Enforcing contraction via data

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***Abstract*—We present data-based conditions for enforcing contractivity via feedback control and obtain desired asymptotic properties of the closed-loop system. We focus on unknown nonlinear control systems whose vector fields are expressible via a dictionary of functions and derive data-dependent semidefinite programs whose solution returns the controller that guarantees contractivity. When data are perturbed by disturbances that are linear combination of sinusoids of known frequencies (but unknown amplitude and phase) and constants, we remarkably obtain conditions for contractivity that do not depend on the magnitude of the disturbances, with imaginable positive consequences for the synthesis of the controller. Finally, we show how to design from data an integral controller for nonlinear systems that achieves constant reference tracking and constant disturbance rejection.**

I. Introduction

There is widespread attention in the control community towards new ways to utilize data collected from dynamical systems in the process of designing feedback control. The interest is motivated by the shared belief that machine learning is revolutionising many branches of science and engineering, including automatic control, and learning to design control from data is a first step towards a thorough cross-fertilization of machine learning and control theory.

Albeit the use of data has always represented the backbone of control design in areas such as adaptive control, recently a new wave of solutions have focused on the so-called direct data-driven control design, which produces control policies from data collected in offline experiments without *explicitly* attempting to identify the system [1], [2]. Among the various proposed solutions, the one that relates control design to data-dependent convex programs is one of the approaches that has caught on the most in recent times. Such approach has mainly focused on linear control systems [3], [4], [5], [6].

In this paper, we are interested to data-driven control for nonlinear systems. Designing feedback control from data is inevitably harder due to the varied complexity of nonlinear dynamical systems. Many of the most successful model-based nonlinear control design techniques have exploited special structures of nonlinear systems [7], which are difficult to work with when the systems to control are hardly known. To date, contributions to data-driven control design for nonlinear systems are sparse. In [4], by taking into account the impact of the remainder as perturbations corrupting the data, a feedback controller that stabilizes an equilibrium of nonlinear systems by the first approximation was designed from data. As for linear systems, the design was based on the solution of semidefinite programs and prompted the observation that this approach might be fruitful for those classes of nonlinear systems whose control design relies on convex programs. These include bilinear systems [8], [9] and polynomial systems for which [10] and [11] independently proposed distinct approaches. Later, [12] remarked that the results of [11] could be extended to rational systems. By Taylor's series expansion, the results on data-driven control for linear and polynomials systems can be used for general nonlinear systems by a careful analysis of the remainder [13], [14], [15]. Another way to deal with general nonlinear systems is to express the system's vector fields via dictionaries of nonlinear functions and then design controllers that makes the closed-loop system dominantly linear [16]. When coupled with coordinates transformations, these techniques do mimic a data-dependent version of nonlinear feedback linearization [16, Section VII.B]. Data-dependent nonlinear feedback linearization has been also investigated in the context of predictive control [17], and using Gaussian process regression [18]. Other approaches consist of looking at the nonlinear system as a linear one with state dependent matrices and recursively designing the feedback controller at each time step via linear control techniques [19] or carrying out an LPV embedding of an incremental representation of the nonlinear system followed by a data-driven control synthesis [20].

Contribution and related works. For autonomous discrete-time dynamical systems whose vector field is a contraction mapping, convergence to a fixed point is a consequence of the contraction mapping theorem. More in general and for both discrete- and continuous-time systems, concepts such as contraction analysis [21], incremental stability [22] and convergent systems [23] (whose relations have been studied in [24], [25]) have established deep connections between asymptotic properties of trajectories with respect to each other and Lyapunov analysis. When specialized to systems that have an equilibrium, establishing contractivity does not require the explicit knowledge of the equilibrium itself, a feature which is useful when dealing with unknown systems. More importantly, contractivity is instrumental in solving various problems such as observer design and output regulation.

The purpose of this paper is to present data-based conditions for enforcing contractivity via feedback control and obtain desired asymptotic properties of the closed-loop system. As in [16], we focus on unknown nonlinear control systems whose vector fields are expressible via a dictionary of functions and aim at deriving data-dependent semidefinite programs whose solution returns the controller that guarantees contractivity. Using semidefinite programs to guarantee contractivity via feed-

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back has been used in the case the system's model is known [26], [27], [28], but the use of data to offset the uncertainty on the model makes the presented results distinctively novel. The presented approach also carries over to the case of data that are perturbed by unknown but bounded process disturbances. Even more, if the disturbances are linear combinations of constant and sinusoidal signals of known frequencies (but unknown amplitude and phase), we present a new result where the data-based conditions for contractivity do not depend on the magnitude of the disturbance affecting the data and remarkably bear similarities with the ones obtainable in the case of noise-free data, with imaginable positive consequences for the synthesis of the controller. As our final contribution, in order to underscore the relevance of contractivity in data-driven control problems different from mere stabilization, we show how to design from data an integral controller for nonlinear systems that achieves constant reference tracking and constant disturbance rejection. For this problem, we follow the idea in [29], [30] of rendering the given system, extended with the integral controller and without forcing inputs, contractive and then analyzing the impact of the reference and the disturbance by relying on the established contractivity. This idea does not rely on special forms of the system and is therefore well suited for data-based control design.

Being a powerful concept to deal with nonlinear systems analysis and control, contraction theory has been already employed in learning-based control. Some existing results have been surveyed in [31], which uses estimation techniques based on neural networks ([31]), while [32, Section 4] uses Gaussian process regression to estimate a closed-loop system and to give a condition for incremental exponential stability in the second moment.

Outline. The framework and problem formulation are discussed in Section II. The main result is given in Section III, which is then expanded to deal with noisy data (Section V). The special case of disturbances given by linear combination of sinusoids and constants is studied in Section VI. The paper ends with the application of the previous results to the design of integral controllers (Section VII). To make the paper self-contained, we added an Appendix that contains a few basic results about contractive and convergent systems that are used in the paper.

Notation. Throughout the paper, $\mathbb{R}$ denotes the set of real numbers, while $\mathbb{R}_{\geq c}$ denotes the set of real numbers greater than or equal to the real number c. Given a symmetric matrix M, $M \succ 0$ ($M \succeq 0$) indicates that M is positive definite (positive semidefinite), while $M \prec 0$ ($M \preceq 0$) indicates that M is negative definite (negative semidefinite). $\mathbb{S}^{n\times n}$ denotes the set of real-valued symmetric matrices. We abbreviate a symmetric matrix $\left[\begin{smallmatrix} M & N \\ N^\top & P \end{smallmatrix}\right]$ as $\left[\begin{smallmatrix} M & \star \\ N^\top & P \end{smallmatrix}\right]$. Finally, we let $|v|$ be the 2-norm of a vector v, and $\|M\|$ be the induced 2-norm of a matrix M. Other, less standard, notions are introduced throughout the paper.

II. Set-up and problem formulation

We consider a continuous-time system of the form

$$\dot{x} = f(x) + Bu \tag{1}$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^m$ is the control input. The vector field $f(x)$ and the input matrix B are unknown. To ease the analysis we focus on the case of a constant input matrix B. We are interested in the problem of making the system contractive via feedback. We will introduce later a problem that motivates the interest in this property.

Without further priors about the vector field f, any control design is challenging. Hence, we introduce the following:

Assumption 1: We know a continuously differentiable vector-valued function $Z : \mathbb{R}^n \to \mathbb{R}^s$ such that $f(x) = AZ(x)$ for some matrix $A \in \mathbb{R}^{n\times s}$. □

Under Assumption 1, system (1) can be equivalently written as

$$\dot{x} = AZ(x) + Bu \tag{2}$$

with A, B unknown.

For control design purposes, it is advantageous to have in $Z(x)$ both the state vector x and the vector of nonlinear functions $Q(x)$, namely we consider $Z(x)$ of the form

$$Z(x) = \begin{bmatrix} x \\ Q(x) \end{bmatrix}, \tag{3}$$

with $Q : \mathbb{R}^n \to \mathbb{R}^{s-n}$ containing only nonlinear functions.

Dataset. The lack of knowledge about A, B are offset by data collected in an experiment, which returns the following dataset to the designer:

$$\mathbb{D} := \{(x_i, u_i, \dot{x}_i)\}_{i=0}^{T-1} \tag{4}$$

where $T > 0$ is an integer equal to the number of samples of the dataset, $x_i := x(t_i), u_i := u(t_i), \dot{x}_i := \dot{x}(t_i)$ and $0 \leq t_0 \leq t_1 \leq \ldots \leq t_{T-1}$ are the sampling times. No requirements are imposed on the sampling times. By construction, the samples satisfy the relation $\dot{x}_i = AZ(x_i) + Bu_i$ for $i = 0, \ldots, T-1$, $T > 0$. Note that, $Z(x_i)$ can be computed from x_i and $Z(x)$, since the latter is known. We also assume that the time derivative of the state $\dot{x}_i$ can be measured. While this is an unrealistic assumption, there are two ways to relax it. One is to consider approximate estimates of the derivatives, regard the estimation error as a perturbation on the data and design the control accordingly (see Section V on how to establish contractivity with noisy data). The other is to use a different data collection scheme that considers integral versions of the relation (2) (see [33, Appendix A] for details). Finally, we begin to study the scenario in which data are not perturbed by disturbances or noise.

Problem. We are interested in the design of a feedback control of the form $u = KZ(x)$, which makes the closed-loop system

$$\dot{x} = (A + BK)Z(x) \tag{5}$$

contractive on a set $\mathcal{X} \subseteq \mathbb{R}^n$. We recall below such a property in a form that is suitable for our purposes:

Definition 1: Consider a set $\mathcal{X} \subseteq \mathbb{R}^n$. System (5) is exponentially contractive on $\mathcal{X}$ if

$$\begin{aligned}&\exists P_1 \succ 0, \beta > 0 \text{ such that } \forall x \in \mathcal{X}\\ &((A + BK)\frac{\partial Z}{\partial x})^\top P_1^{-1} + P_1^{-1}(A + BK)\frac{\partial Z}{\partial x} \preceq -\beta P_1^{-1}.\end{aligned} \tag{6}$$

The set $\mathcal{X}$ is the contraction region with respect to the metric P_1^{-1}. □

This is a property instrumental in several estimation and control problems, whose use in the control literature has been popularized by [21]. In the original definition therein, the metric P_1^{-1} is taken as state- and time-dependent. Since the search for a state- or time-dependent metric is difficult, here we concentrate our efforts on a constant metric. In fact, to complicate the situation here is the lack of knowledge of the matrices A, B in (2) and the presence of the design variable K. Hence, beside P_1 and β, one has to simultaneously seek a K that enforces (6). Finally, in Section VII we will consider a tracking problem with constant reference and disturbance signal of arbitrary magnitude, for which considering a constant metric results in no loss of generality [34, Theorem 2].

III. Control design

A. A data-dependent representation of the closed-loop system

A first step we take toward designing the controller is to obtain a data-dependent representation of the closed-loop system. To this end, we arrange the dataset in the following matrices:

$$U_0 := \begin{bmatrix} u_0 & u_1 & \cdots & u_{T-1} \end{bmatrix} \in \mathbb{R}^{m\times T}, \tag{7a}$$
$$X_0 := \begin{bmatrix} x_0 & x_1 & \cdots & x_{T-1} \end{bmatrix} \in \mathbb{R}^{n\times T}, \tag{7b}$$
$$X_1 := \begin{bmatrix} \dot{x}_0 & \dot{x}_1 & \cdots & \dot{x}_{T-1} \end{bmatrix} \in \mathbb{R}^{n\times T}, \tag{7c}$$
$$Z_0 := \begin{bmatrix} x_0 & x_1 & \cdots & x_{T-1} \\ Q(x_0) & Q(x_1) & \cdots & Q(x_{T-1}) \end{bmatrix} \in \mathbb{R}^{s\times T}. \tag{7d}$$

Note that these matrices satisfy the identity

$$X_1 = AZ_0 + BU_0.$$

The following result, established in [16, Lemma 1] and recalled here without proof, returns the data-dependent representation of the closed-loop system we are interested in.

Lemma 1: Consider any matrices $K \in \mathbb{R}^{m\times s}$, $G \in \mathbb{R}^{T\times s}$ such that

$$\begin{bmatrix} K \\ I_s \end{bmatrix} = \begin{bmatrix} U_0 \\ Z_0 \end{bmatrix} G. \tag{8}$$

Let G be partitioned as $G = \begin{bmatrix} G_1 & G_2 \end{bmatrix}$, where $G_1 \in \mathbb{R}^{T\times n}$ and $G_2 \in \mathbb{R}^{T\times(s-n)}$. Let Assumption 1 hold. Then system (1) under the control law $u = KZ(x)$ results in the closed-loop dynamics

$$\dot{x} = (A + BK)Z(x) = Mx + NQ(x) \tag{9}$$

where $M := X_1G_1$ and $N := X_1G_2$. □

Proof. See [16, Lemma 1]. ■

We note that the condition (8) and the right-hand side of (9) only depend on the matrices of data U_0, Z_0, X_1 and the decision variable G. Hence, we will design G that enforces the desired contractivity property and then obtain the control gain K via (8).

B. Contractivity from data

The first statement gives data-dependent conditions under which there exists P_1, K for which (6) holds.

