THE ZERO DYNAMICS OF A NONLINEAR SYSTEM

1 Relative degree and local normal forms

The purpose of this Section is to show how single-input single-output nonlinear systems can be locally given, by means of a suitable change of coordinates in the state space, a "normal form" of special interest, on which several important properties can be elucidated.

The point of departure of the whole analysis is the notion of relative degree of the system, which is formally described in the following way. The single-input single-output nonlinear system

$$\begin{aligned} \dot{x} &= f(x) + g(x)u \\ y &= h(x) \end{aligned} \tag{1}$$

is said to have *relative degree* r at a point x° if [1] (i) $L_g L_f^k h(x) = 0$ for all x in a neighborhood of x° and all $k < r-1$
(ii) $L_g L_f^{r-1} h(x^\circ) \neq 0$.

Note that there may be points where a relative degree cannot be defined. This occurs, in fact, when the first function of the sequence

$$L_g h(x), L_g L_f h(x), \ldots, L_g L_f^k h(x), \ldots$$

which is not identically zero (in a neighborhood of x°) has a zero exactly at the point $x = x^\circ$. However, the set of points where a relative degree can be defined is clearly an open and dense subset of the set U where the system (1) is defined.

Remark. In order to compare the notion thus introduced with a familiar concept, let us calculate the relative degree of a linear system

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \,. \end{aligned}$$

In this case, since $f(x) = Ax$, $g(x) = B$, $h(x) = Cx$, it easily seen that

$$L_f^k h(x) = CA^k x$$

and therefore

$$L_g L_f^k h(x) = CA^k B \,.$$

Thus, the integer r is characterized by the conditions

$$\begin{aligned} CA^k B &= 0 \qquad \text{for all } k < r-1 \\ CA^{r-1} B &\neq 0 \,. \end{aligned}$$

[1] Let λ be real-valued function and f an n-vector-valued vector, both defined on a subset U of $\mathbb{R}^n$. The function $L_f\lambda$ is the real-valued function defined as

$$L_f\lambda(x) = \sum_{i=1}^{n} \frac{\partial \lambda}{\partial x_i} f_i(x) := \frac{\partial \lambda}{\partial x} f(x) \,.$$

This function is sometimes called *derivative of* λ *along* f.

It is well-known that the integer satisfying these conditions is exactly equal to the *difference* between the degree of the denominator polynomial and the degree of the numerator polynomial of the transfer function

$$T(s) = C(sI - A)^{-1}B$$

of the system. $\triangleleft$

We illustrate now a simple interpretation of the notion of relative degree, which is not restricted to the assumption of linearity considered in the previous Remark. Assume the system at some time t° is in the state $x(t^\circ) = x^\circ$ and suppose we wish to calculate the value of the output $y(t)$ and of its derivatives with respect to time $y^{(k)}(t)$, for $k = 1, 2, \ldots$, at $t = t^\circ$. We obtain

$$\begin{aligned} y(t^\circ) &= h(x(t^\circ)) = h(x^\circ) \\ y^{(1)}(t) &= \frac{\partial h}{\partial x}\frac{dx}{dt} = \frac{\partial h}{\partial x}(f(x(t)) + g(x(t))u(t)) \\ &= L_f h(x(t)) + L_g h(x(t))u(t) \ . \end{aligned}$$

If the relative degree r is larger than 1, for all t such that $x(t)$ is near x°, i.e. for all t near t°, we have $L_g h(x(t)) = 0$ and therefore

$$y^{(1)}(t) = L_f h(x(t)) \ .$$

This yields

$$\begin{aligned} y^{(2)}(t) = \frac{\partial L_f h}{\partial x}\frac{dx}{dt} &= \frac{\partial L_f h}{\partial x}(f(x(t)) + g(x(t))u(t)) \\ &= L_f^2 h(x(t)) + L_g L_f h(x(t))u(t) \ . \end{aligned}$$

Again, if the relative degree is larger than 2, for all t near t° we have $L_g L_f h(x(t)) = 0$ and

$$y^{(2)}(t) = L_f^2 h(x(t)) \ .$$

Continuing in this way, we get

$$\begin{aligned} y^{(k)}(t) &= L_f^k h(x(t)) \qquad \text{for all } k < r \text{ and all } t \text{ near } t^\circ \\ y^{(r)}(t^\circ) &= L_f^r h(x^\circ) + L_g L_f^{r-1} h(x^\circ)u(t^\circ) \ . \end{aligned}$$

Thus, the relative degree r is exactly equal to the number of times one has to differentiate the output $y(t)$ at time $t = t^\circ$ in order to have the value $u(t^\circ)$ of the input explicitly appearing.

