, the output regulation property is insensitive to plant parameter variations.

A nonlinear enhancement of this theory, which uses a combination of geometry and nonlinear dynamical systems theory, was initiated in [19, 16, 15] in the context of solving the problem near an equilibrium, in the presence of exogenous signals which were produced by a Poisson stable system. In particular, in [19] it was shown how the use center manifold theory near an equilibrium determines the necessity of the existence of an internal model whenever one can solve an output regulation problem in spite of (small) parameter uncertainties (see also [3]). Since these pioneering contributions, the theory has experienced a tremendous growth, culminating in the development of design methods able to handle the case of parametric uncertainties affecting the autonomous (linear) system which generates the exogenous signals (such as in [23, 9]), the case of nonlinear exogenous systems (such as in [5]), or a combination thereof (as in [22]). The purpose of these notes is to present a self-contained exposition of the fundamentals of these design methods.

3 The Generalized Tracking Problem

In this article, we address tracking problems that can be cast in the following terms. Consider a finite-dimensional, time-invariant, nonlinear system modelled by equations of the form

$$\begin{aligned} \dot{x} &= f(w,x,u) \\ e &= h(w,x) \\ y &= k(w,x)\,, \end{aligned} \tag{1}$$

in which $x \in \mathbb{R}^n$ is a vector of state variables, $u \in \mathbb{R}^m$ is a vector of inputs used for *control* purposes, $w \in \mathbb{R}^s$ is a vector of inputs which cannot be controlled and include *exogenous* commands, exogenous disturbances and model uncertainties, $e \in \mathbb{R}^p$ is a vector of *regulated* outputs which include tracking errors and any other variable that needs to be steered to 0, $y \in \mathbb{R}^q$ is a vector of outputs that are available for *measurement* and hence used to feed the device that supplies the control action. The problem is to design a controller, which receives $y(t)$ as input and produces $u(t)$ as output, able to guarantee that, in the resulting closed-loop system, $x(t)$ remains bounded and

$$\lim_{t\to\infty} e(t) = 0\,, \tag{2}$$

regardless of what the exogenous input $w(t)$ actually is.

The ability to successfully address this problem very much depends on how much the controller is allowed to know about the exogenous disturbance $w(t)$. In the ideal situation in which $w(t)$ is available to the controller in real-time, the design problem indeed looks much simpler. This is, though, only an extremely optimistic situation which does not represent, in any circumstance, a realistic scenario. The other extreme situation is the one in which nothing is known about $w(t)$. In this, pessimistic, scenario the best result one could hope for is the fulfillment of some prescribed ultimate bound for $|e(t)|$, but certainly not a sharp goal such as (2). A more comfortable, intermediate, situation is the one in which $w(t)$ is only known *to belong to a fixed family* of functions of time, for instance the family of all solutions obtained from a fixed ordinary differential equation of the form

$$\dot{w} = s(w) \tag{3}$$

as the corresponding initial condition $w(0)$ is allowed to vary on a prescribed set. This situation is in fact sufficiently distant from the ideal but unrealistic case of perfect knowledge of $w(t)$ and from the realistic but conservative case of totally unknown $w(t)$. But, above all, this way of thinking at the exogenous inputs covers a number of cases of major practical relevance. There is, in fact, abundance of design problems in which parameter uncertainties, reference commands and/or exogenous disturbances can be modelled as functions of time that satisfy an ordinary differential equation.

The control law is to be provided by a system modelled by equations of the form

$$\begin{aligned} \dot{\xi} &= \varphi(\xi,y) \\ u &= \gamma(\xi,y) \end{aligned} \tag{4}$$

with state $\xi \in \mathbb{R}^\nu$. The initial conditions $x(0)$ of the *plant* (1), $w(0)$ of the *exosystem* (3) and $\xi(0)$ of the *controller* (4) are allowed to range over a fixed *compact* sets $X \subset \mathbb{R}^n$, $W \subset \mathbb{R}^s$ and, respectively $\Xi \subset \mathbb{R}^\nu$. All maps characterizing the of the controlled plant, of the exosystem and of the controller are assumed to be sufficiently differentiable.

The problem which will be studied, known as the *generalized tracking problem* (or *problem of output regulation* or also *generalized servomechanism problem*) is to design a feedback controller of the form (4) so as to obtain a closed loop system in which all trajectories are bounded and the regulated output $e(t)$ asymptotically decays to 0 as $t \to \infty$. More precisely, it is required that the composition of (1), (3) and (4), that is the *autonomous* system

$$\begin{array}{rcl} \dot{w} & = & s(w) \\ \dot{x} & = & f(w, x, \gamma(\xi, k(w, x))) \\ \dot{\xi} & = & \varphi(\xi, k(w, x)) \end{array} \tag{5}$$

with output

$$e = h(w, x)\,,$$

be such that:

- the positive orbit of $W \times X \times \Xi$ is bounded, i.e. there exists a bounded subset S of $\mathbb{R}^s \times \mathbb{R}^n \times \mathbb{R}^\nu$ such that, for any $(w_0, x_0, \xi_0) \in W \times X \times \Xi$, the integral curve $(w(t), x(t), \xi(t))$ of (5) passing through (w_0, x_0, ξ_0) at time $t = 0$ remains in S for all $t \geq 0$.
- $\lim_{t\to\infty} e(t) = 0$, uniformly in the initial condition, i.e., for every $\varepsilon > 0$ there exists a time $\bar{t}$, depending only on ε and *not on* (w_0, x_0, ξ_0) such that the integral curve $(w(t), x(t), \xi(t))$ of (5) passing through (w_0, x_0, ξ_0) at time $t = 0$ yields $\|e(t)\| \leq \varepsilon$ for all $t \geq \bar{t}$.

4 The Steady-State Behavior of a Nonlinear System

4.1 Limit Sets

The generalized tracking problem can be seen as the problem of forcing in the plant, by means of an appropriate control input $u(t)$, a response $x(t)$ that asymptotically compensates the effect, on the regulated variable $e(t)$, of the exogenous input $w(t)$. The classical way in which the problem is addressed for linear, time-invariant systems, when the exosystem is a neutrally stable linear system, is to seek a controller forcing in the associated closed-loop system (5) a (stable) "steady state" behavior entirely contained in the kernel of the map defining the tracking error e. Thus, it is natural to expect that a similar tool should also be effective in the more general setting considered here. It appears, though, that a rigorous investigation of the concept of "steady state", beyond the classical domain of linear system theory, had never been fully pursued.