Theorem 1: Consider the nonlinear system (1). Let Assumption 1 hold and $Z(x)$ be of the form (3). Let $R_Q \in \mathbb{R}^{n\times r}$ be a known matrix and $\mathcal{X} \subseteq \mathbb{R}^n$ a set such that

$$\frac{\partial Q}{\partial x}(x)^\top \frac{\partial Q}{\partial x}(x) \preceq R_Q R_Q^\top \text{ for any } x \in \mathcal{X}. \tag{10}$$

Consider the following semidefinite program (SDP) in the decision variables $P_1 \in \mathbb{S}^{n\times n}$, $Y_1 \in \mathbb{R}^{T\times n}$, $G_2 \in \mathbb{R}^{T\times(s-n)}$ and $\alpha \in \mathbb{R}_{>0}$:

$$P_1 \succ 0, \tag{11a}$$
$$Z_0 Y_1 = \begin{bmatrix} P_1 \\ 0_{(s-n)\times n} \end{bmatrix}, \tag{11b}$$
$$\begin{bmatrix} X_1Y_1 + (X_1Y_1)^\top + \alpha I_n & X_1G_2 & P_1R_Q \\ (X_1G_2)^\top & -I_{s-n} & 0 \\ (P_1R_Q)^\top & 0 & -I_r \end{bmatrix} \preceq 0, \tag{11c}$$
$$Z_0 G_2 = \begin{bmatrix} 0_{n\times(s-n)} \\ I_{s-n} \end{bmatrix}. \tag{11d}$$

If the program is feasible then the control law

$$u = KZ(x) \tag{12}$$

with

$$K = U_0 \begin{bmatrix} Y_1P_1^{-1} & G_2 \end{bmatrix} \tag{13}$$

is such that the closed-loop dynamics (1), (12) is exponentially contractive on $\mathcal{X}$, *i.e.*, (6) holds. □

Proof. If (11) is feasible, then we can define $G_1 := Y_1P_1^{-1}$ and note that (11b), (11d) and the definition of K in (13), give (8). Hence, by Lemma 1, the closed-loop dynamics $\dot{x} = (A + BK)Z(x)$ can be written as $\dot{x} = Mx + NQ(x)$, with $M = X_1G_1$ and $N = X_1G_2$. By Schur complement, (11c) gives

$$\begin{bmatrix} X_1Y_1 + (X_1Y_1)^\top + P_1R_QR_Q^\top P_1 + \alpha I_n & X_1G_2 \\ (X_1G_2)^\top & -I_{s-n} \end{bmatrix} \preceq 0.$$

Another Schur complement returns

$$X_1Y_1 + (X_1Y_1)^\top + \alpha I_n + P_1R_QR_Q^\top P_1 + X_1G_2(X_1G_2)^\top \preceq 0.$$

Left- and right-multiply by P_1^{-1} and recall that $G_1 := Y_1P_1^{-1}$. Then

$$\begin{aligned} P_1^{-1}X_1G_1 + (X_1G_1)^\top P_1^{-1} + \alpha P_1^{-2} + R_QR_Q^\top \\ + P_1^{-1}X_1G_2(X_1G_2)^\top P_1^{-1} \preceq 0. \end{aligned} \tag{14}$$

This shows that there exist a real number $\lambda = 1$ and matrices

$$\begin{aligned} H &:= P_1^{-1}X_1G_1 + (X_1G_1)^\top P_1^{-1} + \alpha P_1^{-2} \\ J &:= P_1^{-1}X_1G_2 \\ L &:= I_n \\ O &:= R_QR_Q^\top \end{aligned}$$

that satisfy $H + \lambda JJ^\top + \lambda^{-1}L^\top OL \preceq 0$. If the solution of (11) is such that $J = 0$, then $(A+BK)\frac{\partial Z}{\partial x} = X_1G_1$ and (14) implies (6) with $\beta := \alpha\lambda_{\min}(P_1^{-1})$, where $\lambda_{\min}(P_1^{-1})$ is the minimum eigenvalue of P_1^{-1}, which ends the proof. On the other hand, if the solution of (11) is such that $J \neq 0$, then by the nonstrict Petersen's lemma [35, Fact 2], we have that $H+\lambda JJ^\top+\lambda^{-1}L^\top OL\preceq 0$ implies $H+JR^\top L+L^\top RJ^\top\preceq 0$ for all $R\in\mathcal{R}:=\{R\colon RR^\top\preceq O\}$, *i.e.*,

$$P_1^{-1}X_1G_1+(X_1G_1)^\top P_1^{-1}+\alpha P_1^{-2} \\ +P_1^{-1}X_1G_2R^\top+RG_2^\top X_1^\top P_1^{-1}\preceq 0,$$

for all $R\in\mathcal{R}:=\{R\colon RR^\top\preceq R_QR_Q^\top\}$.

By (10), for any $x\in\mathcal{X}$, the inequality above gives

$$P_1^{-1}X_1G_1+G_1^\top X_1^\top P_1^{-1}+\alpha P_1^{-2}+P_1^{-1}X_1G_2\frac{\partial Q}{\partial x}(x) \\ +\frac{\partial Q}{\partial x}(x)^\top(X_1G_2)^\top P_1^{-1}\preceq 0, \tag{15}$$

that is, recalling that $\beta:=\alpha\lambda_{\min}(P_1^{-1})$,

$$P_1^{-1}X_1G\frac{\partial Z}{\partial x}+\frac{\partial Z}{\partial x}^\top G^\top X_1^\top P_1^{-1}\preceq-\beta P_1^{-1}\text{ for all }x\in\mathcal{X},$$

which ends the proof. ∎

Having enforced the contraction property on the closed-loop system, we state its asymptotic properties, provided an equilibrium exists.

Corollary 1: Let the conditions of Theorem 1 hold and assume additionally that $Z(x_*)=0$ and $\mathcal{X}$ is a convex set containing x_* in its interior. Then x_* is the unique[1] equilibrium point in $\mathcal{X}$, it is uniformly exponentially stable for $\dot{x}=(A+BK)Z(x)$ and any solution initialized in the set $\mathcal{V}:=\left\{x\in\mathbb{R}^n\colon(x-x_*)^\top P_1^{-1}(x-x_*)\le\delta\right\}$, with $\delta>0$ such that $\mathcal{V}$ is contained in $\mathcal{X}$, converges to x_*.

Proof. It is a consequence of Corollary 7 in the Appendix. It can also be directly derived by the following standard arguments. Note that

$$\begin{aligned}&\tfrac{\partial V}{\partial x}(A+BK)Z(x)\\ &\overset{(a)}{=}2(x-x_*)^\top P_1^{-1}(A+BK)(Z(x)-Z(x_*))\\ &\overset{(b)}{=}2(x-x_*)^\top P_1^{-1}(A+BK)\left[\tfrac{\partial Z}{\partial x}(x_*+\lambda(x-x_*))\right]_{\lambda\in(0,1)}\cdot\\ &\qquad\cdot(x-x_*)\\ &\overset{(c)}{\le}\;-\beta(x-x_*)^\top P_1^{-1}(x-x_*)\end{aligned}$$

where (a) holds thanks to $Z(x_*)=0$, (b) by the mean value theorem for vector valued functions and thanks to the convexity of the set $\mathcal{X}$, (c) by (6). The other claims are immediate. ∎

The assumption $Z(x_*)=0$ necessarily implies that $x_*=0$. The assumption can be lifted under a more stringent condition on $\mathcal{X}$.

Corollary 2: Let the conditions of Theorem 1 hold where $\mathcal{X}$ is either (i) a convex, closed, forward invariant set for $\dot{x}=(A+BK)Z(x)$ and $\dot{x}=(A+BK)Z(x)$ is forward complete on $\mathcal{X}$ or (ii) $\mathcal{X}=\mathbb{R}^n$ and there exists $\overline{x}\in\mathbb{R}^n$ such that $|Z(\overline{x})|<\infty$. Then there exists a unique equilibrium point x_* in $\mathcal{X}$ for $\dot{x}=(A+BK)Z(x)$. Moreover, x_* is uniformly exponentially stable for $\dot{x}=(A+BK)Z(x)$ and any solution of $\dot{x}=(A+BK)Z(x)$ initialized in $\mathcal{X}$ uniformly exponentially converges to x_*.

[1]In fact, it is the unique solution defined and bounded for all $t\in(-\infty,+\infty)$ and contained in $\mathcal{X}$.

Proof. Under the conditions of Theorem 1, (6) holds. If $\mathcal{X}$ satisfies (i), then the thesis follows from Theorem 5 in the Appendix. If $\mathcal{X}$ satisfies (ii), then the thesis follows from Theorem 6 in the Appendix. ∎

A few remarks are in order.

(*On condition* (10)) Theorem 1 holds under the local Lipschitz property (10). The result also holds for nonlinearities satisfying more general conditions such as

$$\frac{\partial Q}{\partial x}(x)^\top R\frac{\partial Q}{\partial x}(x)+S\frac{\partial Q}{\partial x}(x) \\ +\frac{\partial Q}{\partial x}(x)^\top S^\top\preceq W\text{ for any }x\in\mathcal{X} \tag{16}$$

where $W=W^\top\succeq 0$, $R=R^\top\succeq 0$, S are known matrices, provided that the LMI (11c) is replaced by

$$\begin{bmatrix}X_1Y_1+(X_1Y_1)^\top+\alpha I_n & X_1G_2-P_1S & P_1W^{1/2}\\ (X_1G_2-P_1S)^\top & -R & 0\\ (P_1W^{1/2})^\top & 0 & -I_r\end{bmatrix}\preceq 0$$

In fact, the Schur complement applied to the inequality above returns

$$\begin{bmatrix}X_1Y_1+(X_1Y_1)^\top+P_1WP_1+\alpha I_n & X_1G_2-P_1S\\ (X_1G_2-P_1S)^\top & -R\end{bmatrix}\preceq 0\,.$$

By left- and right-multiplying by $\text{block.diag}(P_1^{-1},I_{s-n})$ and setting $G_1:=Y_1P_1^{-1}$, the inequality above is rewritten as

$$\begin{bmatrix}P_1^{-1}X_1G_1+(P_1^{-1}X_1G_1)^\top+W+\alpha P_1^{-2} & \star\\ (P_1^{-1}X_1G_2-S)^\top & -R\end{bmatrix}\preceq 0,$$

Left-multiplying the last inequality by $\left[\begin{smallmatrix}I\\ \frac{\partial Q}{\partial x}\end{smallmatrix}\right]^\top$, right-multiplying it by $\left[\begin{smallmatrix}I\\ \frac{\partial Q}{\partial x}\end{smallmatrix}\right]$, and bearing in mind (16), it follows that

$$\begin{bmatrix}I\\ \frac{\partial Q}{\partial x}\end{bmatrix}^\top\begin{bmatrix}P_1^{-1}X_1G_1+(P_1^{-1}X_1G_1)^\top+\alpha P_1^{-2} & \star\\ (P_1^{-1}X_1G_2)^\top & 0\end{bmatrix}\begin{bmatrix}I\\ \frac{\partial Q}{\partial x}\end{bmatrix}\preceq 0$$

for all $x\in\mathcal{X}$, which is (6), as claimed. Examples of nonlinearities that satisfy (16) include those that satisfy incremental sector bound conditions [36, Lemma 4.4]. For other nonlinearities covered by this analysis see [37], [30].

(*On condition* (11)) Contractivity and the asymptotic properties given above are guaranteed by (11). The conditions (11b), (11d) are used to obtain the data-dependent representation (9) of the closed-loop system: $\dot{x}=Mx+NQ(x)$, where $M=X_1G_1$ and $N=X_1G_2$. On the other hand, to enforce contractivity on $\mathcal{X}$, i.e., to enforce the inequality (cf. (14))

$$P_1^{-1}X_1G_1+(P_1^{-1}X_1G_1)^\top+\alpha P_1^{-2}+P_1^{-1}X_1G_2\frac{\partial Q}{\partial x}(x) \\ +\frac{\partial Q}{\partial x}(x)^\top(X_1G_2)^\top P_1^{-1}\preceq 0,$$

the key property is (11c), which, written as

$$P_1^{-1}X_1G_1+(X_1G_1)^\top P_1^{-1}+\alpha P_1^{-2}+R_QR_Q^\top \\ +P_1^{-1}X_1G_2(X_1G_2)^\top P_1^{-1}\preceq 0\,,$$

reveals the aim of our control design. In fact, the possibility of fulfilling this condition is related to the existence of a matrix $P_1\succ 0$ and for the linear part of the controller $P_1^{-1}X_1G_1+(X_1G_1)^\top P_1^{-1}$ to dominate both $P_1^{-1}X_1G_2(X_1G_2)^\top P_1^{-1}$ and the contribution of the Jacobian $\frac{\partial Q}{\partial x}(x)$, which appears through the term $R_Q R_Q^\top$. Controllability of the pair $(\overline{A}, B)$, where $A = \begin{bmatrix}\overline{A} & \hat{A}\end{bmatrix}$, $\overline{A} \in \mathbb{R}^{n\times n}$, also plays a role in achieving this goal.

C. Comparison with other methods and a numerical example

It is useful to compare this approach with other methods on controller design that are based on convex programming, see [16], [38], [39]. Starting from the data-based representation of the closed-loop dynamics $\dot{x} = X_1 G_1 x + X_1 G_2 Q(x)$, and by assuming that the term $Q(x)$ acts as a "remainder" term, *i.e.*, $\lim_{x\to 0} \frac{|Q(x)|}{|x|} = 0$, [2] a simple approach is to search for matrices $P_1 \in \mathbb{S}^{n\times n}, Y_1 \in \mathbb{R}^{T\times n}$, and a positive scalar α satisfying the following conditions:

$$\text{(11a), (11b), (11d),} \tag{17a}$$

$$X_1 Y_1 + (X_1 Y_1)^\top + \alpha I_n \preceq 0. \tag{17b}$$

If (17) is feasible then $u = Kx$ with $K = U_0 Y_1 P_1^{-1}$ locally stabilizes the origin of the closed-loop system. This approach is inspired by Lyapunov's indirect method, which guarantees stability of the nonlinear dynamics by stabilizing the linearized dynamics around the desired equilibrium point.

Building on (17), an alternative approach is to stabilize the linearized dynamics while simultaneously reducing the impact of the nonlinearities in closed-loop. This amounts to searching for matrices $P_1 \in \mathbb{S}^{n\times n}, Y_1 \in \mathbb{R}^{T\times n}$, $G_2 \in \mathbb{R}^{T\times(s-n)}$ and a positive scalar α satisfying the following program:

$$\text{minimize } \|X_1 G_2\| \tag{18a}$$

$$\text{subject to (11a), (11b), (11d), (17b).} \tag{18b}$$

If the minimum is attained at zero, then $(A+BK)\frac{\partial Z}{\partial x} = X_1 G_1$ and condition (6) holds with $\mathcal{X} = \mathbb{R}^n$. In particular, in this case the closed-loop dynamics becomes linear.

Compared with the method proposed in this paper, (17) and (18) entail milder feasibility conditions since condition (17b) is less demanding than (11c). Furthermore, Lyapunov's indirect method allows to treat the term $X_1 G_2 Q(x)$ that appears in the expression $\dot{x} = X_1 G_1 x + X_1 G_2 Q(x)$ as an *unknown* quantity, see [38] for details. This is somewhat harder with the method proposed in this paper since the control strategy relies on the full nonlinear behaviour to enforce a stable (contracting) behaviour. On the other hand, the proposed method possesses some nice features, notably:

(1) *Stability properties.* The proposed method permits to have *global* stability properties, as long as condition (6) holds with $\mathcal{X} = \mathbb{R}^n$. This is instead not achievable (at least not by design) with Lyapunov's indirect method, and requires exact nonlineary cancellation, *i.e.*, $X_1 G_2 = 0$, with (18), which is possible only when the system to control has a specific structure, see [16], [40].