Note also that if

$$L_g L_f^k h(x) = 0 \qquad \text{for all } x \text{ in a neighborhood of } x^\circ \text{ and all } k \geq 0$$

(in which case no relative degree can be defined at any point around x°) then the output of the system is not affected by the input, for all t near t°. As a matter of fact, if this is the case, the previous calculations show that the Taylor series expansion of $y(t)$ at the point $t = t^\circ$ has the form

$$y(t) = \sum_{k=0}^{\infty} L_f^k h(x^\circ)\frac{(t - t^\circ)^k}{k!}$$

i.e. that $y(t)$ is a function depending only on the initial state and not on the input.

These calculations suggest that the functions $h(x)$, $L_f h(x), \dots, L_f^{r-1} h(x)$ must have a special importance. As a matter of fact, it is possible to show that they can be used in order to define, at least partially, a local coordinates transformation around x°(recall that x° is a point where $L_g L_f^{r-1} h(x^\circ) \neq 0$). This fact is based on the following property.

Lemma 1 *The row vectors* [2]

$$dh(x^\circ), dL_f h(x^\circ), \dots, dL_f^{r-1} h(x^\circ)$$

are linearly independent.

Lemma 1 shows that necessarily $r \leq n$ and that the r functions $h(x), L_f h(x), \dots, L_f^{r-1} h(x)$ qualify as a partial set of new coordinate functions around the point x°. As we shall see in a moment, the choice of these new coordinates entails a particularly simple structure for the equations describing the system. However, before doing this, it is convenient to summarize the results discussed so far in a formal statement, that also illustrates a way in which the set of new coordinates can be completed in case the relative degree r is strictly less than n.

Proposition 1 *Suppose the system has relative degree r at x°. Then $r \leq n$. If r is strictly less than n, it is always possible to find $n - r$ more functions $\psi_1(x), \dots, \psi_{r-r}(x)$ such that the mapping*

$$\Phi(x) = \begin{pmatrix} \psi_1(x) \\ \cdots \\ \psi_{n-r}(x) \\ h(x) \\ L_f h(x) \\ \cdots \\ L_f^{r-1} h(x) \end{pmatrix}$$

has a jacobian matrix which is nonsingular at x° and therefore qualifies as a local coordinates transformation in a neighborhood of x°. The value at x° of these additional functions can be fixed arbitrarily. Moreover, it is always possible to choose $\psi_1(x), \dots, \psi_{n-r}(x)$ in such a way that

$$L_g \psi_i(x) = 0 \qquad \textit{for all } 1 \leq i \leq n - r \textit{ and all } x \textit{ around } x^\circ.$$

The description of the system in the new coordinates is found very easily. Set

$$z = \begin{pmatrix} \psi_1(x) \\ \psi_2(x) \\ \cdots \\ \psi_{n-r}(x) \end{pmatrix}, \quad \xi = \begin{pmatrix} h(x) \\ L_f h(x) \\ \cdots \\ L_f^{r-1} h(x) \end{pmatrix}$$

[2]Let λ be a real-valued function defined on a subset U of $\mathbb{R}^n$. Its *differential*, denote $d\lambda(x)$ is the row vector

$$d\lambda(x) = \begin{pmatrix} \frac{\partial \lambda}{\partial x_1} & \frac{\partial \lambda}{\partial x_2} & \cdots & \frac{\partial \lambda}{\partial x_n} \end{pmatrix} := \frac{\partial \lambda}{\partial x}.$$

and

$$\tilde{x} = (z, \xi)\,.$$

Looking at the calculations already carried out at the beginning, we obtain for $\xi_1, \dots, \xi_r$

$$\begin{aligned}
\frac{d\xi_1}{dt} &= \frac{\partial h}{\partial x}\frac{dx}{dt} = L_f h(x(t)) = \xi_2(t) \\
&\cdots \\
\frac{d\xi_{r-1}}{dt} &= \frac{\partial (L_f^{r-2}h)}{\partial x}\frac{dx}{dt} = L_f^{r-1}h(x(t)) = \xi_r(t)\,.
\end{aligned}$$