Motivated by the current practice in linear system theory, the "steady state" behavior of a dynamical system can be viewed as a kind *limit* behavior, approached either as the *actual* time t tends to $+\infty$ or, alternatively, as the *initial* time t_0 tends to $-\infty$. Relevant, in this regard, are certain concepts introduced by G.D.Birkhoff in his classical 1927 essay, where he asserts that "with an arbitrary dynamical system ... there is associated always a closed set of 'central motions' which do possess this property of regional recurrence, towards which all other motions of the system in general tend asymptotically" [2, page 190]. In particular, a fundamental role is played by the concept of ω-limit set of a given point, which is defined as follows. Consider an *autonomous* ordinary differential equation

$$\dot{x} = f(x) \tag{6}$$

with $x \in \mathbb{R}^n$, $t \in \mathbb{R}$. It is well known that, if $f : \mathbb{R}^n \to \mathbb{R}^n$ is locally Lipschitz, for any $x_0 \in \mathbb{R}^n$, the solution of (6) with initial condition $x(0) = x_0$, denoted by $x(t, x_0)$, exists on some open interval of the point $t = 0$ and is unique.

Assume, in particular, that $x(t, x_0)$ is defined for all $t \geq 0$. A point x is said to be an ω-limit *point* of the motion $x(t, x_0)$ if there exists a sequence of times $\{t_k\}$, with $\lim_{k\to\infty} t_k = \infty$, such that

$$\lim_{k\to\infty} x(t_k, x_0) = x\,.$$

The ω-limit *set* of a point x_0, denoted $\omega(x_0)$, is *the union* of all ω-limit points of the motion $x(t, x_0)$.

It is obvious from this definition that an ω-limit point *is not* necessarily a limit of $x(t, x_0)$ as $t \to \infty$, as the solution in question may not admit any limit as $t \to \infty$. It happens though, that if the motion $x(t, x_0)$ is *bounded*, then $x(t, x_0)$ asymptotically approaches *the set* $\omega(x_0)$. This property is precisely described in what follows [2, page 198].

Lemma 1 Suppose there is a number M such that $\|x(t, x_0)\| \leq M$ for all $t \geq 0$. Then, $\omega(x_0)$ is a nonempty compact connected set, invariant under (6). Moreover, the distance of $x(t, x_0)$ from $\omega(x_0)$ tends to 0 as $t \to \infty$.

One of the remarkable features of $\omega(x_0)$, as indicated in this Lemma, is the fact that this set is *invariant* for (6). Invariance means that for any initial condition $\bar{x}_0 \in \omega(x_0)$ the solution $x(t, \bar{x}_0)$ of (6) exists *for all* $t \in (-\infty, +\infty)$ and that $x(t, \bar{x}_0) \in \omega(x_0)$ for all such t. Put in different terms, the set $\omega(x_0)$ is filled by motions of (6) which are bounded backward and forward in time. The other remarkable feature is that $x(t, x_0)$ *approaches* $\omega(x_0)$ as $t \to \infty$, in the sense that the distance *of the point* $x(t, x_0)$ (the value at time t of the solution of (6) starting in x_0 at time $t = 0$) *from the set* $\omega(x_0)$ tends to 0 as $t \to \infty$.

Since any motion $x(t, x_0)$ which is bounded in positive time asymptotically approaches the ω-limit set $\omega(x_0)$ as $t \to \infty$, one may be tempted to look, for a system (6) in which *all* motions are bounded in positive time, at the *union* of the limit sets of all points x_0, i.e. at the set

$$\Omega = \bigcup_{x_0 \in \mathbb{R}^n} \omega(x_0)$$

and to say that the system is in steady state if its state $x(t)$ evolves in the (invariant) set Ω. There is a major drawback, though, in taking this as definition of "steady state" behavior of a nonlinear system: the convergence of $x(t, x_0)$ to Ω is not guaranteed to be *uniform* in x_0, even if the latter ranges over a compact set (see, e.g. [7]).

One of the main motivations for looking into the concept of steady state is the aim *to shape* the steady state response of a system to a given (or to a given family of) forcing inputs. But this motivation looses much of its meaning if the time needed to get within an ε-distance from the steady state may grow unbounded as the initial state changes (even when the latter is picked within a fixed *bounded* set). In other words, *uniform* convergence to the steady state (which is automatically guaranteed in the case of linear systems) is an indispensable feature to be required in a nonlinear version of this notion. The set Ω, the union of all ω-limit points of all points in the state space does not have this property of uniform convergence, but there is a larger set which does have this property. This larger set, known as the ω limit set *of a set*, is precisely defined as follows.

Consider again system (6), let B be a subset of $\mathbb{R}^n$ and suppose $x(t, x_0)$ is defined for all $t \geq 0$ and all $x_0 \in B$. The ω-limit set of B, denoted $\omega(B)$, is the set of all points x for which there exists a sequence of pairs $\{x_k, t_k\}$, with $x_k \in B$ and $\lim_{k\to\infty} t_k = \infty$ such that

$$\lim_{k\to\infty} x(t_k, x_k) = x\,.$$

It is clear from the definition that if B consists of only one single point x_0, all x_k's in the definition above are necessarily equal to x_0 and the definition in question reduces to the definition of ω-limit set of a point, given earlier. It is also clear form this definition that, if for some $x_0 \in B$ the set $\omega(x_0)$ is nonempty, all points of $\omega(x_0)$ are points of $\omega(B)$. In fact, all such points have the property indicated in the definition, if all the x_k's are taken equal to x_0. Thus, in particular, if all motions with $x_0 \in B$ are bounded in positive time,

$$\bigcup_{x_0 \in B} \omega(x_0) \subset \omega(B) .$$

However, the converse inclusion is not true in general.

The relevant properties of the ω-limit set of a set, which extend those presented earlier in Lemma 1, can be summarized as follows [14, page 8].

Lemma 2 *Let B be a nonempty bounded subset of $\mathbb{R}^n$ and suppose there is a number M such that $\|x(t, x_0)\| \leq M$ for all $t \geq 0$ and all $x_0 \in B$. Then $\omega(B)$ is a nonempty compact set, invariant under (6). Moreover, the distance of $x(t, x_0)$ from $\omega(B)$ tends to 0 as $t \to \infty$, uniformly in $x_0 \in B$. If B is connected, so is $\omega(B)$.*

Thus, as it is the case for the ω-limit set of a point, we see that the ω-limit set of a bounded set, being compact and invariant, is filled with motions which exist for all $t \in (-\infty, +\infty)$ and are bounded backward and forward in time. But, above all, we see that the set in question is *uniformly* approached by motions with initial state $x_0 \in B$, a property that the set Ω does not have. Note also that the set of all such trajectories is a "behavior", in the sense of J.C. Willems [25].

The set $\omega(B)$, as shown in the previous Lemma, asymptotically attracts, as $t \to \infty$, all motions that start in B. Since the convergence to $\omega(B)$ is uniform in x_0, it is also true that, whenever $\omega(B)$ is contained in the interior of B, the set $\omega(B)$ is *asymptotically stable*, in the sense of Lyapunov.

4.2 The Steady State Behavior of a Nonlinear System

Consider now again system (6), with initial conditions in a closed subset $X \subset \mathbb{R}^n$. Suppose the set X is *positively invariant*, which means that for any initial condition $x_0 \in X$, the solution $x(t, x_0)$ exists for all $t \geq 0$ and $x(t, x_0) \in X$ for all $t \geq 0$. The motions of this system are said to be *ultimately bounded* if there is a bounded subset B with the property that, for every compact subset X_0 of X, there is a time $T > 0$ such that $\|x(t, x_0)\| \in B$ for all $t \geq T$ and all $x_0 \in X_0$. In other words, if the motions of the system are ultimately bounded, every motion eventually enters and remains in the bounded set B.