(2) *Knowledge of the equilibrium.* As detailed in Corollary 2, there are case of practical interest in which the proposed method does not require the knowledge of the equilibrium point, which follows because contractive systems have the incremental stability property that all the trajectories converge to a unique solution, see Theorems 5 and 6 in the Appendix. In contrast, neither (17) nor (18) possess this property. For these methods, *exact* knowledge of the equilibrium point is needed to transform the problem into the standard form of a zero-regulation problem. (This is not surprising considering that both (17) and (18) rest on Lyapunov's indirect method.)

[2] This implies that $\overline{x} = 0$ is a known equilibrium of the system.

We recall the definition of region of attraction used in the following numerical results.

Definition 2: ([16, Def. 1]) Let $\overline{x}$ be an asymptotically stable equilibrium point for the system $\dot{x} = f(x)$. A set $\mathcal{R}$ defines a region of attraction (ROA) for the system relative to $\overline{x}$ if for every $x(0) \in \mathcal{R}$ we have $\lim_{t\to\infty} x(t) = \overline{x}$.

We illustrate the foregoing considerations via a numerical example. [3]

Example 1: Consider the dynamics of a one-link robot arm [7, Section 4.10]

$$\dot{x}_1 = x_2 \tag{19a}$$

$$\dot{x}_2 = -\frac{K_c}{J_2}x_1 - \frac{F_2}{J_2}x_2 + \frac{K_c}{J_2 N_c}x_3 - \frac{mgd}{J_2}\cos x_1 \tag{19b}$$

$$\dot{x}_3 = x_4 \tag{19c}$$

$$\dot{x}_4 = -\frac{K_c}{J_1 N_c}x_1 + \frac{K_c}{J_1 N_c^2}x_3 - \frac{F_1}{J_1}x_4 + \frac{1}{J_1}u. \tag{19d}$$

where x_1, x_3 denote the angular positions of the link and of the actuator shaft, respectively, and u is the torque produced at the actuator axis. We will collect data from the system setting the parameters as $K_c = 0.4$, $F_2 = 0.15$, $J_2 = 0.2$, $N_c = 2$, $F_1 = 0.1$, $J_1 = 0.15$, $m = 0.4$, $g = 9.8$ and $d = 0.1$.

We introduce $Q(x) = \cos x_1$. Note that

$$\frac{\partial Q}{\partial x}(x)^\top \frac{\partial Q}{\partial x}(x) = \begin{bmatrix} \sin(x_1)^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

If we set $\mathcal{X} = \mathbb{R}^4$, then (10) is satisfied with

$$R_Q = R_Q^\top = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \tag{20}$$

We collect $T = 10$ samples by running an experiment with input uniformly distributed in $[-0.1, 0.1]$, and with an initial state within the same interval. The SDP (11) is feasible and returns the controller K and the closed-loop dynamics in (21).

$$K = \begin{bmatrix} -3.1639 & -4.6751 & -3.9299 & -0.7614 & 0.0106 \end{bmatrix} \tag{21a}$$

$$\dot{x} = \begin{bmatrix} 0.0000 & 1.0000 & 0.0000 & 0.0000 & 0.0000 \\ -2.0000 & -0.7500 & 1.0000 & 0.0000 & -1.9600 \\ 0.0000 & 0.0000 & 0.0000 & 1.0000 & 0.0000 \\ -22.4261 & -31.1671 & -25.5328 & -5.7430 & 0.0708 \end{bmatrix} Z(x) \tag{21b}$$

Since $\mathcal{X} = \mathbb{R}^4$ and $|Z(\overline{x})| < \infty$ for any $\overline{x} \in \mathbb{R}^n$, by Corollary 2, the equilibrium point $x_* = \begin{bmatrix} -0.5718 & 0 & 0.5046 & 0 \end{bmatrix}^\top$

[3] The code to reproduce the examples is available at https://github.com/zjhurug/codes-for-contraction-paper.git

$$K = \begin{bmatrix}-280.8884 & -257.1662 & -91.3493 & -8.2326 & 0.1591 & -0.0032 & -0.0030\end{bmatrix} \tag{22a}$$

$$\dot{x} = \begin{bmatrix} 0.0000 & 1.0000 & 0.0000 & 0.0000 & 0.0000 & -0.0000 & -0.0000 \\ -2.0000 & -0.7500 & 1.0000 & 0.0000 & -1.9600 & 0.0000 & 0.0000 \\ 0.0000 & 0.0000 & 0.0000 & 1.0000 & -0.0000 & 0.0000 & 0.0000 \\ -1873.9225 & -1714.4416 & -608.3286 & -55.5510 & 1.0603 & -0.0213 & -0.0201 \end{bmatrix} Z(x) \tag{22b}$$

obtained from the simulation experiment or analytically from (21b) is unique in $\mathbb{R}^4$ for (21b). Moreover, x_* is globally uniformly exponentially stable for (21b).

Next, in order to test the effectiveness of Theorem 1 for other choices of $Q(x)$, we add more nonlinearities in $Q(x)$ and let $Q(x) = \begin{bmatrix}\cos x_1 & x_1^2 & \sin x_2\end{bmatrix}^\top$. Note that

$$\frac{\partial Q}{\partial x}(x)^\top \frac{\partial Q}{\partial x}(x) = \begin{bmatrix} \sin(x_1)^2 + 4x_1^2 & 0 & 0 & 0 \\ 0 & \cos(x_2)^2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

If we set $\mathcal{X} = [-w, w] \times \mathbb{R}^3$, where $w \in \mathbb{R}_{>0}$, then (10) is satisfied with

$$R_Q = R_Q^\top = \begin{bmatrix} \sqrt{4w^2+1} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Here, we set $w = 1$. We collect $T = 10$ samples under the same experiment setup. The SDP (11) is feasible and returns the controller K and the closed-loop dynamics in (22). The unique equilibrium point obtained from the simulation experiment or analytically from (22) is $x_* = \begin{bmatrix}-0.3607 & 0 & 1.1126 & 0\end{bmatrix}^\top$. Note that $x_* \in \text{int}(\mathcal{X})$. By Corollary 7 in the Appendix, any solution of (22b) initialized in any sub-level set of $V(x) = (x - x_*)^\top P_1^{-1}(x - x_*)$ contained in $\mathcal{X}$ uniformly exponentially converges to x_*. We observe the following: (i) from numerical results, the solutions of (22b) initialized at the points outside the largest Lyapunov sublevel set contained in $\mathcal{X}$ still converge to x_*, which means the exact ROA is much larger and the obtained controller (22b) is possibly a global controller. (ii) (11) remains feasible up to $w = 100$, and as w increases, the magnitude of the coefficients for the linear part of the obtained controller tends to increase, which is consistent with the comments about condition (11). With $w = 100$, the magnitude is of order 10^5, which results in a high-gain controller.

We do not compare (11) with (18), since the lack of a specific structure of the systems does not allow (18) to exactly cancel the nonlinearity. It is more interesting to compare (11) with the Lyapunov's indirect method [38, Theorem 2]. Denote system (19) as $\dot{x} = f(x, u)$ and assume that the desired equilibrium point (x_*, u_*) is known, i.e. $x_* = \begin{bmatrix}-0.3607 & 0 & 1.1126 & 0\end{bmatrix}^\top$ and $u_* = -0.1834$. Linearize system (19) at (x_*, u_*) which yields

$$\dot{\tilde{x}} = A\tilde{x} + B\tilde{u} + r(\tilde{x}, \tilde{u})$$

where $\tilde{x} = x - x_*$, $\tilde{u} = u - u_*$ are the shifted state and input variables,

$$A = \left.\frac{\partial f}{\partial x}\right|_{(x,u)=(x_*,u_*)}, \quad B = \left.\frac{\partial f}{\partial u}\right|_{(x,u)=(x_*,u_*)}$$

are unknown constant matrices and $r\colon \mathbb{R}^4 \times \mathbb{R} \to \mathbb{R}^4$ denotes the unknown remainder. Collect $\{(x_i, u_i, \dot{x}_i)\}_{i=0}^{T-1}$ from system (19) and compute the samples $\{(\tilde{x}_i, \tilde{u}_i, \dot{\tilde{x}}_i)\}_{i=0}^{T-1}$, which are used to define the data matrices

$$\tilde{U}_0 := \begin{bmatrix}\tilde{u}_0 & \tilde{u}_1 & \dots & \tilde{u}_{T-1}\end{bmatrix} \in \mathbb{R}^{m\times T}, \tag{23a}$$

$$\tilde{X}_0 := \begin{bmatrix}\tilde{x}_0 & \tilde{x}_1 & \dots & \tilde{x}_{T-1}\end{bmatrix} \in \mathbb{R}^{n\times T}, \tag{23b}$$

$$\tilde{X}_1 := \begin{bmatrix}\dot{\tilde{x}}_0 & \dot{\tilde{x}}_1 & \dots & \dot{\tilde{x}}_{T-1}\end{bmatrix} \in \mathbb{R}^{n\times T}, \tag{23c}$$

$$\tilde{Z}_0 := \begin{bmatrix} \tilde{x}_0 & \tilde{x}_1 & \dots & \tilde{x}_{T-1} \\ Q(\tilde{x}_0) & Q(\tilde{x}_1) & \dots & Q(\tilde{x}_{T-1}) \end{bmatrix} \in \mathbb{R}^{(n+2)\times T}. \tag{23d}$$

These matrices satisfy the identity $\tilde{X}_1 = A\tilde{X}_0 + B\tilde{U}_0 + R_0$, where $R_0 := \begin{bmatrix}r_0 & r_1 & \cdots & r_{T-1}\end{bmatrix} \in \mathbb{R}^{n\times T}$ is the (unknown) data matrix of samples of the reminder r. Under the control law $\tilde{u} = K\tilde{x}$, the data-based representation of the closed-loop dynamics is

$$\dot{\tilde{x}} = (\tilde{X}_1 - R_0)G\tilde{x} + r(\tilde{x}, \tilde{u}). \tag{24}$$

Let $\mathcal{R} := \{R \in \mathbb{R}^{n\times T}\colon RR^\top \preceq \Delta\Delta^\top\}$ where $\Delta \in \mathbb{R}^{n\times p}$ is chosen by the designer, and assume $R_0 \in \mathcal{R}$. Similar to (17), a simple approach to obtain a stabilizing controller K is to search for matrices $P_1 \in \mathbb{S}^{n\times n}, Y_1 \in \mathbb{R}^{T\times n}$, and positive scalars α and μ satisfying the following program [38]:

$$\text{(11a), (11b), (11d),} \tag{25a}$$

$$\begin{bmatrix} X_1Y_1 + (X_1Y_1)^\top + \alpha I_n + \mu\Delta\Delta^\top & Y_1^\top \\ Y_1 & -\mu I_T \end{bmatrix} \preceq 0. \tag{25b}$$

If SDP (25) with X_1 replaced by $\tilde{X}_1$ and Z_0 replaced by $\tilde{Z}_0$ is feasible then $\tilde{u} = K\tilde{x}$ with $K = \tilde{U}_0 Y_1 P_1^{-1}$ locally stabilizes the origin of the closed-loop system (24). We collect $T = 10$ samples from system (19). In this example, the reminder term is $r(\tilde{x}, \tilde{u}) = \begin{bmatrix}0 & 1 & 0 & 0\end{bmatrix}^\top \overline{r}(\tilde{x})$ with $\overline{r}(\tilde{x}) := -\frac{mgd\cos x_{*1}}{J_2}(\cos\tilde{x}_1 - 1) + \frac{mgd\sin x_{*1}}{J_2}(\sin\tilde{x}_1 - \tilde{x}_1)$. Let $\delta(\tilde{x}) := 4|\cos\tilde{x}_1 - 1| + 2|\sin\tilde{x}_1 - \tilde{x}_1|$, which over-approximates $\overline{r}(\tilde{x})$ by more than 100%. Then, we set $\Delta = \sqrt{T}\text{diag}(0, c, 0, 0)$, with c equal to the maximum of $\delta(\tilde{x})$ over the experimental data, such that $R_0 \in \mathcal{R}$ is satisfied. Finally, we set $\Delta = \text{diag}(0, 0.0145, 0, 0)$. Then the SDP (25) is feasible and returns the controller $K = \begin{bmatrix}2.3662 & -0.2030 & -10.7730 & -1.4698\end{bmatrix}$. For this controller, we numerically determine the set $\mathcal{H} = \{\tilde{x} : \dot{V}(\tilde{x}) <$

$0\}$, with $\dot{V}(\tilde{x}) := 2\tilde{x}^\top P_1^{-1}\dot{\tilde{x}}$, over which the Lyapunov function $V(\tilde{x}) = \tilde{x}^\top P_1^{-1}\tilde{x}$ decreases and any sub-level set $\mathcal{R}_\gamma := \{\tilde{x} : V(\tilde{x}) \le \gamma\}$ of V with $\gamma > 0$ contained in $\mathcal{H} \cup \{0\}$ gives an estimate of the ROA for the closed-loop system. Planar projections of the set $\mathcal{H}$ and of a sub-level set of V are displayed in Figure 1. Hence, the controller $u = K(x - x_*) + u_*$ renders (x_*, u_*) a locally asymptotically stable equilibrium point for system (19).

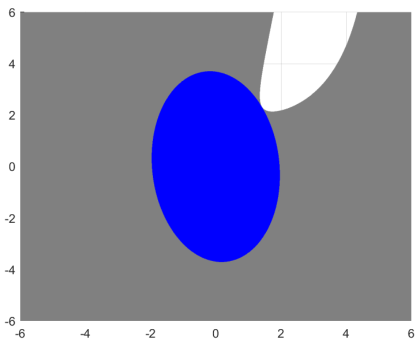

Fig. 1. The grey set represents the projection of the set $\mathcal{H}$ onto the plane $\{\tilde{x} : \tilde{x}_3 = \tilde{x}_4 = 0\}$. The blue set is the projection onto the same plane of a Lyapunov sub-level set $\mathcal{R}_\gamma$ contained in $\mathcal{H}$; here, $P_1^{-1} = \begin{bmatrix} 4.2206 & 0.2028 & -2.7615 & -0.1858 \\ 0.2028 & 1.1915 & 0.9613 & 0.0219 \\ -2.7615 & 0.9613 & 11.4795 & 0.6800 \\ -0.1858 & 0.0219 & 0.6800 & 0.1742 \end{bmatrix}$ and $\gamma = 16.3$.