For ξ_r we obtain

$$\frac{d\xi_r}{dt} = L_f^r h(x(t)) + L_g L_f^{r-1} h(x(t)) u(t).$$

On the right-hand side of this equation we must now replace $x(t)$ with its expression as a function of $\tilde{x}(t)$, which will be written as $x(t) = \Phi^{-1}(z(t), \xi(t))$. Thus, setting

$$\begin{aligned}
q(z,\xi) &= L_f^r h(\Phi^{-1}(z,\xi)) \\
b(z,\xi) &= L_g L_f^{r-1} h(\Phi^{-1}(z,\xi))
\end{aligned}$$

the equation in question can be rewritten as

$$\frac{d\xi_r}{dt} = q(z(t),\xi(t)) + b(z(t),\xi(t))u(t)\,.$$

Note that at the point $(z^\circ, \xi^\circ) = \Phi(x^\circ)$, $b(z^\circ, \xi^\circ) \neq 0$ by definition. Thus, the coefficient $a(z,\xi)$ is nonzero for all (z,ξ) in a neighborhood of (z°, ξ°).

As far as the other new coordinates are concerned, we cannot expect any special structure for the corresponding equations, if nothing else has been specified. However, if $\psi_1(x), \dots, \psi_{n-r}(x)$ have been chosen in such a way that $L_g\psi_i(x) = 0$, then

$$\frac{dz_i}{dt} = \frac{\partial \psi_i}{\partial x}(f(x(t)) + g(x(t))u(t)) = L_f\psi_i(x(t)) + L_g\psi_i(x(t))u(t) = L_f\psi_i(x(t)).$$

Setting

$$f_0(z,\xi) = \begin{pmatrix} L_f\psi_1(\Phi^{-1}(z,\xi)) \\ \cdots \\ L_f\psi_{n-r}(\Phi^{-1}(z,\xi)) \end{pmatrix}$$

the latter can be rewritten as

$$\frac{dz}{dt} = f_0(z(t),\xi(t))\,.$$

Thus, in summary, the state-space description of the system in the new coordinates will be as follows

$$\begin{aligned}
\dot{z} &= f_0(z,\xi) \\
\dot{\xi}_1 &= \xi_2 \\
\dot{\xi}_2 &= \xi_3 \\
&\cdots \\
\dot{\xi}_{r-1} &= \xi_r \\
\dot{\xi}_r &= q(z,\xi) + b(z,\xi)u\,.
\end{aligned} \tag{2}$$

In addition to these equations one has to specify how the output of the system is related to the new state variables. But, being $y = h(x)$, it is immediately seen that

$$y = \xi_1 \ . \tag{3}$$

The equations thus defined are said to be in *normal form*. They are useful in understanding how certain control problems can be solved.

2 The Zero Dynamics

In this section we introduce and discuss an important concept, that in many instances plays a role exactly similar to that of the "zeros" of the transfer function in a linear system. We have already seen that the relative degree r of a linear system can be interpreted as the difference between the number of poles and the number of zeros in the transfer function. In particular, any linear system in which r is strictly less than n has zeros in its transfer function. On the contrary, if $r = n$ the transfer function has no zeros; thus, a nonlinear system having relative degree $r = n$ in some sense analogue to a linear systems without zeros. We shall see in this section that this kind of analogy can be pushed much further.

Consider a nonlinear system with r strictly less than n and look at its normal form. Recall that, if x° is such that $f(x^\circ) = 0$ and $h(x^\circ) = 0$, then necessarily the set ξ of the last r new coordinates is 0 at x°. Note also that it is always possible to choose arbitrarily the value at x° of the first $n-r$ new coordinates, thus in particular being 0 at x°. Therefore, without loss of generality, one can assume that $\xi = 0$ and $z = 0$ at x°. Thus, if x° was an equilibrium for the system in the original coordinates, its corresponding point $(z, \xi) = (0, 0)$ is an equilibrium for the system in the new coordinates and from this we deduce that

$$\begin{aligned} q(0,0) &= 0 \\ f_0(0,0) &= 0 \ . \end{aligned}$$

Suppose now we want to analyze the following problem, called the *Problem of Zeroing the Output*. Find, if any, pairs consisting of an initial state x° and of an input function $u^\circ(\cdot)$, defined for all t in a neighborhood of $t = 0$, such that the corresponding output $y(t)$ of the system is identically zero for all t in a neighborhood of $t = 0$. Of course, we are interested in finding *all* such pairs (x°, u°) and not simply in the trivial pair $x^\circ = 0, u^\circ = 0$ (corresponding to the situation in which the system is initially at rest and no input is applied). We perform this analysis on the normal form of the system.