Suppose the motions of (6) are ultimately bounded and let $B' \neq B$ be any other bounded subset with the property that, for every compact subset X_0 of X, there is a time $T > 0$ such that $\|x(t, x_0)\| \in B'$ for all $t \geq T$ and all $x_0 \in X_0$. Then, it is easy to check that $\omega(B') = \omega(B)$. Thus, in view of the properties described in Lemma 2 above, the following definition can be adopted (see [7]).

Definition. Suppose the motions of system (6), with initial conditions in a closed and positively invariant set X, are ultimately bounded. A *steady state* motion is any motion with initial condition in $x(0) \in \omega(B)$. The set $\omega(B)$ is the *steady state locus* of (6) and the *restriction* of (6) to $\omega(B)$ is the *steady state behavior* of (6). $\triangleleft$

The notion introduced in this way recaptures the classical notion of steady state for linear systems and provides a new powerful tool to deal with similar issues in the case of nonlinear systems.

Example. In order to see how this notion includes the classical viewpoint, consider an n-dimensional, single-input, *asymptotically stable* linear system

$$\dot{z} = Fz + Gu \tag{7}$$

forced by the harmonic input $u(t) = u_0 \sin(\omega t + \phi_0)$. A simple method to determine the periodic motion of (7) consists in viewing the forcing input $u(t)$ as provided by an autonomous "signal generator" of the form

$$\dot{w} = Sw \qquad u = Qw$$

in which

$$S = \begin{pmatrix} 0 & \omega \\ -\omega & 0 \end{pmatrix} \qquad Q = \begin{pmatrix} 1 & 0 \end{pmatrix}$$

and in analyzing the state state behavior of the associated "augmented" system

$$\begin{aligned} \dot{w} &= Sw \\ \dot{z} &= Fz + GQw\,. \end{aligned} \tag{8}$$

As a matter of fact, let Π be the unique solution of the Sylvester equation $\Pi S = F\pi + GQ$ and observe that the graph of the linear map

$$\begin{array}{rcccc} \pi & : & \mathbb{R}^2 & \to & \mathbb{R}^n \\ & & w & \mapsto & \Pi w \end{array}$$

is an invariant subspace for the system (8). Since all trajectories of (8) approach this subspace as $t \to \infty$, the limit behavior of (8) is determined by the restriction of its motion to this invariant subspace.

Revisiting this analysis from the viewpoint of the more general notion of steady state introduce above, let $W \subset \mathbb{R}^2$ be a set of the form

$$W = \{w \in \mathbb{R}^2 : \|w\| \leq c\} \tag{9}$$

in which c is a fixed number, and suppose the set of initial conditions for (8) is $W \times \mathbb{R}^n$. This is in fact the case when the problem of evaluating the periodic response of (7) to harmonic inputs whose amplitude does not exceed a fixed number c is addressed. The set W is compact and invariant for the upper subsystem of (8) and, as it is easy to check, the ω-limit set of W under the motion of the upper subsystem of (8) is the subset W itself.

The set $W \times \mathbb{R}^n$ is closed and positively invariant for the full system (8) and, moreover, since the lower subsystem of (8) is a linear asymptotically stable system driven by a bounded input, it is immediate to check that the motions of system (8), with initial conditions taken in $W \times \mathbb{R}^n$, are ultimately bounded. As a matter of fact, any bounded set B of the form

$$B = \{(w, z) \in \mathbb{R}^2 \times \mathbb{R}^n : w \in W, \|z - \Pi w\| \leq d\}$$

in which d is any positive number, has the property indicated in the definition of ultimate boundedness. It is easy to check that

$$\omega(B) = \{(w, z) \in \mathbb{R}^2 \times \mathbb{R}^n : w \in W, z = \Pi w\}\,,$$

i.e. $\omega(B)$ is the graph of the restriction of the map π to the set W. The restriction of (8) to the invariant set $\omega(B)$ characterizes the steady state behavior of (7) under the family of all harmonic inputs of fixed angular frequency ω, and amplitude not exceeding c. $\triangleleft$

Example. A similar result, namely the fact that the *steady state locus* is the *graph* of a map, can be reached if the "signal generator" is any nonlinear system, with initial conditions chosen in a compact invariant set W. More precisely, consider an augmented system of the form

$$\begin{aligned} \dot{w} &= s(w) \\ \dot{z} &= Fz + Gq(w)\,, \end{aligned} \tag{10}$$

in which $w \in W \subset \mathbb{R}^r$, $x \in \mathbb{R}^n$, and assume that: (i) all eigenvalues of F have negative real part, (ii) the set W is a compact set, invariant for the the upper subsystem of (10).

As in the previous example, the ω-limit set of W under the motion of the upper subsystem of (10) is the subset W itself. Moreover, since the lower subsystem of (10) is a linear asymptotically stable system driven by the bounded input $u(t) = q(w(t, w_0))$, the motions of system (10), with initial conditions taken in $W \times \mathbb{R}^n$, are ultimately bounded.

It is easy to check that the steady state locus of (10) is the graph of the map

$$\begin{array}{rccc} \pi & : & W & \to & \mathbb{R}^n \\ & & w & \mapsto & \pi(w)\,, \end{array}$$

defined by

$$\pi(w) = \lim_{T\to\infty} \int_{-T}^{0} e^{-F\tau} Gq(w(\tau,w))d\tau\,. \tag{11}$$

To see why this is the case, pick any initial condition (w_0, z_0) for (10) on the graph of π and compute the solution $z(t)$ of the lower equation of (10) by means of the classical variation of constants formula, to obtain

$$z(t) = e^{Ft}z_0 + \int_0^t e^{F(t-\tau)}Gq(w(\tau,w_0))d\tau$$

Since by hypothesis $z_0 = \pi(w_0)$, using (11) one obtains

$$\begin{array}{rcl} z(t) & = & e^{Ft}\displaystyle\int_{-\infty}^{0} e^{-F\tau}Gq(w(\tau,w_0))d\tau + \int_0^t e^{F(t-\tau)}Gq(w(\tau,w_0))d\tau \\ & = & \displaystyle\int_{-\infty}^{t} e^{F(t-\tau)}Gq(w(\tau,w_0))d\tau = \int_{-\infty}^{0} e^{-F\theta}Gq(w(\theta+t,w_0))d\theta \\ & = & \displaystyle\int_{-\infty}^{0} e^{-F\theta}Gq(w(\theta,w(t,w_0)))d\theta = \pi(w(t,w_0)) = \pi(w(t)) \end{array}$$

which proves the invariance of the graph of π for (10). It is deduced from this that that any point of the graph of π is necessarily a point of the steady state locus of (10). To complete the proof of the claim it remains to show that no other point of $W \times \mathbb{R}^n$ can be a point of the steady state locus. But this is a straightforward consequence of the fact that F has eigenvalues with negative real part. $\triangleleft$