IV. An extension to general nonlinear systems

The results of the paper can be extended to a more general class of nonlinear systems, which contains as a special case not only the system (2) but also the important class of input-affine systems. In fact, here we consider systems of the form

$$\dot{x} = f(x, u) \tag{26}$$

We introduce the stacked vector of states and inputs $\xi := \left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right]$, and consistently with the previous section we assume the following:

Assumption 2: A continuously differentiable function $\mathcal{Z} : \mathbb{R}^{n+m} \to \mathbb{R}^R$ is known such that $f(x, u) = A\mathcal{Z}(\xi)$ for some matrix $A \in \mathbb{R}^{n \times R}$. □

Under this assumption the system dynamics (26) is written as

$$\dot{x} = A\mathcal{Z}(\xi) \tag{27}$$

We partition $\mathcal{Z}(\xi)$ into subvectors ξ and $\mathcal{Q} : \mathbb{R}^{n+m} \to \mathbb{R}^{R-n-m}$, where $\mathcal{Q}(\xi)$ collects all the entries of $\mathcal{Z}(\xi)$ that are nonlinear functions of the extended state ξ, namely we consider

$$\mathcal{Z}(\xi) = \begin{bmatrix} \xi \\ \mathcal{Q}(\xi) \end{bmatrix}. \tag{28}$$

Finally, we regard ξ as the new state variable and extend the dynamics (27) with the integral control $\dot{u} = v$, where $v \in \mathbb{R}^m$ is a new control input. Other uses of the integral control will be considered in Section VII. We obtain the overall system

$$\dot{\xi} = \mathcal{A}\mathcal{Z}(\xi) + \mathcal{B}v, \tag{29}$$

where

$$\mathcal{A} := \begin{bmatrix} \overline{A} & \hat{A} \\ 0_{m\times(n+m)} & 0_{m\times(R-n-m)} \end{bmatrix}, \quad \mathcal{B} := \begin{bmatrix} 0_{n\times m} \\ I_m \end{bmatrix} \tag{30}$$

having partitioned A as $A = \begin{bmatrix} \overline{A} & \hat{A} \end{bmatrix}$ with $\overline{A} \in \mathbb{R}^{n\times(n+m)}$. We then aim at the design of the feedback control

$$v = \mathcal{K}\mathcal{Z}(\xi)$$

where $\mathcal{K} = \begin{bmatrix} \overline{\mathcal{K}} & \hat{\mathcal{K}} \end{bmatrix}$ is to enforce desired properties on $\dot{\xi} = (\mathcal{A} + \mathcal{B}\mathcal{K})\mathcal{Z}(\xi)$. The design can be pursued as in the previous section *mutatis mutandis*. Below we concisely present the main result.

Collect the dataset $\{x(k), u(k), v(k)\}_{k=0}^T$ from the system (29) and define the data matrices

$$\begin{aligned} V_0 &:= \begin{bmatrix} v(0) & v(1) & \dots & v(T-1) \end{bmatrix} \in \mathbb{R}^{m\times T} \\ \Xi_0 &:= \begin{bmatrix} \xi(0) & \xi(1) & \dots & \xi(T-1) \end{bmatrix} \in \mathbb{R}^{(n+m)\times T} \\ \Xi_1 &:= \begin{bmatrix} \xi(1) & \xi(2) & \dots & \xi(T) \end{bmatrix} \in \mathbb{R}^{(n+m)\times T} \\ \mathcal{Z}_0 &:= \begin{bmatrix} \xi(0) & \xi(1) & \dots & \xi(T-1) \\ \mathcal{Q}(\xi(0)) & \mathcal{Q}(\xi(1)) & \dots & \mathcal{Q}(\xi(T-1)) \end{bmatrix} \in \mathbb{R}^{R\times T} \end{aligned}$$

which satisfy the identity $\Xi_1 = \mathcal{A}\mathcal{Z}_0 + \mathcal{B}V_0$.

Corollary 3: Consider the nonlinear system (26). Let Assumption 2 hold and $\mathcal{Z}(\xi)$ be of the form (28), with $\xi = \left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right]$. Let $\mathcal{R}_Q \in \mathbb{R}^{(n+m)\times r}$ be a known matrix and $\mathcal{W} \subseteq \mathbb{R}^n \times \mathbb{R}^m$ a set such that

$$\frac{\partial \mathcal{Q}}{\partial \xi}(\xi)^\top \frac{\partial \mathcal{Q}}{\partial \xi}(\xi) \preceq \mathcal{R}_Q \mathcal{R}_Q^\top \text{ for any } \xi \in \mathcal{W} \tag{31}$$

Consider the following SDP in the decision variables $\mathcal{P}_1 \in \mathbb{S}^{(n+m)\times(n+m)}$, $\mathcal{Y}_1 \in \mathbb{R}^{T\times(n+m)}$, $\mathcal{G}_2 \in \mathbb{R}^{T\times(R-n-m)}$ and $\alpha \in \mathbb{R}_{>0}$:

$$\mathcal{P}_1 \succ 0, \tag{32a}$$

$$\mathcal{Z}_0\mathcal{Y}_1 = \begin{bmatrix} \mathcal{P}_1 \\ 0_{(R-n-m)\times(n+m)} \end{bmatrix}, \tag{32b}$$

$$\begin{bmatrix} \Xi_1\mathcal{Y}_1 + (\Xi_1\mathcal{Y}_1)^\top + \alpha I_{n+m} & \Xi_1\mathcal{G}_2 & \mathcal{P}_1\mathcal{R}_Q \\ (\Xi_1\mathcal{G}_2)^\top & -I_{R-n-m} & 0 \\ \mathcal{R}_Q^\top\mathcal{P}_1^\top & 0 & -I_r \end{bmatrix} \preceq 0, \tag{32c}$$

$$\mathcal{Z}_0\mathcal{G}_2 = \begin{bmatrix} 0_{(n+m)\times(R-n-m)} \\ I_{R-n-m} \end{bmatrix}. \tag{32d}$$

If the program is feasible then the dynamic feedback control

$$\dot{u} = \mathcal{K}\mathcal{Z}(\xi) \tag{33}$$

with

$$\mathcal{K} = V_0 \begin{bmatrix} \mathcal{Y}_1\mathcal{P}_1^{-1} & \mathcal{G}_2 \end{bmatrix} \tag{34}$$

is such that the closed-loop dynamics (1), (12) is exponentially contractive on $\mathcal{W}$.

Proof. It is similar to the proof of Theorem 1 and is omitted. ■

The asymptotic properties of the closed-loop system can be established similarly to Corollaries 1 and 2. For the sake of brevity, we do not formally state them.

Example 2: Consider the dynamics of a continuous stirred tank reactor [41]

$$\dot{x}_1 = 4.25x_1 + x_2 - 0.25u - x_1 u \tag{35a}$$
$$\dot{x}_2 = -6.25x_1 - 2x_2 \tag{35b}$$

where x_1 and x_2 are the deviations from the steady-state output temperature and concentration, respectively, and the control input u is the effect of coolant flow on the chemical reaction. Note that the open-loop system is unstable. The control objective is to maintain the output temperature and concentration close to their steady-state values by regulating the flow of coolant fluid.

We introduce $\xi := \left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right]$ and $Q(\xi) = \xi_1\xi_3$, and then the overall system is given as

$$\dot{\xi} = \mathcal{A}\mathcal{Z}(\xi) + \mathcal{B}v, \tag{36}$$

where

$$\mathcal{A} := \begin{bmatrix} 4.25 & 1 & -0.25 & -1 \\ -6.25 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \mathcal{B} := \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}. \tag{37}$$

Note that $\frac{\partial Q}{\partial \xi}(\xi)^\top \frac{\partial Q}{\partial \xi}(\xi) = \begin{bmatrix} \xi_3^2 & 0 & \xi_1\xi_3 \\ 0 & 0 & 0 \\ \xi_1\xi_3 & 0 & \xi_1^2 \end{bmatrix}$. Let $\mathcal{W} = [-w, w] \times \mathbb{R} \times [-w, w]$, where $w = 0.05$ and let $\mathcal{R}_Q = \mathcal{R}_Q^\top = \begin{bmatrix} 0.1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0.1 \end{bmatrix}$ such that (31) is satisfied.

We collect $T = 10$ samples by running an experiment on system (36) with input uniformly distributed in $[-0.1, 0.1]$, and with an initial state within the same interval. The SDP (32) is feasible and returns the following controller $\mathcal{K}$ and closed-loop dynamics

$$\mathcal{K} = \begin{bmatrix} 851.2634 & 169.3755 & -27.9383 & -0.9400 \end{bmatrix} \tag{38a}$$
$$\dot{\xi} = \begin{bmatrix} 4.2500 & 1.0000 & -0.2500 & -1.0000 \\ -6.2500 & -2.0000 & 0.0000 & 0.0000 \\ 851.2634 & 169.3755 & -27.9383 & -0.9400 \end{bmatrix} \mathcal{Z}(\xi) \tag{38b}$$

and by Corollary 3, (38b) is exponentially contractive on $\mathcal{W}$.

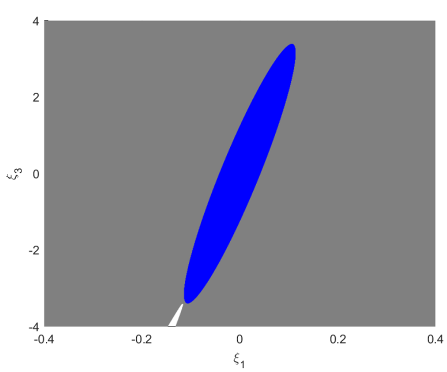

Fig. 2. The grey set represents the projection of the set $\mathcal{H}$ onto the plane $\{\xi : \xi_2 = 0\}$. The blue set is the projection onto the same plane of a Lyapunov sub-level set $\mathcal{R}_\gamma$ contained in $\mathcal{H}$; here, $\mathcal{P}_1^{-1} = \left[\begin{smallmatrix} 32.9160 & 6.5218 & -1.0322 \\ 6.5218 & 1.3114 & -0.2042 \\ -1.0322 & -0.2042 & 0.0376 \end{smallmatrix}\right]$ and $\gamma = 0.06$.

Note that $\mathcal{Z}(0) = 0$ and $\mathcal{W}$ is a convex set containing the origin. By Corollary 1, any solution of (38b) initialized in any sub-level set of $V(\xi) = \xi^\top \mathcal{P}_1^{-1} \xi$ contained in $\mathcal{W}$ uniformly exponentially converges to the origin. For this controller, we numerically determine the set $\mathcal{H} = \{\xi : \dot{V}(\xi) < 0\}$, with $\dot{V}(\xi) := 2\xi^\top \mathcal{P}_1^{-1}\dot{\xi}$, over which the Lyapunov function $V(\xi) = \xi^\top \mathcal{P}_1^{-1}\xi$ computed along the solutions of (38b) decreases and any sub-level set $\mathcal{R}_\gamma := \{\xi : V(\xi) \le \gamma\}$ of V with $\gamma > 0$ contained in $\mathcal{H} \cup \{0\}$ gives an estimate of the ROA for the closed-loop system (38b). Planar projections of the set $\mathcal{H}$ and of a sub-level set of V are displayed in Figure 2. Note that $\mathcal{W}$ is determined by w, i.e., the range of ξ_1 and ξ_3, and thus we project the set $\mathcal{H}$ onto the plane $\{\xi : \xi_2 = 0\}$ to observe the Lyapunov sub-level set contained in $\mathcal{W}$. Moreover, we observe that (32) remains feasible up to $w = 0.1$, and as w increases, the magnitude of the coefficients for the linear part of the obtained controller tends to increase. With $w = 0.1$ and $\mathcal{R}_Q = \mathcal{R}_Q^\top = \begin{bmatrix} 0.2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0.2 \end{bmatrix}$ such that (31) is satisfied, the magnitude is of order 10^3 in this case.

V. Establishing contractivity with noisy data

In this section we extend the previous results to the setup where an additive process disturbance affects the dynamics both during the data acquisition phase and the execution of the control task. Beside the interest per se, establishing contractivity in the presence of disturbance and from noisy data have interesting implications in the solution of other control tasks than stabilization, as we will investigate later.

In the presence of process disturbances, system (1) becomes

$$\dot{x} = f(x) + Bu + Ed \tag{39}$$

which, under Assumption 1, can be written as

$$\dot{x} = AZ(x) + Bu + Ed \tag{40}$$

where $d \in \mathbb{R}^q$ is an unknown signal representing a process disturbance and $E \in \mathbb{R}^{n \times q}$ is a known matrix that indicates which parts of the dynamics are affected by d.

Similar to the noiseless case, we perform an experiment to collect the dataset $\mathbb{D}$ in (4) and arrange the data into the matrices U_0, X_0, X_1, Z_0 defined in (7). Note that now these matrices are related by the equation $X_1 = AZ_0 + BU_0 + ED_0$, where

$$D_0 := \begin{bmatrix} d_0 & d_1 & \cdots & d_{T-1} \end{bmatrix} \in \mathbb{R}^{q \times T} \tag{41}$$

is an unknown matrix.

A data-dependent representation of system (39) in closed-loop with the controller $u = KZ(x)$ can also be given in the presence of disturbances. In fact, the following holds:

Lemma 2: Consider any matrices $K \in \mathbb{R}^{m \times s}$, $G \in \mathbb{R}^{T \times s}$ such that (8) holds. Let G be partitioned as $G = \begin{bmatrix} G_1 & G_2 \end{bmatrix}$, where $G_1 \in \mathbb{R}^{T \times n}$ and $G_2 \in \mathbb{R}^{T \times (s-n)}$. Let Assumption 1 hold. Then system (39) under the control law (12) results in the closed-loop dynamics

$$\begin{aligned} \dot{x} &= (A + BK)Z(x) + Ed \\ &= (X_1 - ED_0)G_1 x + (X_1 - ED_0)G_2 Q(x) + Ed. \ \square \end{aligned} \tag{42}$$

Proof. See [16, Lemma 2]. ■

To further the design and the analysis, we impose some requirements on D_0, namely we introduce the following assumption:

Assumption 3: For a given Δ, $D_0 \in \mathcal{D} = \{D \in \mathbb{R}^{q\times T} \colon DD^\top \preceq \Delta\Delta^\top\}$. □

This assumption on the disturbance has been shown to be flexible enough to capture various classes of disturbances, such as disturbances satisfying energy and pointwise bounds as well as stochastic disturbances. We refer the reader to [4], [35], [42], [16] for details.

Arguments similar to those used to prove Theorem 1 lead to the following:

Theorem 2: Consider the nonlinear system (39), let Assumption 1 hold and $Z(x)$ be of the form (3). Let $R_Q \in \mathbb{R}^{n\times r}$ be a known matrix and $\mathcal{X} \subseteq \mathbb{R}^n$ a set such that (10) holds. Let Assumption 3 hold. Consider the following SDP in the decision variables $P_1 \in \mathbb{S}^{n\times n}$, $Y_1 \in \mathbb{R}^{T\times n}$, $G_2 \in \mathbb{R}^{T\times(s-n)}$, $\alpha \in \mathbb{R}_{>0}$, $\mu \in \mathbb{R}_{>0}$:

$$P_1 \succ 0, \tag{43a}$$

$$Z_0 Y_1 = \begin{bmatrix} P_1 \\ 0_{(s-n)\times n} \end{bmatrix}, \tag{43b}$$

$$\begin{bmatrix} M(Y_1,\alpha,\mu) & X_1G_2 & P_1R_Q & Y_1^\top \\ (X_1G_2)^\top & -I_{s-n} & 0_{n\times r} & G_2^\top \\ R_Q^\top P_1^\top & 0_{r\times n} & -I_r & 0_{r\times T} \\ Y_1 & G_2 & 0_{T\times r} & -\mu I_T \end{bmatrix} \preceq 0\,, \tag{43c}$$

$$Z_0 G_2 = \begin{bmatrix} 0_{n\times(s-n)} \\ I_{s-n} \end{bmatrix} \tag{43d}$$

where $M(Y_1,\alpha,\mu) := X_1Y_1 + (X_1Y_1)^\top + \alpha I_n + \mu E\Delta\Delta^\top E^\top$. If the program is feasible then the control law (12) with K as in (13) is such that the closed-loop dynamics (39), (12) is exponentially contractive on $\mathcal{X}$, *i.e.*, (6) holds. □

Proof. If (43) is feasible, then we can define $G_1 := Y_1 P_1^{-1}$, and (11b), (11d) as well as the definition of K in (13), imply that (8) holds. Hence, by Lemma 2, the closed-loop dynamics $\dot{x} = (A+BK)Z(x) + Ed$ can be written as in (42). By Schur complement, (43c) is equivalent to[4]

$$\begin{bmatrix} X_1Y_1 + (X_1Y_1)^\top + \alpha I_n & X_1G_2 & P_1R_Q \\ (X_1G_2)^\top & -I_{s-n} & 0 \\ R_Q^\top P_1^\top & 0 & -I_r \end{bmatrix} + \mu \begin{bmatrix} -E \\ 0 \\ 0 \end{bmatrix} \Delta\Delta^\top \begin{bmatrix} -E \\ 0 \\ 0 \end{bmatrix}^\top + \mu^{-1} \begin{bmatrix} Y_1^\top \\ G_2^\top \\ 0 \end{bmatrix} \begin{bmatrix} Y_1^\top \\ G_2^\top \\ 0 \end{bmatrix}^\top \preceq 0$$

which is compactly written as $H + \mu L^\top \Delta\Delta^\top L + \mu^{-1} JJ^\top \preceq 0$, with obvious definition of H, J, L. Trivially, $L \neq 0$, whereas $\Delta\Delta^\top \succeq 0$. Moreover, $J \neq 0$, for otherwise $Y_1 = 0$, which would contradict (43b). Hence, by the nonstrict Petersen's lemma [35, Fact 2], we have that $H + JD^\top L + L^\top DJ^\top \preceq 0$ for all $D \in \mathcal{D}$, *i.e.*,

$$\begin{bmatrix} X_1Y_1 + (X_1Y_1)^\top + \alpha I_n & X_1G_2 & P_1R_Q \\ (X_1G_2)^\top & -I_{s-n} & 0 \\ R_Q^\top P_1^\top & 0 & -I_r \end{bmatrix} + \begin{bmatrix} -E \\ 0 \\ 0 \end{bmatrix} D \begin{bmatrix} Y_1 & G_2 & 0 \end{bmatrix} + \begin{bmatrix} Y_1^\top \\ G_2^\top \\ 0 \end{bmatrix} D^\top \begin{bmatrix} -E^\top & 0 & 0 \end{bmatrix} \preceq 0, \forall D \in \mathcal{D},$$

which can be rewritten as

$$\begin{bmatrix} (X_1-ED)Y_1 + Y_1^\top (X_1-ED)^\top + \alpha I_n & (X_1-ED)G_2 & P_1R_Q \\ G_2^\top (X_1-ED)^\top & -I_{s-n} & 0 \\ R_Q^\top P_1^\top & 0 & -I_r \end{bmatrix} \preceq 0, \quad \forall D \in \mathcal{D}.$$

Observe now that this inequality is the same as (11c) provided that X_1Y_1 and X_1G_2 in the latter are replaced by $(X_1 - ED)Y_1$ and $(X_1 - ED)G_2$. Hence, retracing the arguments of the proof of Theorem 1, one arrives at showing that

$$P_1^{-1}(X_1-ED)G\frac{\partial Z}{\partial x} + \frac{\partial Z}{\partial x}^\top G^\top (X_1-ED)^\top P_1^{-1} \preceq -\beta P_1^{-1} \quad \text{for all } x \in \mathcal{X} \text{ and } D \in \mathcal{D}.$$

Since $(X_1 - ED_0)G\frac{\partial Z}{\partial x}$ is the Jacobian of the closed-loop vector field in (42) and $D_0 \in \mathcal{D}$ by Assumption 3, the thesis is proven. ■

To state the asymptotic properties of the closed-loop system, we need to impose assumptions on the entire signal d and not only on the samples collected during the experiment. Thus, we introduce the following:

Assumption 4: d is a continuous function of time that, for some known $\delta > 0$ and for all $t \in \mathbb{R}$, satisfies $|d(t)| \le \delta$. □

A disturbance d that satisfies Assumption 4 also satisfies Assumption 3 with $\Delta := \delta\sqrt{T} I_q$.

The following result underscores that, by enforcing (6), one obtains that the closed-loop system is convergent (see Definition 3 in the Appendix) and as such it enjoys certain properties in the presence of forcing signals.

Corollary 4: Let the conditions of Theorem 2 hold, with Assumption 3 replaced by Assumption 4 and Δ in (43c) replaced by $\delta\sqrt{T} I_q$, and where $\mathcal{X} = \mathbb{R}^n$ and there exists $\overline{x} \in \mathbb{R}^n$ such that $|Z(\overline{x})| < \infty$. Then system $\dot{x} = (A+BK)Z(x) + Ed$ is convergent. Moreover, if $d(t)$ is periodic of period T then $x_*(t)$, the unique solution that is defined and bounded for all $t \in (-\infty, +\infty)$ and globally uniformly exponentially stable, is periodic of period T. If $d(t)$ is constant, then $x_*(t) = x_*$.

Proof. Under the stated conditions, the Jacobian of the right-hand side of the closed-loop dynamics $\dot{x} = (A+BK)Z(x) + Ed$, i.e., $(A+BK)\frac{\partial Z}{\partial x}$, satisfies (6). The thesis is implied by Theorem 6 in the Appendix. ■

Corollary 4 gives conditions under which all the solutions of the closed-loop system designed from data converge to a unique solution, which is periodic or constant depending on the nature of the disturbance. In fact, if the disturbance is a linear combination of constant and sinusoidal signals of known

[4]To avoid burdening the notation, in the following expressions we omit the size of the zero matrices.

frequencies, then Corollary 4 holds with a simpler form of the SDP, as we show in the next section.

VI. Data generated under disturbances of known frequencies

In this subsection, we investigate the case in which the disturbance perturbing the system is a linear combination of constant and sinusoidal signals of known frequencies. Formally, we replace Assumption 4 with the following one:

Assumption 5: d is a function of time solution of the autonomous system

$$\dot{w} = \Psi w, \quad d = \Gamma w \tag{44}$$

where $w = \mathrm{col}(w_1, \ldots, w_{\sigma_1}, w_{\sigma_1+1}, \ldots, w_{\sigma_1+\sigma_2})$,

$$\Psi = \mathrm{block.diag}(\begin{bmatrix} 0 & \psi_1 \\ -\psi_1 & 0 \end{bmatrix}, \ldots, \begin{bmatrix} 0 & \psi_{\sigma_1} \\ -\psi_{\sigma_1} & 0 \end{bmatrix}, \underbrace{0, \ldots, 0}_{\sigma_2 \text{ times}}),$$

with $\psi_1, \ldots, \psi_{\sigma_1} \in \mathbb{R}_{>0}$ known frequencies, and $\Gamma \in \mathbb{R}^{q \times (2\sigma_1+\sigma_2)}$ an unknown matrix. □

We denote by

$$w(0) = \begin{bmatrix} w_1(0)^\top \ldots w_{\sigma_1}(0)^\top \; w_{\sigma_1+1}(0) \; \ldots \; w_{\sigma_1+\sigma_2}(0) \end{bmatrix}^\top$$

the initial condition of the system (44) generating the disturbance during the data collection phase, where $w_i(0) = [w_{i1}(0) \;\; w_{i2}(0)]^\top \in \mathbb{R}^2$ for $i = 1, \ldots, \sigma_1$ and $w_i(0) \in \mathbb{R}$ for $i = \sigma_1 + 1, \ldots, \sigma_1 + \sigma_2$. Under Assumption 5 the matrix of disturbance samples D_0 in (41) becomes

$$D_0 = \Gamma \begin{bmatrix} w_1(t_0) & \ldots & w_1(t_{T-1}) \\ \vdots & \ddots & \vdots \\ w_{\sigma_1+\sigma_2}(t_0) & \ldots & w_{\sigma_1+\sigma_2}(t_{T-1}) \end{bmatrix}$$

where, for $i = 1, 2, \ldots, \sigma_1$,

$$\begin{aligned} & \begin{bmatrix} w_i(t_0) & \ldots & w_i(t_{T-1}) \end{bmatrix} \\ = \; & \underbrace{\begin{bmatrix} w_{i1}(0) & w_{i2}(0) \\ w_{i2}(0) & -w_{i1}(0) \end{bmatrix}}_{=:L_i} \underbrace{\begin{bmatrix} \cos(\psi_i t_0) & \ldots & \cos(\psi_i t_{T-1}) \\ \sin(\psi_i t_0) & \ldots & \sin(\psi_i t_{T-1}) \end{bmatrix}}_{=:M_i} \end{aligned}$$

and, for $i = \sigma_1 + 1, \sigma_1 + 2, \ldots, \sigma_1 + \sigma_2$,

$$\begin{bmatrix} w_i(t_0) & \ldots & w_i(t_{T-1}) \end{bmatrix} = \underbrace{w_i(0)}_{=:L_i} \underbrace{\begin{bmatrix} 1 & \ldots & 1 \end{bmatrix}}_{=:M_i}$$

Set

$$\begin{array}{rl} L := & \mathrm{block.diag}(L_1, \ldots, L_{\sigma_1+\sigma_2}) \\ M := & \mathrm{block.col}(M_1, \ldots, M_{\sigma_1+\sigma_2}) \\ N := & \Gamma L \end{array} \tag{45}$$

and note that M is known while N is unknown. Bearing in mind the system (40), and the expression of D_0 given above, namely $D_0 = \Gamma L M = NM$, we realize that the matrices of data satisfy

$$X_1 = AZ_0 + BU_0 + ENM.$$

Hence, we have the following interesting data-dependent representation of the closed-loop system

Lemma 3: Let Assumption 1 and 5 hold. Consider any matrices $K \in \mathbb{R}^{m \times s}$, $G \in \mathbb{R}^{T \times s}$ such that

$$\begin{bmatrix} K \\ I_s \\ 0_{(2\sigma_1+\sigma_2)\times s} \end{bmatrix} = \begin{bmatrix} U_0 \\ Z_0 \\ M \end{bmatrix} G \tag{46}$$

holds. Let G be partitioned as $G = \begin{bmatrix} G_1 & G_2 \end{bmatrix}$, where $G_1 \in \mathbb{R}^{T \times n}$ and $G_2 \in \mathbb{R}^{T \times (s-n)}$. Then system (39) under the control law (12) results in the closed-loop dynamics

$$\begin{array}{rl} \dot{x} = & (A + BK)Z(x) + Ed \\ = & X_1 G_1 x + X_1 G_2 Q(x) + Ed. \end{array} \tag{47}$$

□

Proof. Left-multiply both sides of (46) by $[\, B \;\; A \;\; EN \,]$ and obtain

$$A + BK = (AZ_0 + BU_0 + ENM)G = X_1 G.$$

This ends the proof. ■

The remarkable feature of this result is that, for systems affected by disturbances generated by (44), a data-dependent representation that is *independent* of the unknown matrix D_0 is derived, provided that the additional condition $0_{q \times s} = MG$ holds. The result allows us to derive contractivity and asymptotic properties of the closed-loop system similar to those obtained in the case of noise-free data. These results are introduced below.

Theorem 3: Consider the nonlinear system (39), let Assumption 1 hold and $Z(x)$ be of the form (3). Let $R_Q \in \mathbb{R}^{n \times r}$ be a known matrix and $\mathcal{X} \subseteq \mathbb{R}^n$ a set such that (10) holds. Let Assumption 5 hold. Consider the following SDP in the decision variables $P_1 \in \mathbb{S}^{n \times n}$, $Y_1 \in \mathbb{R}^{T \times n}$, $G_2 \in \mathbb{R}^{T \times (s-n)}$, $\alpha \in \mathbb{R}_{>0}$:

$$(11), \; 0_{(2\sigma_1+\sigma_2)\times s} = M \begin{bmatrix} Y_1 & G_2 \end{bmatrix}, \tag{48}$$

where M is the matrix defined in (45). If the program is feasible then the control law (12) with K as in (13) is such that the closed-loop dynamics (39), (12) is exponentially contractive on $\mathcal{X}$, *i.e.*, (6) holds. □

Proof. If (48) is feasible then (11) is feasible. Then we can define $G_1 := Y_1 P_1^{-1}$ and note that (11b), (11d), the definition of K in (13) and the constraint $0_{q \times s} = M \begin{bmatrix} Y_1 & G_2 \end{bmatrix}$ give (46). Hence, by Lemma 3, the closed-loop dynamics $\dot{x} = (A + BK)Z(x) + Ed$ can be written as $\dot{x} = Mx + NQ(x) + Ed$, with $M = X_1 G_1$ and $N = X_1 G_2$. By (11c) and (10), the proof of Theorem 1 shows that (6) holds. ■

The proof of the following result is identical to the proof of Corollary 4 and is omitted.