There are various ways in which the result discussed in the previous example can be generalized. For instance, it can be extended to describe the steady state response of a nonlinear system

$$\dot{z} = f(z,u) \tag{12}$$

in the neighborhood of a locally exponentially stable equilibrium point. To this end, suppose that $f(0,0) = 0$ and that the matrix

$$F = \left[\frac{\partial f}{\partial z}\right](0,0)$$

has all eigenvalues with negative real part. Then, it is well known (see e.g. [13, page 275]) that it is always possible to find a compact subset $Z \subset \mathbb{R}^n$, which contains $z = 0$ in its interior and a number $\sigma > 0$ such that, if $\|z_0\| \in Z$ and $\|u(t)\| \leq \sigma$ for all $t \geq 0$, the solution of (12) with initial condition $z(0) = z_0$ satisfies $\|z(t)\| \in Z$ for all $t \geq 0$. Suppose that the input u to (12) is produced, as before, by a signal generator of the form

$$\begin{array}{rcl} \dot{w} & = & s(w) \\ u & = & q(w) \end{array} \tag{13}$$

with initial conditions chosen in a compact invariant set W and, moreover, suppose that, $\|q(w)\| \leq \sigma$ for all $w \in W$. If this is the case, the set $W \times Z$ is positively invariant for

$$\begin{array}{rcl} \dot{w} & = & s(w) \\ \dot{z} & = & f(z,q(w))\,, \end{array} \tag{14}$$

and the motions of the latter are ultimately bounded, with $B = W \times Z$. The set $\omega(B)$ may have a complicated structure but it is possible to show, by means arguments similar to those which are used in the proof of the Center Manifold theorem, that if Z and B are small enough the set in question can still be expressed as the graph of a map $z = \pi(w)$. In particular, the graph in question is precisely the center manifold of (14) at $(0, 0)$ if $s(0) = 0$ and the matrix

$$S = \left[\frac{\partial s}{\partial w}\right](0)$$

has all eigenvalues on the imaginary axis.

A common feature of the examples discussed above is the fact that the steady state locus of a system of the form (14) can be expressed as the graph of a map $z = \pi(w)$. This means that, so long as this is the case, a system of this form has a *unique* well defined *steady state response* to the input $u(t) = q(w(t))$. As a matter of fact, the response in question is precisely $z(t) = \pi(w(t))$. Of course, this may not always be the case and *multiple* steady state responses to a given input may occur. In general, the following property holds.

Lemma 3 *Let W be a compact set, invariant under the flow of (13). Let Z be a closed set and suppose that the motions of (14) with initial conditions in $W \times Z$ are ultimately bounded. Then, the steady state locus of (14) is the graph of a set-valued map defined on the whole of W.*

5 Necessary Conditions for Output Regulation

Taking advantage of the notions introduced in the previous section, we are now in a position to highlight some general properties that any controller that solves a problem of output regulation must necessarily have. Recall that, as defined earlier, the problem of output regulation is solved if, in the composite system (5):

- the positive orbit of $W \times X \times \Xi$ is bounded,
- $\lim_{t\to\infty} e(t) = 0$, uniformly in the initial condition.

The notions introduced in the previous section are instrumental to prove the following, elementary – but fundamental – result, which is a nonlinear enhancement of a Lemma of [10] on which all the theory of output regulation for linear systems is based.

Lemma 4 *Suppose the positive orbit of $W \times X \times \Xi$ is bounded. Then*

$$\lim_{t\to\infty} e(t) = 0$$

if and only if

$$\omega(W \times X \times \Xi) \subset \{(w, x, \xi) : h(w, x) = 0\}. \tag{15}$$

It is seen from this simple result that the problem of output regulation can be simply cast as the problem of *shaping the steady state locus of the closed loop system*, in such a way that property (15) holds.

To proceed with the analysis in a more concrete fashion, we consider from now on the special case in which the controlled plant (4) is modelled by equations *in normal form*

$$\begin{aligned}
\dot{z} &= f_0(w, z) + f_1(w, z, e_1)e_1 \\
\dot{e}_1 &= e_2 \\
&\vdots \\
\dot{e}_{r-1} &= e_r \\
\dot{e}_r &= q(w, z, e_1, \dots, e_r) + b(w, z, e_1, \dots, e_r)u \\
e &= e_1 \\
y &= \mathrm{col}(e_1, \dots, e_r)\,,
\end{aligned} \tag{16}$$

with state $(z, e_1, \ldots, e_r) \in \mathbb{R}^{n-r} \times \mathbb{R}^r$, control input $u \in \mathbb{R}$, regulated output $e \in \mathbb{R}$, measured output $y \in \mathbb{R}^r$. The functions $f_0(\cdot), f_1(\cdot), q(\cdot), b(\cdot), s(\cdot)$ in (16) and (3) are assumed to be at least continuously differentiable. It is also assumed that

$$b(w, z, e_1, \ldots, e_r) \neq 0 \qquad \forall (w, z, e_1, \ldots, e_r)\,.$$

The initial conditions of (16) range on a set $Z \times E$, in which Z is a fixed *compact* subset of $\mathbb{R}^{n-r}$ and $E = \{(e_1, \ldots, e_r) \in \mathbb{R}^r : |e_i| \leq c\}$, with c a fixed number.

Suppose that a controller of the form (4) solves the problem of output regulation. Then Lemma 2 applies and, since $e = e_1$, we deduce that the steady state locus of the closed loop system (5) is necessarily a subset of the set of all states in which $e_1 = 0$. This being the case, it is seen from the form of the equations (16) that, when the closed loop system (5) is in steady state, necessarily also

$$e_2 = e_3 = \cdots = e_r = 0\,.$$

As a consequence, the following conclusions hold:

- *The steady state locus $\omega(W \times Z \times E \times \Xi)$ of the closed-loop system is a subset of the set $W \times \mathbb{R}^{n-r} \times \{0\} \times \mathbb{R}^\nu$.*
- *The restriction of the closed-loop system to its steady state locus $\omega(W \times Z \times E \times \Xi)$ reduces to*

$$\begin{array}{rcl} \dot{w} &=& s(w) \\ \dot{z} &=& f_0(w, z) \\ \dot{\xi} &=& \varphi(\xi, 0)\,. \end{array} \tag{17}$$

- *For each $(w, z, 0, \ldots, 0, \xi) \in \omega(W \times Z \times E \times \Xi)$*

$$0 = q(w, z, 0, \ldots, 0) + b(w, z, 0, \ldots, 0)\gamma(\xi, 0)\,. \tag{18}$$

The prior analysis implicitly assumes that the positive orbit of W under the flow of exosystem is bounded, i.e. that the motions of the exosystem asymptotically approach the its own steady state locus $\omega(W)$. In principle, $\omega(W)$ may differ from W but there is no loss of generality in assuming from the very beginning that the two sets coincide. After all, the problem in question is a problem concerning how the closed-loop system behaves in steady state and there is no special interest in considering exosystems that are not "in steady state". We make this assumption precise as follows.