Corollary 5: Let the conditions of Theorem 3 with $\mathcal{X} = \mathbb{R}^n$ and let $\overline{x} \in \mathbb{R}^n$ be such that $|Z(\overline{x})| < \infty$. Then system $\dot{x} = (A + BK)Z(x) + Ed$ is convergent. Moreover, if $d(t)$ is periodic of period T then $x_*(t)$ is periodic of period T. If $d(t)$ is constant, then $x_*(t) = x_*$. □

The requirement for $d(t)$ to be periodic is needed since the sum of sinusoidal signals of different frequencies might not be periodic. On the other hand, if $d(t)$ is a constant disturbance, then the result above guarantees convergence to the unique equilibrium under conditions that are simpler than the ones required by Corollary 4.

VII. Data driven integral control for nonlinear systems

In this section we discuss how establishing contractivity via data paves the way for the data-driven design of an integral controller for nonlinear systems of the form

$$\begin{array}{rl} \dot x = & f(x)+Bu+Ed \\ y = & h(x) \\ e = & y-r \end{array} \tag{49}$$

where $d \in \mathbb{R}^q, r \in \mathbb{R}^p$ are a *constant* disturbance and a *constant* reference signal, $y \in \mathbb{R}^p$ is the regulated output and $e \in \mathbb{R}^p$ is the tracking error. We assume that y is also available for measurements, in addition to the state x. We are interested in the problem of designing the integral controller

$$\begin{array}{rl} \dot \xi = & e \\ u = & k(x,\xi) \end{array} \tag{50}$$

such that the solutions $x(t), \xi(t)$ of the closed-loop system

$$\begin{array}{rl} \dot x = & f(x)+Bk(x,\xi)+Ed \\ \dot \xi = & e \\ e = & h(x)-r \end{array} \tag{51}$$

are bounded for all $t \geq 0$ and satisfy $e(t) \to 0$ as $t \to +\infty$.

The following assumption on the system replaces Assumption 1.

Assumption 6: A continuously differentiable vector-valued function $Z : \mathbb{R}^n \to \mathbb{R}^s$ is known such that $f(x) = AZ(x)$ and $h(x) = CZ(x)$ for some matrices $A \in \mathbb{R}^{n\times s}$, $C \in \mathbb{R}^{p\times s}$. □

Hence, the system (49) considered in this section takes the form

$$\begin{array}{rl} \dot x = & AZ(x)+Bu+Ed \\ e = & CZ(x)-r \end{array} \tag{52}$$

We let $Z(x)$ be of the form (3) and partition A, C accordingly as

$$\begin{bmatrix} A \\ C \end{bmatrix} = \begin{bmatrix} \overline{A} & \hat A \\ \overline{C} & \hat C \end{bmatrix}$$

After stacking $Z(x)$ and the controller state variable ξ in the vector

$$\mathcal{Z}(x,\xi) = \begin{bmatrix} x \\ \xi \\ Q(x) \end{bmatrix} \tag{53}$$

the system (49) extended with the integral action $\dot\xi = e$ can be written as

$$\begin{bmatrix} \dot x \\ \dot \xi \end{bmatrix} = \mathcal{A}\mathcal{Z}(x,\xi) + \mathcal{B}u + \mathcal{E}d - \mathcal{I}r \tag{54}$$

where

$$\mathcal{A} = \begin{bmatrix} \overline{A} & 0_{n\times p} & \hat A \\ \overline{C} & 0_{p\times p} & \hat C \end{bmatrix}, \; \mathcal{B} = \begin{bmatrix} B \\ 0_{p\times m} \end{bmatrix}$$
$$\mathcal{E} = \begin{bmatrix} E \\ 0_{p\times q} \end{bmatrix}, \; \mathcal{I} = \begin{bmatrix} 0_{n\times p} \\ I_p \end{bmatrix}.$$

The constant disturbance d is generated by the system (44) with $\Sigma = 0_{\sigma\times\sigma}$, and $\Gamma \in \mathbb{R}^{q\times\sigma}$ unknown.

We collect data

$$\mathbb{D} := \{(x_i, u_i, y_i, \dot x_i, \xi_i)\}_{i=0}^{T-1} \tag{55}$$

from systems (49), (50) through an experiment, where, as before, $x_i := x(t_i)$, etc., and define, in addition to the data matrix U_0 defined in (7), the matrices

$$\mathcal{Z}_1 := \begin{bmatrix} \dot x(t_0) & \dot x(t_1) & \dots & \dot x(t_{T-1}) \\ y(t_0) & y(t_1) & \dots & y(t_{T-1}) \end{bmatrix}$$

$$\mathcal{Z}_0 := \begin{bmatrix} x(t_0) & x(t_1) & \dots & x(t_{T-1}) \\ \xi(t_0) & \xi(t_1) & \dots & \xi(t_{T-1}) \\ Q(x(t_0)) & Q(x(t_1)) & \dots & Q(x(t_{T-1})) \end{bmatrix}$$

Bearing in mind the analysis of Section VI, the matrix of disturbance samples D_0 in (41) is equal to NM where $N = \Gamma L$,

$$L = \mathrm{diag}(w_1(0), \dots, w_\sigma(0)) \in \mathbb{R}^{\sigma\times\sigma} \tag{56}$$

$$M = \begin{bmatrix} \mathbb{1}_{1\times T} \\ \vdots \\ \mathbb{1}_{1\times T} \end{bmatrix} \in \mathbb{R}^{\sigma\times T} \tag{57}$$

and $\mathbb{1}_{1\times T}$ denotes a $1\times T$ matrix of all ones. The matrices of data satisfy the identity

$$\mathcal{Z}_1 = \mathcal{A}\mathcal{Z}_0 + \mathcal{B}U_0 + \mathcal{E}NM$$

Note that to derive this identity, we are using the relation

$$y(t_i) = e(t_i) + r = CZ(x(t_i)), \; i = 0, 1, \dots, T-1.$$

We can give a data-dependent representation of system (54) in feedback with

$$u = k(x,\xi) := \begin{bmatrix} \overline{K} & \widetilde{K} & \hat K \end{bmatrix} \begin{bmatrix} x \\ \xi \\ Q(x) \end{bmatrix} =: \mathcal{K}\mathcal{Z}(x,\xi) \tag{58}$$

Lemma 4: Consider any matrices $\mathcal{K} \in \mathbb{R}^{m\times(s+p)}$, $\mathcal{G} \in \mathbb{R}^{T\times(s+p)}$ such that

$$\begin{bmatrix} \mathcal{K} \\ I_{s+p} \\ 0_{\sigma\times(s+p)} \end{bmatrix} = \begin{bmatrix} U_0 \\ \mathcal{Z}_0 \\ M \end{bmatrix} \mathcal{G} \tag{59}$$

holds. Let $\mathcal{G}$ be partitioned as $\mathcal{G} = \begin{bmatrix} \mathcal{G}_1 & \mathcal{G}_2 \end{bmatrix}$, where $\mathcal{G}_1 \in \mathbb{R}^{T\times(n+p)}$ and $\mathcal{G}_2 \in \mathbb{R}^{T\times(s-n)}$. Then, the dynamics of the system (49), (50), where $k(x,\xi)$ is as in (58), results in the closed-loop dynamics

$$\begin{aligned} \begin{bmatrix} \dot x \\ \dot \xi \end{bmatrix} &= (\mathcal{A}+\mathcal{B}\mathcal{K})\mathcal{Z}(x,\xi) + \mathcal{E}d - \mathcal{I}r \\ &= \mathcal{Z}_1\mathcal{G}_1 \begin{bmatrix} x \\ \xi \end{bmatrix} + \mathcal{Z}_1\mathcal{G}_2 Q(x) + \mathcal{E}d - \mathcal{I}r. \end{aligned} \tag{60}$$

Proof. The proof is similar to the proof of Lemma 3 and is omitted. ■

The result below shows contractivity of the closed-loop dynamics.

Theorem 4: Consider the nonlinear system (49), let Assumption 6 hold and $\mathcal{Z}(x,\xi)$ be of the form (53). Let $R_Q \in \mathbb{R}^{n\times r}$ be a known matrix and $\mathcal{X} \subseteq \mathbb{R}^n$ a set such that (10) holds. Set $\mathcal{R}_Q := \begin{bmatrix} R_Q \\ 0_{p\times r} \end{bmatrix}$. Consider the following SDP in the decision variables $\mathcal{P}_1 \in \mathbb{S}^{(n+p)\times(n+p)}$, $\mathcal{Y}_1 \in \mathbb{R}^{T\times(n+p)}$, $\mathcal{G}_2 \in \mathbb{R}^{T\times(s-n)}$, $\alpha \in \mathbb{R}_{>0}$:

$$\mathcal{P}_1 \succ 0, \tag{61a}$$

$$\mathcal{Z}_0\mathcal{Y}_1 = \begin{bmatrix} \mathcal{P}_1 \\ 0_{(s-n)\times(n+p)} \end{bmatrix}, \tag{61b}$$

$$\begin{bmatrix} \mathcal{Z}_1\mathcal{Y}_1 + (\mathcal{Z}_1\mathcal{Y}_1)^\top + \alpha I_{n+p} & \mathcal{Z}_1\mathcal{G}_2 & \mathcal{P}_1\mathcal{R}_Q \\ (\mathcal{Z}_1\mathcal{G}_2)^\top & -I_{s-n} & 0 \\ \mathcal{R}_Q^\top P_1^\top & 0 & -I_r \end{bmatrix} \preceq 0\,, \tag{61c}$$

$$\mathcal{Z}_0\mathcal{G}_2 = \begin{bmatrix} 0_{(n+p)\times(s-n)} \\ I_{s-n} \end{bmatrix}, \tag{61d}$$

$$\mathbb{1}_{1\times T}\begin{bmatrix}\mathcal{Y}_1 & \mathcal{G}_2\end{bmatrix} = 0_{1\times(s+p)}\,. \tag{61e}$$

If the program is feasible then the control law (58) with $\mathcal{K}$ defined as

$$\mathcal{K} = U_0\begin{bmatrix}\mathcal{Y}_1\mathcal{P}_1^{-1} & \mathcal{G}_2\end{bmatrix} \tag{62}$$

is such that the closed-loop dynamics (49), (50) is exponentially contractive on $\mathcal{X}\times\mathbb{R}^p$, *i.e.*,

$$\begin{aligned}&\exists\beta>0 \text{ such that for all } (x,\xi)\in\mathcal{X}\times\mathbb{R}^p\\ &\Big[(\mathcal{A}+\mathcal{B}\mathcal{K})\tfrac{\partial\mathcal{Z}}{\partial(x,\xi)}\Big]^\top\mathcal{P}_1^{-1} + \mathcal{P}_1^{-1}\Big[(\mathcal{A}+\mathcal{B}\mathcal{K})\tfrac{\partial\mathcal{Z}}{\partial(x,\xi)}\Big] \preceq -\beta\mathcal{P}_1^{-1}.\end{aligned}$$

□

Proof. If (61) is feasible and we set $\mathcal{G}_1 = \mathcal{Y}_1\mathcal{P}_1^{-1}$, then it holds that

$$\begin{aligned}&\mathcal{P}_1^{-1}\mathcal{Z}_1\mathcal{G}_1 + \mathcal{G}_1^\top\mathcal{Z}_1^\top\mathcal{P}_1^{-1} + \alpha\mathcal{P}_1^{-2} + \mathcal{R}_Q\mathcal{R}_Q^\top\\ &\quad+\mathcal{P}_1^{-1}\mathcal{Z}_1\mathcal{G}_2\mathcal{G}_2^\top\mathcal{Z}_1\mathcal{P}_1^{-1} \preceq 0\end{aligned} \tag{63}$$

If $\mathcal{Z}_1\mathcal{G}_2 = 0$, then $(\mathcal{A}+\mathcal{B}\mathcal{K})\frac{\partial\mathcal{Z}}{\partial(x,\xi)} = \mathcal{Z}_1\mathcal{G}_1\left[\begin{smallmatrix}x\\\xi\end{smallmatrix}\right]$, and (63) implies the exponential contractivity property with $\beta := \alpha\lambda_{\min}(P_1^{-1})$, as claimed. If $\mathcal{Z}_1\mathcal{G}_2 \neq 0$, (63) implies that

$$\begin{aligned}&\mathcal{P}_1^{-1}\mathcal{Z}_1\mathcal{G}_1 + \mathcal{G}_1^\top\mathcal{Z}_1^\top\mathcal{P}_1^{-1} + \alpha\mathcal{P}_1^{-2}\\ &\quad+\mathcal{P}_1^{-1}\mathcal{Z}_1\mathcal{G}_2\mathcal{R}^\top + \mathcal{R}\mathcal{G}_2^\top\mathcal{Z}_1^\top\mathcal{P}_1^{-1} \preceq 0,\\ &\quad\text{for all } \mathcal{R}\in\{\mathcal{R}\colon \mathcal{R}\mathcal{R}^\top \preceq \mathcal{R}_Q\mathcal{R}_Q^\top\}.\end{aligned} \tag{64}$$

By (10), for any $(x,\xi)\in\mathcal{X}\times\mathbb{R}^p$, we have

$$\frac{\partial Q}{\partial(x,\xi)}^\top\frac{\partial Q}{\partial(x,\xi)} = \begin{bmatrix}\frac{\partial Q}{\partial x}^\top\\ \frac{\partial Q}{\partial\xi}^\top\end{bmatrix}\begin{bmatrix}\frac{\partial Q}{\partial x} & \frac{\partial Q}{\partial\xi}\end{bmatrix} \preceq \begin{bmatrix}R_QR_Q^\top & 0\\ 0 & 0\end{bmatrix} = \mathcal{R}_Q\mathcal{R}_Q^\top$$

that is, $\frac{\partial Q}{\partial(x,\xi)}^\top \in \{\mathcal{R}\colon \mathcal{R}\mathcal{R}^\top \preceq \mathcal{R}_Q\mathcal{R}_Q^\top\}$. Therefore

$$\begin{aligned}&\mathcal{P}_1^{-1}\mathcal{Z}_1\mathcal{G}_1 + \mathcal{G}_1^\top\mathcal{Z}_1^\top\mathcal{P}_1^{-1} + \alpha\mathcal{P}_1^{-2}\\ &+\mathcal{P}_1^{-1}\mathcal{Z}_1\mathcal{G}_2\tfrac{\partial Q}{\partial(x,\xi)} + \tfrac{\partial Q}{\partial(x,\xi)}^\top\mathcal{G}_2^\top\mathcal{Z}_1^\top\mathcal{P}_1^{-1} \preceq 0\end{aligned}$$

that is, recalling that $\beta := \alpha\lambda_{\min}(P_1^{-1})$,

$$\mathcal{P}_1^{-1}\mathcal{Z}_1\mathcal{G}\tfrac{\partial\mathcal{Z}}{\partial(x,\xi)} + \tfrac{\partial\mathcal{Z}}{\partial(x,\xi)}^\top G^\top\mathcal{Z}_1^\top\mathcal{P}_1^{-1} \preceq -\beta\mathcal{P}_1^{-1}$$

for all $(x,\xi)\in\mathcal{X}\times\mathbb{R}^p$. Bearing in mind that, by (61a), (61b), (61d), (61e), (62) and Lemma 4,

$$(\mathcal{A}+\mathcal{B}\mathcal{K})\mathcal{Z}(x,\xi) = \mathcal{Z}_1\mathcal{G}_1\begin{bmatrix}x\\\xi\end{bmatrix} + \mathcal{Z}_1\mathcal{G}_2Q(x)$$

the thesis is complete. ∎

The application to the data-driven integral control problem is immediate.