Assumption (i): the compact set W is invariant for (3). $\triangleleft$

With this in mind we observe that, by Lemma 3, if the positive orbit of $W \times Z \times E \times \Xi$ under the flow of (5) is bounded, then $\omega(W \times Z \times E \times \Xi)$ is the graph of a (possibly set-valued) map defined on the whole of W. Consider now the set

$$\mathcal{A}_{\rm ss} = \{(w, z) : (w, z, 0, \ldots, 0, \xi) \in \omega(W \times Z \times E \times \Xi),\ \text{for some } \xi \in \mathbb{R}^\nu\}$$

and define the map

$$\begin{array}{rccl} u_{\rm ss}: & \mathcal{A}_{\rm ss} & \to & \mathbb{R} \\ & (w, z) & \mapsto & -\dfrac{q(w, z, 0, \ldots, 0)}{b(w, z, 0, \ldots, 0)}\,. \end{array}$$

By construction, the set $\mathcal{A}_{\rm ss}$ is the graph of a (possibly set-valued) map defined on the whole of W, which is invariant for the dynamics of

$$\begin{array}{rcl} \dot{w} &=& s(w) \\ \dot{z} &=& f_0(w, z)\,, \end{array} \tag{19}$$

that are precisely *the zero dynamics of the "augmented system"* (3) – (16), while the map $u_{\rm ss}(\cdot)$ is the control that forces the motion of (3) – (16) to evolve on $\mathcal{A}_{\rm ss}$.

With this in mind, the conclusions reached above can be rephrased in the following terms. Suppose that a controller of the form (4) solves the problem of output regulation for (16) with exosystem (3). Then, there exists a (possibly set-valued) map defined on the whole of W whose graph $\mathcal{A}_{\rm ss}$ is invariant for the autonomous system (19). Moreover, for each $(w_0, z_0) \in \mathcal{A}_{\rm ss}$ there is a point $\xi_0 \in \mathbb{R}^\nu$ such that the integral curve of (19) issued from (w_0, z_0) and the integral curve of

$$\dot{\xi} = \varphi(\xi, 0)$$

issued from ξ_0 satisfy

$$u_{\rm ss}(w(t), z(t)) = \gamma(\xi(t))\,, \qquad \forall t \in \mathbb{R}\,.$$

This is a nonlinear version of the celebrated *internal model principle* of [11].

6 Sufficient Conditions for Output Regulation

6.1 The Control Structure

On the basis of the ideas presented in the previous section we proceed now with the construction of a controller that solves the problem of output regulation. The "steady state" features of this controller are those identified at the end of the section, namely this controller has to be able to "generate" all controls of the form $u_{\rm ss}(w(t), z(t))$ for any "steady state" trajectory $w(t), z(t)$ of (19). The controller should incorporate a device that generates all such trajectories (the *internal model*), thus making sure that the "appropriate" state-state behavior takes place, and a device guaranteeing that convergence to this specific steady state behavior occurs. It is here that additional assumptions are needed.

Note that, since W is invariant for $\dot{w} = s(w)$, the closed cylinder

$$\mathcal{C} := W \times \mathbb{R}^{n-r}$$

is locally invariant for (19). Hence, it is natural regard (19) as a system defined on $\mathcal{C}$ and endow the latter with the subset topology.

Assumption (ii): there exists a bounded subset B of $\mathcal{C}$ which contains the positive orbit of the set $W \times Z$ under the flow of (19) and the resulting omega-limit set

$$\mathcal{A} := \omega(W \times Z)$$

satisfies

$$(w, z) \in \mathcal{C}\,, \quad |(w, z)|_{\mathcal{A}} \le d_0 \quad \Rightarrow \quad z \in Z \tag{20}$$

where d_0 is a positive number. $\triangleleft$

While in the analysis of the necessity we have only identified the existence of a compact set (actually, the graph of a map defined on W) which is invariant for (19), the new assumption (ii) implies, in its first part, the existence of a compact set $\mathcal{A}$ (still the graph of a map defined on W) which is not only invariant but also uniformly attractive of all trajectories of (19) issued from points of $W \times Z$. The second part of the assumption, in turn, guarantees that this set is also stable in the sense of Lyapunov. In the next assumption we strengthen this property by also requiring the set $\mathcal{A}$ is locally exponentially stable (this assumption is useful to straighten the subsequent analysis, but is not essential).

Assumption (iii): there exist $M \ge 1$, $\lambda > 0$ such that

$$(w_0, z_0) \in \mathcal{C}\,, \quad |(w_0, z_0)|_{\mathcal{A}} \le d_0 \quad \Rightarrow \quad |(w(t), z(t))|_{\mathcal{A}} \le M e^{-\lambda t} |(w_0, z_0)|_{\mathcal{A}}$$

in which $(w(t), z(t))$ denotes the solution of (19) passing through (w_0, z_0) at time $t = 0$. $\triangleleft$

To simplify the exposition, we address the special case in which the controlled system (16) has relative degree 1, and in which the coefficient $b(w, z, e_1)$ is identically equal to 1. In other words, we consider a system is modelled by equations of the form

$$\begin{array}{rcl} \dot{z} & = & f_0(w,z) + f_1(w,z,e)e \\ \dot{e} & = & q(w,z,e) + u \\ y & = & e\,. \end{array} \tag{21}$$

There is no loss of generality in considering a system having this simple form (21) because, as shown for instance in [9, 22], the case of a more general system of the form (16) can be easily reduced, by appropriate manipulations, to this one.

For convenience, rewrite the augmented system (3) – (21) as

$$\begin{array}{rcl} \dot{\mathbf{z}} & = & \mathbf{f}_0(\mathbf{z}) + \mathbf{f}_1(\mathbf{z},e)e \\ \dot{e} & = & \mathbf{q}_0(\mathbf{z}) + \mathbf{q}_1(\mathbf{z},e)e + u \end{array} \tag{22}$$

having set $\mathbf{z} = (w, z)$. Consistently let $\mathbf{Z} := W \times Z$ denote the compact set where the initial condition $\mathbf{z}(0)$ is supposed to range. In these notations, assumptions (i) – (ii) - (iii) express the property that, in the autonomous system

$$\dot{\mathbf{z}} = \mathbf{f}_0(\mathbf{z})\,, \tag{23}$$

the set $\mathcal{A}$ is asymptotically and locally exponentially stable, with a domain of attraction that contains the set $\mathbf{Z}$.

Suppose now that the control u is chosen as $u = -ke$. The closed-loop system thus obtained, can be regarded as a feedback interconnection of

$$\dot{\mathbf{z}} = \mathbf{f}_0(\mathbf{z}) + \mathbf{f}_1(\mathbf{z},e)e \tag{24}$$

viewed as a system with input e and state $\mathbf{z}$, and

$$\dot{e} = \mathbf{q}_0(\mathbf{z}) + \mathbf{q}_1(\mathbf{z},e)e - ke \tag{25}$$

viewed as a system with input $\mathbf{z}$ and state e.

By assumption, system (24) possesses, when $e = 0$, an invariant set $\mathcal{A}$ which is asymptotically and locally exponentially stable, with a domain of attraction that contains the set $\mathbf{Z}$ of all admissible initial conditions. Thus, standard arguments (see, e.g. [6]) can be invoked to claim that, if k is large enough, all trajectories of the interconnection (24) – (25) with initial conditions in $\mathbf{Z} \times E$ remain bounded and the state $(\mathbf{z}, e)$ can be steered to an arbitrary small neighborhood of the set $\mathcal{A} \times \{0\}$. This does not solve the problem at issue, though, because the variable $e(t)$ is not guaranteed to converge to zero (but only to converge to a neighborhood of zero, whose size can be made arbitrarily small by increasing the gain coefficient k). The condition for having $e(t) \to 0$ as $t \to \infty$ is simply that the "coupling" term $\mathbf{q}_0(\mathbf{z})$ vanishes on the set $\mathcal{A}$, but there is no reason for this to occur (see again, e.g.[6]). This is why a more elaborate, internal-model-based, controller is needed.

System (16) being affine in the control input u, it seems natural to look for a controller having a similar structure, namely a controller of the form

$$\begin{array}{rcl} \dot{\xi} & = & \varphi(\xi) + Gv \\ u & = & \gamma(\xi) + v \end{array} \tag{26}$$

with state $\xi \in \mathbb{R}^\nu$, in which v is a residual control input, to be eventually chosen as a function of the measured output y. Here $\varphi(\cdot)$, G and $\gamma(\cdot)$ are functions to be determined. we will show in what follows that, if the triplet $\{\varphi(\xi), G, \gamma(\xi)\}$ possesses what we will define as *asymptotic internal model* property, the choice of the residual control v in (26) as

$$v = ke$$

solves the problem of output regulation, provided that the gain coefficient k is sufficiently high.

6.2 The Internal Model

Controlling this system by means of (26) yields a closed-loop system

$$\begin{aligned}\dot{\mathbf{z}} &= \mathbf{f}_0(\mathbf{z}) + \mathbf{f}_1(\mathbf{z}, e)e \\ \dot{e} &= \mathbf{q}_0(\mathbf{z}) + \mathbf{q}_1(\mathbf{z}, e)e + \gamma(\xi) + v \\ \dot{\xi} &= \varphi(\xi) + Gv\end{aligned} \tag{27}$$

which, regarded as a system with input v and output e, has relative degree 1 and zero dynamics given by

$$\begin{aligned}\dot{\mathbf{z}} &= \mathbf{f}_0(\mathbf{z}) \\ \dot{\xi} &= \varphi(\xi) - G[\gamma(\xi) + \mathbf{q}_0(\mathbf{z})]\,.\end{aligned} \tag{28}$$

System (27) can be put in normal form by means of the change of variables

$$x = \xi - Ge$$

which yields

$$\begin{aligned}\dot{\mathbf{z}} &= \mathbf{f}_0(\mathbf{z}) + \mathbf{f}_1(\mathbf{z}, e)e \\ \dot{x} &= \varphi(x + Ge) - G\gamma(x + Ge) - G\mathbf{q}_0(\mathbf{z}) - G\mathbf{q}_1(\mathbf{z}, e)e \\ \dot{e} &= \mathbf{q}_0(\mathbf{z}) + \mathbf{q}_1(\mathbf{z}, e)e + \gamma(x + Ge) + v\,.\end{aligned} \tag{29}$$

Setting $\mathbf{x} = (\mathbf{z}, x)$, this system can be further rewritten in the form

$$\begin{aligned}\dot{\mathbf{x}} &= f(\mathbf{x}) + \ell(\mathbf{x}, e)e \\ \dot{e} &= q(\mathbf{x}) + r(\mathbf{x}, e)e + v\end{aligned} \tag{30}$$

in which

$$f(\mathbf{x}) = \begin{pmatrix} \mathbf{f}_0(\mathbf{z}) \\ \varphi(x) - G[\gamma(x) + \mathbf{q}_0(\mathbf{z})] \end{pmatrix}$$

$$q(\mathbf{x}) = \mathbf{q}_0(\mathbf{z}) + \gamma(x)$$

and $\ell(\mathbf{x}, e)$, $r(\mathbf{x}, e)$ are suitable continuous functions.

Suppose now that the residual control v is chosen as $v = -ke$. This yields a closed-loop system having a structure similar to the one considered in the previous subsection, namely a feedback interconnection of

$$\dot{\mathbf{x}} = f(\mathbf{x}) + \ell(\mathbf{x}, e)e \tag{31}$$

viewed as a system with input e and state $\mathbf{x}$, and

$$\dot{e} = q(\mathbf{x}) + r(\mathbf{x}, e)e - ke \tag{32}$$

viewed as a system with input $\mathbf{x}$ and state e. As claimed earlier, a high-gain control on e, namely a control $v = -ke$ with large k, would succeed in steering $e(t)$ to zero if two conditions are fulfilled:

(P1) the dynamics (28) possesses a compact invariant set which is asymptotically (and locally exponentially) stable, with a domain of attraction that contains the set $\mathbf{Z} \times \Xi$ of all admissible initial conditions, and

(P2) the function $\mathbf{q}_0(\mathbf{z}) + \gamma(\xi)$ vanishes on this invariant set.

These are the properties that will be sought in what follows. Note that the fulfillment of these is determined by properties of the autonomous system (23) and of the function

$$\rho = \mathbf{q}_0(\mathbf{z}) \tag{33}$$

which, in the composite system (28) can be viewed as the output of (23) driving a system of the form

$$\dot{\xi} = \varphi(\xi) - G[\gamma(\xi) + \mathbf{q}_0(\mathbf{z})]\,. \tag{34}$$

For convenience, we will say that triplet $\{\varphi(\xi), G, \gamma(\xi)\}$ *is an asymptotic internal model of the pair* (23) – (33) if properties (P1) and (P2) are satisfied. In this terminology, we can summarize as follows the conclusion obtained so far.