Corollary 6: Let the conditions of Theorem 4 with $\mathcal{X} = \mathbb{R}^n$ hold and let $\overline{x}\in\mathbb{R}^n$ be such that $|Z(\overline{x})| < \infty$. Then the control law (58) with $\mathcal{K}$ defined as in (62) is such that the solutions of the closed-loop dynamics (49), (50) are bounded for all $t\geq 0$ and satisfy $e(t)\to 0$ as $t\to+\infty$.

Proof. By Theorem 4, the closed-loop dynamics (49), (50), where $k(x,\xi)$ is as in (58) and $\mathcal{K}$ is as in (62), is exponentially contractive on $\mathbb{R}^n\times\mathbb{R}^p$. Moreover, it is time invariant. By Theorem 6 in the Appendix, all its solutions are defined for all $t\in\mathbb{R}_{\geq 0}$ and for all $(x(0),\xi(0))\in\mathbb{R}^n\times\mathbb{R}^p$. There exists a unique equilibrium $(x_\star,\xi_\star)$, which is globally uniformly exponentially stable. Hence, all the solutions are bounded. Finally, since $\dot{\xi} = e$, at the equilibrium $0 = h(x_*) - r$, hence by continuity of h, $e(t)\to 0$ as $t\to+\infty$. ∎

The result just stated gives conditions for the regulation of the output of a nonlinear system to a constant reference. Given the nature of the result, one would expect some conditions on the feasibility on the problem to be present in the statement. For instance, given that the constant reference value r can be any vector in $\mathbb{R}^p$, it would be natural to require that the matrix C appearing in the output function $y = h(x)$ has full row rank. Yet, not such an assumption *explicitly* appears in the statement. We show below that, in fact, the condition is *implicitly* encoded in the SDP (61). To show this, assume by contradiction that C has not full row rank and that (61) is feasible. This implies in particular that there exists $\mathcal{Y}_1$ such that $\mathcal{Z}_1\mathcal{Y}_1 + (\mathcal{Z}_1\mathcal{Y}_1)^\top + \alpha I_{n+p}\preceq 0$. We rewrite the matrix on the left-hand side of the inequality as

$$\begin{bmatrix}X_1\\ C\mathcal{Z}_0\end{bmatrix}\mathcal{Y}_1 + \mathcal{Y}_1^\top\begin{bmatrix}X_1^\top & \mathcal{Z}_0^\top C^\top\end{bmatrix} + \alpha I_{n+p}\preceq 0$$

where X_1 is defined in (7c). Let $v\neq 0$ be such that $v^\top C = 0$ and multiply the inequality above by $\left[\begin{smallmatrix}0^\top & v^\top\end{smallmatrix}\right]$ on the left and by $\left[\begin{smallmatrix}0\\ v\end{smallmatrix}\right]$ on the right. Then

$$v^\top C\mathcal{Z}_0\mathcal{Y}_1 v + v^\top\mathcal{Y}_1^\top\mathcal{Z}_0^\top C^\top v + \alpha|v|^2 \leq 0$$

which is a contradiction. We conclude that C not fulfilling a necessary condition for the feasibility of the regulation problem consistently yields infeasibility of the SDP that is at the basis of the design.

Finally, before considering a numerical example, we observe that, by Corollary 7 in the Appendix, a local version of Corollary 6 can be given where $\mathcal{X}$ is a convex subset of $\mathbb{R}^n$ and the condition that there exist $\overline{x}\in\mathbb{R}^n$ such that $|Z(\overline{x})| < \infty$ is replaced by the existence of an equilibrium $(\overline{x},\overline{\xi})$ of the closed-loop system (60) such that $\overline{x}\in\mathrm{int}(\mathcal{X})$. The challenge of applying this result lies in the determination of $\overline{x}$ due to the unknown disturbance d.

Example 3: We consider again the one-link robot arm of Example 1, here assuming that the system is corrupted by the constant disturbance $d = \begin{bmatrix}0.1 & 0.2 & 0.3 & 0.4\end{bmatrix}^\top$ and $E = I_4$. The regulated output is $y = x_1$ and the constant reference signal is $r = \frac{\pi}{3}$. Note that matrix C satisfies the full row rank condition. The control objective is to regulate the angular position of the link x_1 to a desired constant reference r. We introduce $Q(x) = \cos x_1$. If we set $\mathcal{X} = \mathbb{R}^4$, then (10) is satisfied with R_Q as in (20). We collect $T = 10$ samples, i.e. $\mathbb{D} := \{(x_i, u_i, y_i, \dot{x}_i, \xi_i)\}_{i=0}^{9}$, by running an experiment with input uniformly distributed in $[-0.1, 0.1]$, and with an initial state within the same interval. Note that $M = \mathbb{1}_{1\times 9}$. The SDP (61) is feasible and returns the controller K and the closed-loop dynamics in (65). The state evolution of (65b) with initial state uniformly distributed in $[-1, 1]$ is displayed in Figure 3, and the closed-loop trajectories converge to an equilibrium point where the tracking error is zero.

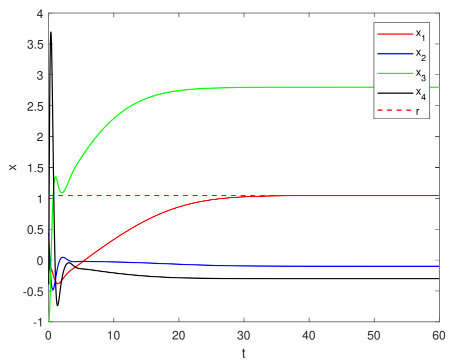

Fig. 3. State response of system (65b) with initial state uniformly distributed in $[-1, 1]$.

Next, in order to test the effectiveness of Theorem 4 for other choices of $Q(x)$, let $Q(x) = \begin{bmatrix}\cos x_1 & \sin x_2\end{bmatrix}^\top$. Note that

$$\frac{\partial Q}{\partial x}(x)^\top \frac{\partial Q}{\partial x}(x) = \begin{bmatrix} \sin(x_1)^2 & 0 & 0 & 0 \\ 0 & \cos(x_2)^2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

If we set $\mathcal{X} = \mathbb{R}^4$, then (10) is satisfied with

$$R_Q = R_Q^\top = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

We collect $T = 10$ samples under the same experiment setup. The SDP (61) is feasible and returns the controller K and the closed-loop dynamics in (66). The state evolution of (66b) with initial state uniformly distributed in $[-1, 1]$ is displayed in Figure 4, and the closed-loop trajectories converge to an equilibrium point where the tracking error is zero.

VIII. Conclusions

We have derived data-based semidefinite programs that once solved return nonlinear feedback controllers that guarantee contractivity. For data perturbed by a class of periodic disturbances, these semidefinite programs have been shown to be independent of the magnitude of the disturbances, a remarkable feature. We have also studied the design of data-dependent integral controllers for tracking and disturbance rejection problems. We have chosen to focus on the data-based representation of closed loop systems introduced in [4], but a

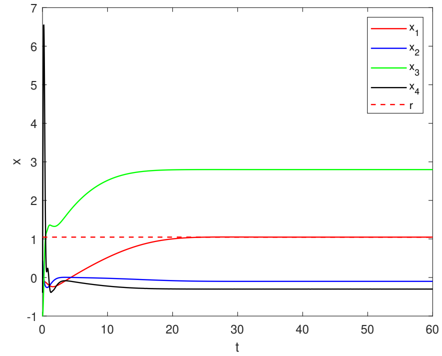

Fig. 4. State response of system (66b) with initial state uniformly distributed in $[-1, 1]$.

similar analysis can be carried out with other representations, such as those studied in [35]. Although we have considered continuous-time systems, results close to the ones presented here can also be derived for discrete-time systems. There are manifold options for future work, which could aim at designing controllers for more sophisticated output regulation problems [43], [44], working with dictionaries of functions $Z(x)$ generated via kernels [45], [46] establishing connections with Gaussian process regression [32], and using non-Euclidean norms [47], [37], to name a few.

Appendix

This appendix collects a few basic results on contraction borrowed from the existing literature. For the sake of a self-contained exposition, some of the proofs are also reported.

A. Convergence results for periodic and time-invariant contractive systems

We consider the system

$$\dot{x} = f(t, x) \tag{67}$$

defined for $(t, x) \in \mathbb{R} \times \mathcal{X}$, where $\mathcal{X}$ is a subset of $\mathbb{R}^n$. The notation $\phi(t, \tau, \xi)$ indicates the solution of $\dot{x} = f(t, x)$, $x(\tau) = \xi$. It is assumed that $f(t, x)$ is differentiable on x and $f(t, x)$ and $\frac{\partial f}{\partial x}(t, x)$ are continuous in t, x. It is also assumed that the set $\mathcal{X}$ is forward invariant, *i.e.*, for any $(t_0, x_0) \in \mathbb{R} \times \mathcal{X}$, the solution $\phi(t, t_0, x_0)$ exists for all $t \in \mathbb{R}_{\geq t_0}$ that belong to the domain of existence of the solution, and system (67) is forward complete on $\mathcal{X}$,[5] *i.e.*, for any $(t_0, x_0) \in \mathbb{R} \times \mathcal{X}$, the solution of (67) initialized at $x(t_0) = x_0$ exists for all $t \in \mathbb{R}_{\geq t_0}$.

If in addition to the previous conditions, other properties hold, then the following results can be stated.

Lemma 5: For a given $T \in \mathbb{R}_{>0}$, assume that f is T-periodic, *i.e.*, $f(t + T, x) = f(t, x)$ for all $(t, x) \in \mathbb{R} \times \mathcal{X}$. Then, for all integers $k \in \mathbb{Z}_{\geq 0}$, for all $t_0 \in \mathbb{R}_{\geq 0}$, for all $t \in \mathbb{R}_{\geq t_0}$, for all $\xi \in \mathcal{X}$, the following holds

1) $\phi(t + kT, t_0 + kT, \xi) = \phi(t, t_0, \xi)$;
2) $\phi(t + kT, t_0, \xi) = \phi(t + kT, t_0 + kT, \phi(t_0 + kT, t_0, \xi)) = \phi(t, t_0, \phi(t_0 + kT, t_0, \xi))$.

[5] Forward completeness is guaranteed if $\mathcal{X}$ is closed and bounded.

$$\mathcal{K} = \begin{bmatrix} -4.1190 & -4.9995 & -4.3688 & -0.8453 & -0.9417 & 0.0030 \end{bmatrix} \tag{65a}$$

$$\begin{bmatrix} \dot{x} \\ \dot{\xi} \end{bmatrix} = \begin{bmatrix} 0.0000 & 1.0000 & 0.0000 & 0.0000 & 0.0000 & -0.0000 \\ -2.0000 & -0.7500 & 1.0000 & 0.0000 & 0.0000 & -1.9600 \\ -0.0000 & -0.0000 & -0.0000 & 1.0000 & -0.0000 & -0.0000 \\ -28.7936 & -33.3298 & -28.4588 & -6.3019 & -6.2778 & 0.0197 \\ 1.0000 & -0.0000 & -0.0000 & -0.0000 & -0.0000 & 0.0000 \end{bmatrix} \begin{bmatrix} x \\ \xi \\ \cos x_1 \end{bmatrix} + \begin{bmatrix} 0.1 \\ 0.2 \\ 0.3 \\ 0.4 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \frac{\pi}{3} \end{bmatrix} \tag{65b}$$

$$\mathcal{K} = \begin{bmatrix} -23.4198 & -46.1397 & -20.1243 & -1.6394 & -5.9556 & -0.0580 & 0.0041 \end{bmatrix} \tag{66a}$$

$$\begin{bmatrix} \dot{x} \\ \dot{\xi} \end{bmatrix} = \begin{bmatrix} -0.0000 & 1.0000 & -0.0000 & -0.0000 & -0.0000 & 0.0000 & 0.0000 \\ -2.0000 & -0.7500 & 1.0000 & 0.0000 & -0.0000 & -1.9600 & -0.0000 \\ -0.0000 & 0.0000 & 0.0000 & 1.0000 & -0.0000 & 0.0000 & -0.0000 \\ -157.4652 & -307.5983 & -133.4954 & -11.5961 & -39.7042 & -0.3869 & 0.0274 \\ 1.0000 & -0.0000 & -0.0000 & -0.0000 & 0.0000 & 0.0000 & 0.0000 \end{bmatrix} \begin{bmatrix} x \\ \xi \\ \cos x_1 \\ \sin x_2 \end{bmatrix} + \begin{bmatrix} 0.1 \\ 0.2 \\ 0.3 \\ 0.4 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \frac{\pi}{3} \end{bmatrix} \tag{66b}$$

Proof. 1) See [48, Remark 1]. 2) Recall the semigroup property (e.g., [49, p. 26]), namely, for any $t > t_1 > t_0$ $\phi(t, t_0, x_0) = \phi(t, t_1, \phi(t_1, t_0, x_0))$ and apply it to $\phi(t + kT, t_0, \xi)$ with $t \to t + kT$, $t_1 \to t_0 + kT$, $t_0 \to t_0$ and $x_0 \to \xi$, to obtain

$$\phi(t + kT, t_0, \xi) = \phi(t + kT, t_0 + kT, \phi(t_0 + kT, t_0, \xi))$$

which is the first identity in 2). Then apply 1) to $\phi(t+kT, t_0+kT, \phi(t_0 + kT, t_0, \xi))$, to obtain $\phi(t + kT, t_0 + kT, \phi(t_0 + kT, t_0, \xi)) = \phi(t, t_0, \phi(t_0 + kT, t_0, \xi))$, which is the second identity in 2). ∎

The following theorem shows the existence of an attractive solution for periodic systems that are contractive on a subset of the state space, provided that the latter is forward invariant with respect to the system's dynamics and the system's dynamics is forward complete on the set. Claims similar to the one below are given in [26, Property 3] and [50, Theorem 3.8] but here we have mostly followed [48, Theorem 5].

Theorem 5: Assume that

1) $\mathcal{X}$ is a closed convex subset of $\mathbb{R}^n$;
2) there exist $P_1 \succ 0, \beta > 0$ such that for all $(t, x) \in \mathbb{R}\times\mathcal{X}$,
$$\frac{\partial f(t, x)}{\partial x}^\top P_1^{-1} + P_1^{-1}\frac{\partial f(t, x)}{\partial x} \preceq -\beta P_1^{-1};$$
3) for a given $T \in \mathbb{R}_{>0}$, f is T-periodic, *i.e.*, $f(t+T, x) = f(t, x)$ for all $(t, x) \in \mathbb{R} \times \mathcal{X}$;
4) $c\mathrm{e}^{-\beta T/2} < 1$, where $c = \left(\frac{\lambda_{\max}(P_1^{-1})}{\lambda_{\min}(P_1^{-1})}\right)^{1/2}$.

Then there exists a unique periodic solution $x_*(t)$ of (67) of period T, which is uniformly exponentially stable. Furthermore, if 3) is replaced by

3') $f(t, x)$ is time invariant, *i*.e., $f(t, x) = f(x)$ for all $(t, x) \in \mathbb{R} \times \mathcal{X}$,

and 4) is removed, then there exists a unique equilibrium x_*, and every solution $\xi(t)$ of (67) which is uniformly exponentially stable.

Proof. Consider any two solutions $\phi(t, t_0, \xi), \phi(t, t_0, \zeta)$ of (67) from the initial conditions $\xi, \zeta \in \mathcal{X}$. These solutions exist for all $t \in \mathbb{R}_{\geq t_0}$ and belong to $\mathcal{X}$, by forward completeness and forward invariance. By 1) and 2), they satisfy $|\phi(t, t_0, \xi)-\phi(t, t_0, \zeta)| \leq c\mathrm{e}^{-\beta(t-t_0)/2}|\xi - \zeta|$, for all $t \in \mathbb{R}_{\geq t_0}$. Define $\mathcal{T}(\xi) := \phi(t_0+T, t_0, \xi)$. $\mathcal{T}$ maps $\mathcal{X}$ into $\mathcal{X}$ because of forward invariance of $\mathcal{X}$. By 4), $|\mathcal{T}(\xi) - \mathcal{T}(\zeta)| \leq \rho|\xi - \zeta|$ for all $\xi, \zeta \in \mathcal{X}$, where $\rho = c\mathrm{e}^{-\beta T/2} < 1$. Consider the sequence $\xi_{k+1} = \mathcal{T}(\xi_k)$, which is defined for all[6] $k \in \mathbb{Z}_{\geq 0}$. Since $\mathcal{X}$ is a closed subset of $\mathbb{R}^n$, $\mathcal{X}$ is a complete metric space. Hence, by the Contraction Mapping theorem, there exists a unique $\xi_* \in \mathcal{X}$ such that $\xi_* = \mathcal{T}(\xi_*)$ and $\xi_k \to \xi_*$ as $k \to \infty$. Define $x_*(t) = \phi(t, t_0, \xi_*)$. Then $x_*(t_0 + T) = \phi(t_0 + T, t_0, \xi_*) = \mathcal{T}(\xi_*) = \xi_* = x_*(t_0)$. Assume that $x_*(t_0 + jT) = x_*(t_0)$ for all $j = 0, 1, \ldots, k$, for some $k \in \mathbb{Z}_{\geq 0}$. Then

$$\begin{aligned} &x_*(t_0 + (k+1)T) = \phi(t_0 + (k+1)T, t_0, \xi_*) \\ &= \phi(t_0 + T, t_0, \phi(t_0 + kT, t_0, \xi_*)) \\ &= \phi(t_0 + T, t_0, x_*(t_0 + kT)) = \phi(t_0 + T, t_0, x_*(t_0)) \\ &= \phi(t_0 + T, t_0, \xi_*) = x_*(t_0 + T) = x_*(t_0) \end{aligned}$$

where the first equation holds by definition of $x_*(t)$, the second by 2) in Lemma 5, the third by the definition of $x_*(t)$, the fourth by the inductive hypothesis, the fifth by the identity $\xi_* = x_*(t_0)$, the sixth by the definition of $x_*(t)$ and the last one by the inductive hypothesis. Hence, $x_*(t_0+kT) = x_*(t_0)$ for all $k \in \mathbb{Z}_{\geq 0}$, *i*.e., $x_*(t)$ is a periodic solution. Uniqueness is due to the periodicity of the solution (see [48, p. 223]). It was shown before that any two solutions $\phi(t, t_0, \xi), \phi(t, t_0, \zeta)$ satisfy $|\phi(t, t_0, \xi) - \phi(t, t_0, \zeta)| \leq c\mathrm{e}^{-\beta(t-t_0)/2}|\xi - \zeta|$, for all $t \in \mathbb{R}_{\geq t_0}$. In particular, for any $\xi \in \mathcal{X}$, the solution $\phi(t, t_0, \xi)$ satisfies $|\phi(t, t_0, \xi) - \phi(t, t_0, \xi_*)| \leq c\mathrm{e}^{-\beta(t-t_0)/2}|\xi - \xi_*|$, for all $t \in \mathbb{R}_{\geq t_0}$, that is, $x_*(t)$ is uniformly exponentially stable.

Assume now that $f(t, x)$ is time invariant, *i.e.* $f(t, x) = f(x)$ for all $(t, x) \in \mathbb{R} \times \mathcal{X}$. $f(x)$ is periodic of any period $T > 0$. By any $T > T_* := 2 \ln c/\beta$, condition 4) is satisfied. Hence, there exists a unique solution $x_*(t)$ of $\dot{x} = f(x)$ that is periodic of any period $T > T_*$ and every solution $\xi(t)$ of $\dot{x} = f(x)$ converges to it. This implies that for any $t \in \mathbb{R}$ and any $T > T_*$, $x_*(t) = x_*(t + T)$. Fixing $t = t_0$, this implies that, for any $T > T_*$, $x_*(t_0) = x_*(t_0 + T)$, that is, after time $t_0 + T_*$, $x_*(t)$ is constant and equal to $x_*(t_0)$. Take now any $t \in (t_0, t_0 + T_*]$ and recall that $x_*(t) = x_*(t + T)$

[6]By definition, $\xi_1 = T(\xi_0) = \phi(t_0 + T, t_0, \xi_0)$. Assume now that $\xi_k = \phi(t_0 + kT, t_0, \xi_0)$ for some $k \in \mathbb{Z}_{\geq 1}$. Then $\xi_{k+1} = \phi(t_0 + T, t_0, \phi(t_0 + kT, t_0, \xi_0))$. Applying the identity $\phi(t + kT, t_0, \xi) = \phi(t, t_0, \phi(t_0 + kT, t_0, \xi))$ in 1) of Lemma 5, the expression on the right-hand side becomes $\phi(t_0 + T + kT, t_0, \xi_0)$. The inductive argument shows that $\xi_k = \phi(t_0 + kT, t_0, \xi_0)$ for all $k \in \mathbb{Z}_{\geq 0}$. Note that the righ-hand side is defined for all $k \in \mathbb{Z}_{\geq 0}$ by forward completeness of the solution.

for any $T > T_*$. But for any $t \in (t_0, t_0 + T_*]$ and $T > T_*$, $t + T > t_0 + T_*$, therefore $x_*(t) = x_*(t+T) = x_*(t_0)$, and we conclude that $x_*(t) = x_*(t_0)$ for all $t \in \mathbb{R}_{\geq t_0}$. That is, $x_*(t)$ is a constant. Since by construction x_* is a solution of $\dot{x} = f(x)$ (see the first part of the proof) and it is constant, then it must be an equilibrium of $\dot{x} = f(x)$. ■

B. Convergent systems

We recall here the class of convergent systems, which exhibits a solution defined and bounded for all time, to which all the solutions uniformly exponentially converge [23]. Compared to the results in the previous subsection, here $\mathcal{X} = \mathbb{R}^n$.

The system of interest is (67) defined for $(t, x) \in \mathbb{R} \times \mathbb{R}^n$, with $f(t, x)$ differentiable on x and $f(t, x)$ and $\frac{\partial f}{\partial x}(t, x)$ continuous in t, x.

Definition 3: The system (67) is convergent if

1) all its solutions $x(t)$ are defined for all $t \in [t_0, +\infty)$ and all initial conditions $t_0 \in \mathbb{R}$, $x(t_0) \in \mathbb{R}^n$;
2) there exists a unique solution $x_*(t)$ defined and bounded for all $t \in (-\infty, +\infty)$;
3) the solution $x_*(t)$ is globally uniformly exponentially stable.

Theorem 6: Assume that

1) $\mathcal{X} = \mathbb{R}^n$;
2) there exist $P_1 \succ 0, \beta > 0$ such that for all $(t, x) \in \mathbb{R} \times \mathbb{R}^n$,
$$\frac{\partial f(t,x)}{\partial x}^\top P_1^{-1} + P_1^{-1} \frac{\partial f(t,x)}{\partial x} \preceq -\beta P_1^{-1};$$
3) there exists $\overline{x} \in \mathcal{X}$ such that, for all $t \in \mathbb{R}$, $|f(t, \overline{x})| \leq \bar{f} < +\infty$.

Then the system is convergent. In addition, if, for a given $T \in \mathbb{R}_{>0}$, f is T-periodic, then $x_*(t)$ is periodic of period T. If $f(t, x)$ is time-invariant, then $x_*(t) = x_*$.

*P*roof. See [23, Theorem 1] and [26, Property 3]. ■

The proof shows that, denoted by $V(x)$ the function $(x - \overline{x})^\top P_1^{-1}(x - \overline{x})$, it holds that $\dot{V}(x) := \frac{\partial V}{\partial x} f(t, x) < 0$ for all $x \in \{x \in \mathbb{R}^n : V(x) > \gamma\}$, where $\gamma = (2\beta^{-1}\bar{f}\|P_1^{-1/2}\|)^2$. This implies that the set $\mathcal{R}_\gamma = \{x \in \mathbb{R}^n : V(x) \leq \gamma\}$ is a closed bounded set of $\mathbb{R}^n$ that is forward invariant for $\dot{x} = f(t, x)$. Hence, it must contain at least one solution $x_*(t)$ defined for all $t \in (-\infty, +\infty)$ [23, p. 260] (its uniqueness can be shown as in [23, p. 261] and uniform exponential stability is straightforwardly proven).

The possibility of restricting the result to a convex subset $\mathcal{X}$ of $\mathbb{R}^n$, without explicitly asking for the forward invariance of $\mathcal{X}$ and forward completeness of the system on $\mathcal{X}$, which are the standing assumptions of Theorem 5, depends on whether or not the set $\mathcal{R}_\gamma$ is contained in $\mathcal{X}$, which cannot be in general established, as it depends on $\beta, P_1, \bar{f}$. A special case in which this is true is when $\overline{x}$ is an equilibrium of the system, i.e., $\bar{f} = 0$, and the set $\mathcal{X}$ contains $\overline{x}$, for in this case $\bar{f} = 0$ implies $\mathcal{R}_\gamma = \{\overline{x}\}$, and $x_*(t) = \overline{x}$. Hence, the following consequence of Theorem 6 holds:

Corollary 7: Assume that

1) $\mathcal{X}$ is a convex subset of $\mathbb{R}^n$;
2) there exist $P_1 \succ 0, \beta > 0$ such that for all $(t, x) \in \mathbb{R} \times \mathcal{X}$,
$$\frac{\partial f(t,x)}{\partial x}^\top P_1^{-1} + P_1^{-1} \frac{\partial f(t,x)}{\partial x} \preceq -\beta P_1^{-1};$$
3) there exists $\overline{x} \in \text{int}(\mathcal{X})$ such that, for all $t \in \mathbb{R}$, $f(t, \overline{x}) = 0$.

Then, there exists a subset $\mathcal{V}$ of $\mathcal{X}$ containing $\overline{x}$ such that

1) all the solutions $x(t)$ of $\dot{x} = f(t, x)$ are defined for all $t \in [t_0, +\infty)$ and all initial conditions $t_0 \in \mathbb{R}$, $x(t_0) \in \mathcal{V}$;
2) there exists a unique solution $x_*(t)$ defined and bounded for all $t \in (-\infty, +\infty)$ and contained in $\mathcal{X}$, and it satisfies $x_*(t) = \overline{x}$;
3) the solution $x_*(t) = \overline{x}$ is uniformly exponentially stable.

Proof. Note that the function $V(x) = (x - \overline{x})^\top P_1^{-1}(x - \overline{x})$ satisfies $\dot{V}(x) := \frac{\partial V}{\partial x} f(t, x) < 0$ for all $x \in \mathcal{X}$. Let $\delta \in \mathbb{R}_{>0}$ and define $\mathcal{V} := \{x \in \mathbb{R}^n : (x - \overline{x})^\top P_1^{-1}(x - \overline{x}) \leq \delta\}$. Fix δ such that $\mathcal{V} \subseteq \mathcal{X}$. The set $\mathcal{V}$ is forward invariant with respect to $\dot{x} = f(t, x)$ and contains a unique solution $x_*(t)$ defined and bounded for all $t \in (-\infty, +\infty)$, which satisfies $x_*(t) = \overline{x}$. To prove uniqueness, we use the argument of [23, p. 261]. Suppose there exists another solution $x'(t)$ defined and bounded for all $t \in (-\infty, +\infty)$ and contained in $\mathcal{X}$. Then such a solution satisfies $|x'(t) - \overline{x}| \leq c\mathrm{e}^{-\beta(t-t_0)/2}|x'(t_0) - \overline{x}|$ for all $t \in \mathbb{R}_{\geq t_0}$. Letting $t_0 \to -\infty$ yields $|x'(t) - \overline{x}| = 0$, which leads to a contradiction by the arbitrariness of t. Finally, the solution $x_*(t) = \overline{x}$ is uniformly exponentially stable as any solution $x(t)$ with initial condition in $\mathcal{V}$ satisfies $|x(t) - \overline{x}| \leq c\mathrm{e}^{-\beta(t-t_0)/2}|x(t_0) - \overline{x}|$ for all $t \in \mathbb{R}_{\geq t_0}$. ■

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