Proposition 1 *Pick compact sets* $\mathbf{Z}$*,* E *and* Ξ *for the initial conditions of the closed-loop system (3), (21), (26). Assume that (i)-(ii)-(iii) hold and that the triplet* $\{\varphi(\xi), G, \gamma(\xi)\}$ *is an asymptotic internal model property of (23) – (33). Then there exists* $k^\star > 0$ *such that for all* $k \geq k^\star$ *the controller (26) with* $v = -ke$ *solves the generalized tracking problem.*

The notion of steady state provides a useful interpretation of the properties in question. In fact, recall that, by Assumption A, all trajectories of system (23) with initial conditions in $\mathbf{Z}$ asymptotically converge to the compact invariant set $\mathcal{A}$, and the latter is also locally exponentially stable. If property (P1) holds, all trajectories of the composite system (28) with initial conditions in $\hat{Z} \times \Xi$ asymptotically converge to the limit set $\omega(\mathbf{Z} \times \Xi)$. Since (28) is a triangular system, its is readily seen, that the set $\omega(\mathbf{Z} \times \Xi)$ is the graph of a set-valued map defined on $\mathcal{A}$, i.e. that there exists a map

$$\tau : \quad \mathbf{z} \in \mathcal{A} \quad \mapsto \quad \tau(\mathbf{z}) \subset \mathbb{R}^\nu \,,$$

such that

$$\omega(\mathbf{Z} \times \Xi) = \{(\mathbf{z}, x) : \mathbf{z} \in \mathcal{A}, \xi \in \tau(\mathbf{z})\} := \mathrm{gr}(\tau)\,.$$

The set $\mathrm{gr}(\tau)$ is the steady state locus of (28) and the restriction of the latter to this invariant set characterizes its steady state behavior. Property (P2), on the other hand, expresses the property that at each point of $(\hat{z}, \xi) \in \mathrm{gr}(\tau)$

$$\hat{q}_0(z) = -\gamma(\xi)\,. \tag{35}$$

Thus, looking again at system (28), it is realized that $\mathrm{gr}(\tau)$ is in fact invariant for

$$\begin{aligned} \dot{\mathbf{z}} &= \mathbf{f}_0(\mathbf{z}) \\ \dot{\xi} &= \varphi(\xi)\,. \end{aligned} \tag{36}$$

Note that, if the map $\tau(\mathbf{z})$ is single-valued and C^1, its invariance for (36) is expressed by the property that

$$\frac{\partial \tau(\mathbf{z})}{\partial \mathbf{z}} \mathbf{f}_0(\mathbf{z}) = \varphi(\tau(\mathbf{z})) \qquad \forall\, \mathbf{z} \in \mathcal{A}\,, \tag{37}$$

while the fact that (35) holds at each point of $(\mathbf{z}, \xi) \in \mathrm{gr}(\tau)$ is expressed by the property that

$$\mathbf{q}_0(\mathbf{z}) = -\gamma(\tau(\mathbf{z})) \qquad \forall\, \mathbf{z} \in \mathcal{A}\,. \tag{38}$$

6.3 The Design of an Internal Model

As we have seen in the earlier sections, the proposed controller, if the asymptotic internal model property holds, is able to force – in the closed loop system – convergence to a steady state in which the regulated variable is identically zero. As a consequence, the controller solves the generalized tracking problem. It remains to be shown, therefore, how the asymptotic internal model property can be obtained. To this end, it is convenient to observe that the properties required in (P1) and (P2) are quite similar to properties that are usually sought in the design of *state observers.* As a matter of fact it is seen from (37) and (38) that, for each $\mathbf{z}_0 \in \mathcal{A}$, the function of time

$$\hat{\xi}(t) = \tau(\mathbf{z}(t, \mathbf{z}_0))$$

which is defined (and bounded) for all $t \in \mathbb{R}$ satisfies

$$\frac{\mathrm{d}\hat{\xi}(t)}{\mathrm{d}t} = \varphi(\hat{\xi}(t)) \tag{39}$$

and, moreover

$$\gamma(\hat{\xi}(t)) = -\mathbf{q}_0(\mathbf{z}(t, \mathbf{z}_0))\,.$$

In view of the latter, system (34) can be rewritten in the form

$$\dot{\xi} = \varphi(\xi) + G[\gamma(\hat{\xi}) - \gamma(\xi)] \tag{40}$$

and interpreted as *a copy of the dynamics* (39) *of* $\hat{\xi}$ *corrected by an "innovation term"* $[\gamma(\hat{\xi}) - \gamma(\xi)]$ *weighted by an "output injection gain"* G. This is the classical structure on an *observer* and the requirement in (P1) expresses the property that the difference $\xi(t) - \hat{\xi}(t)$ (the "observation error", in our interpretation) should asymptotically decay to zero (with ultimate exponential decay).

This interpretation is at the basis of a number of major recent advances in the design of regulators. In fact, in a number of recent papers, this interpretation has been pursued and, taking into consideration various approaches to the design of nonlinear observers, has lead to effective design methods (see [5],[9], [22]). Two of such design methods are highlighted in the remaining part of this section.

6.3.1 The high-gain observer as an internal model (see [5])

The construction summarized in this section relies upon the following additional hypothesis.

Assumption (iv). Suppose there exist an integer $d > 0$ and a locally Lipschitz function $f : \mathbb{R}^d \to \mathbb{R}$ such that, for any $\mathbf{z}_0 \in \mathcal{A}$, the solution $\mathbf{z}(t)$ of passing through $\mathbf{z}_0$ at time $t = 0$ is such that the function $\rho(t) := \mathbf{q}_0(\mathbf{z}(t))$ satisfies

$$\rho^{(d)}(t) = f(\rho(t), \rho^{(1)}(t), \ldots, \rho^{(d-1)}(t))$$

for all $t \in \mathbb{R}$. $\lhd$

Let $\tau : W \times \mathbb{R}^{n-1}\mathbb{R}^d$ be the map defined as

$$\tau(\mathbf{z}) := \mathrm{col}(\mathbf{q}_0(\mathbf{z}), L_{\mathbf{f}_0}\mathbf{q}_0(\mathbf{z}), \ldots, L_{\mathbf{f}_0}^{d-1}\mathbf{q}_0(\mathbf{z})) \tag{41}$$

and let $f_c : \mathbb{R}^d \to \mathbb{R}$ be a C^1 function with compact support which agrees with $f(\cdot)$ on $\tau(\mathcal{A})$. Then, it easy to check that the properties indicated in (37) and (38) are fulfilled by choosing

$$\varphi(\xi) = \begin{pmatrix} \xi_2 \\ \vdots \\ \xi_d \\ f_c(\xi_1, \xi_2, \ldots, \xi_d) \end{pmatrix}, \qquad \gamma(\xi) = \xi_1 \,. \tag{42}$$

Comparing this construction with the remark after Definition we observe, in particular, that system

$$\begin{array}{rcl} \dot{\mathbf{z}} & = & \mathbf{f}_0(\mathbf{z}) \\ \rho & = & \mathbf{q}_0(\mathbf{z}) \end{array} \tag{43}$$

is *immersed into a system which is uniformly observable*, in the sense of [12] (even though system (43) might not have had such a property). It is precisely this that makes it possible to choose G in such a way that also the property indicated in (P1) can be achieved.

As a matter of fact, the property in question is achieved by choosing

$$G = D_g \begin{pmatrix} c_0 \\ \vdots \\ c_{d-1} \end{pmatrix}$$

where $D_g = \mathrm{diag}(g, g^2, \cdots, g^d)$, g is a design parameter, and the c_i's are such that the polynomial $\lambda^d + c_0\lambda^{d-1} + \cdots + c_{d-1} = 0$ is Hurwitz, as formally proved in Lemmas 1 and 2 of [5] to which the interested reader is referred for details.

It is worth noting that the assumption in question clearly covers the interesting (and widely addressed in the recent past literature, see [15]) case in which the function $f(\cdot)$ is linear, namely the case in which (43) is immersed into a linear observable system. In this case, although the choice indicated above is clearly still valid, a more direct way of designing the regulator is to use $f(\cdot)$ instead of $f_c(\cdot)$ in the definition of $\varphi(\xi)$, and simply choose G in such a way that $\dot{\xi} = \varphi(\xi) - G\gamma(\xi)$ is a stable linear system.

6.3.2 The Andrieu-Praly's observer as an internal model (see [22])

In this subsection we exploit certain results of the theory presented in [1] to weaken (and, to some extent, suppress) the Assumption (iv) presented at the beginning of the earlier subsection.

Let $(F, G) \in \mathbb{R}^{d\times d} \times \mathbb{R}^{d\times 1}$ be a controllable pair and set

$$\varphi(\xi) = F\xi + G\gamma(\xi), \tag{44}$$

with $\gamma : \mathbb{R}^d \to \mathbb{R}$ a continuous function to be chosen determined later. If this is the case, the composite system (28) becomes

$$\begin{array}{rcl} \dot{\mathbf{z}} & = & \mathbf{f}_0(\mathbf{z}) \\ \dot{\xi} & = & F\xi - G\mathbf{q}_0(\mathbf{z})\,. \end{array} \tag{45}$$

which is precisely a system of the form (10), considered in Example 4.2. If the matrix F is Hurwitz, and we restrict $\mathbf{z}$ to belong to the set $\mathcal{A}$, this system has a well defined steady state behavior, which is the graph of the map

$$\tau(\mathbf{z}) = \int_{-\infty}^{0} e^{-Fs} G\mathbf{q}_0(\mathbf{z}(s,\mathbf{z}))ds\,. \tag{46}$$

As shown in Example 4.2, the graph in question is invariant for system (45), is asymptotically (and locally exponentially) stable and with a domain of attraction that coincides with $W \times \mathbb{R}^d$. Moreover, it can also be shown that there exists a number $\ell > 0$ such that, if the eigenvalues of F have real part which is less ℓ, the map (46) is C^1 (see, e.g. [22]). If this is the case, to say that the graph is invariant for (45) is equivalent to say that

$$\frac{\partial \tau}{\partial \mathbf{z}}\mathbf{f}_0(\mathbf{z}) = F\mathbf{z} - G\mathbf{q}_0(\mathbf{z}) \qquad \forall\, \mathbf{z} \in \mathcal{A} \tag{47}$$

This being the case, it is immediate to check that properties (P1) and (P2) will be fulfilled if a function $\gamma(\xi)$ can be found that renders (38) satisfied. As a matter of fact, bearing in mind (44), condition (37) becomes

$$\frac{\partial \tau}{\partial \mathbf{z}}\mathbf{f}_0(\mathbf{z}) = F\mathbf{z} - G\gamma \circ \tau(\mathbf{z}) \qquad \forall\, \mathbf{z} \in \mathcal{A}$$

which, if condition (38) holds, reduces to (47). This, shows that a triplet having the asymptotic internal model property can be found if a function $\gamma(\cdot)$ exist which satisfies (38). It is here that the dimension d of the pair (F, G) plays a role, as formalized in the next proposition whose proof can be found in [22].

Proposition 2 *Suppose*

$$d \geq 2(s + n - r) + 2\,.$$

Then for almost all choices (see [22] for details) of a controllable pair (F, G), with F a Hurwitz matrix whose eigenvalues have real part which is less ℓ, the map (46) satisfies

$$\tau(\mathbf{z}_1) = \tau(\mathbf{z}_2) \qquad \Rightarrow \qquad \mathbf{q}_0(\mathbf{z}_1) = \mathbf{q}_0(\mathbf{z}_2)\,.$$

As a consequence there exist a continuous map $\gamma : \tau(\mathcal{A}) \to \mathbb{R}$ fulfilling (38).

The map $\tau(\cdot)$ in (46) is defined only on $\mathcal{A}$, but is not difficult to extend it to a C^1 map defined on the whole set $W \times \mathbb{R}^{n-r}$, as shown in [22]. Also the map $\gamma(\cdot)$ that makes (38) true can be extended to the whole $\mathbb{R}^d$, but this extension is only known to be continuous.

We have shown in this way that the existence of triplet $\psi(\xi), G, \gamma(\xi)$ which has the internal model property can always be achieved, so long as the integer d is large enough. This result shows that the general design procedure outlined earlier in the article is always applicable (so long as the standing hypotheses (i) – (ii) – (iii) are applicable). From the constructive viewpoint, though, is must be observed that the result indicated in Proposition 2 is only an existence result and that the function $\gamma(\xi)$ whose existence is guaranteed in only known to be continuous. Obtaining continuous differentiability of such $\gamma(\xi)$ and a constructive procedure are likely to require further hypotheses, which should be in any case weaker than Assumption (iv) considered earlier, whose study is subject of current investigation.

7 Future Directions

One of the basic hypotheses of the theory described in the previous sections is that the zero dynamics of the augmented system consisting of the controlled plant and of the exosystem possesses a compact attractor which is also locally asymptotically stable. This assumption, with an acceptable abuse of terminology, is usually referred to as the “minimum-phase” property. Another standing hypothesis is that the regulated variable coincides with the measured variable. The future directions of the research in this area research are aimed at the removal of these assumptions. There are several directions in which the problem can be tackled. One way is the use of control structures by means of which the controlled system (plus a part of the controller) can be interpreted with systems having a different zero dynamics (in particular a zero dynamics possessing a compact attractor). This is the nonlinear equivalent of certain design procedures for linear systems based on the assignment of zeros. This technique has proven to be powerful in the stabilization of certain classes of nonlinear systems (see [18]) and is expected to be successful, with appropriate enhancements, in the design of regulators. The analysis of the necessary conditions for output regulation has also shown that, if the generalized tracking problem is solvable, the inverse dynamics of the augmented system consisting of the controlled plant and of the exosystem possesses a compact invariant set which to which all initial conditions are asymptotically controllable (by means of the regulated variable viewed as a control). This set is not necessarily asymptotically stable (as it would be under the “minimum-phase” hypothesis) but could be asymptotically stabilizable (by either full-state or measurement-based feedback). Thus, another direction in which the research will evolve, in the development of control schemes for possibly non “minimum-phase” nonlinear systems, is based on the exploitation of the (weaker) assumption that in the zero dynamics of the augmented system there is a compact set that can be made invariant and asymptotically stable by means of feedback. This will yield a design procedure in which a virtual control (either full-state or measurements-based) is designed to stabilize that compact invariant set. The (possibly dynamic) controller obtained in this way will then be embedded into a regulator designed according to the principles developed in [24].

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