Recalling that in the normal form

$$y(t) = \xi_1(t) \ ,$$

we see that the constraint $y(t) = 0$ for all t implies

$$\xi_1(t) = \xi_2(t) = \ldots = \xi_r(t) = 0 \ ,$$

that is $\xi(t) = 0$ for all t.

Thus, we see that when the output of the system is identically zero its state is constrained to evolve in such a way that also $\xi(t)$ is identically zero. In addition, the input $u(t)$ must necessarily be the unique solution of the equation

$$0 = q(z(t), 0) + b(z(t), 0)u(t)$$

(recall that $b(z(t), 0) \neq 0$ if $z(t)$ is close to 0). As far as the variable $z(t)$ is concerned, it is clear that, being $\xi(t)$ identically zero, its behavior is governed by the differential equation

$$\dot{z}(t) = f_0(z(t), 0) \ . \tag{4}$$

From this analysis we deduce the following facts. If the output $y(t)$ has to be zero, then necessarily the initial state of the system must be set to a value such that $\xi(0) = 0$, whereas $z(0) = z^\circ$ can be chosen arbitrarily. According to the value of z°, the input must be set as

$$u(t) = -\frac{q(z(t), 0)}{b(z(t), 0)}$$

where $z(t)$ denotes the solution of the differential equation

$$\dot{z}(t) = f_0(z(t), 0) \qquad \text{with initial condition } z(0) = z^\circ.$$

Note also that for each set of initial data $\xi = 0$ and $z = z^\circ$ the input thus defined is the *unique* input capable to keep $y(t)$ identically zero for all times.

The dynamics of (4) correspond to the dynamics describing the "internal" behavior of the system when input and initial conditions have been chosen in such a way as to constrain the output to remain identically zero. These dynamics, which are rather important in many of our developments, are called the *zero dynamics* of the system.

Remark. In order to understand why we used the terminology "zero" dynamics in dealing with the dynamical system (4), it is convenient to examine how these dynamics are related to the zeros of the transfer function in a linear system. Let

$$T(s) = K \frac{b_0 + b_1 s + \cdots + b_{n-r-1}s^{n-r-1} + s^{n-r}}{a_0 + a_1 s + \cdots + a_{n-1}s^{n-1} + s^n}$$

denote the transfer function of a linear system (where r characterizes, as expected, the relative degree). Suppose the numerator and denominator polynomials are relatively prime and consider a minimal realization of $T(s)$

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned}$$

with

$$A = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-1} \end{pmatrix} \qquad B = \begin{pmatrix} 0 \\ 0 \\ \cdots \\ 0 \\ K \end{pmatrix}$$

$$C = \begin{pmatrix} b_0 & b_1 & \cdots & b_{n-r-1} & 1 & 0 & \cdots & 0 \end{pmatrix}.$$

Let us now calculate its normal form. For the last r new coordinates we know we have to take

$$\begin{aligned}
\xi_1 &= Cx = b_0x_1 + b_1x_2 + \cdots + b_{n-r-1}x_{n-r} + x_{n-r+1} \\
\xi_2 &= CAx = b_0x_2 + b_1x_3 + \cdots + b_{n-r-1}x_{n-r+1} + x_{n-r+2} \\
& \cdots \\
\xi_r &= CA^{r-1}x = b_0x_r + b_1x_{r+1} + \cdots + b_{n-r-1}x_{n-1} + x_n .
\end{aligned}$$

For the first $n-r$ new coordinates we have some freedom of choice (provided that the conditions stated in Proposition 1 are satisfied), but the simplest one is

$$\begin{aligned}
z_1 &= x_1 \\
z_2 &= x_2 \\
& \cdots \\
z_{n-r} &= x_{n-r} .
\end{aligned}$$

This is indeed an admissible choice because the corresponding coordinates transformation $\tilde{x} = \Phi(x)$ has a jacobian matrix with the following structure

$$\frac{\partial \Phi}{\partial x} = \begin{pmatrix} \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & \cdots & 1 \end{pmatrix} & \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & \cdots & 0 \end{pmatrix} \\ (\cdots) & \begin{pmatrix} 1 & 0 & \cdots & 0 \\ \star & 1 & \cdots & 0 \\ \cdot & \cdot & \cdots & \cdot \\ \star & \star & \cdots & 1 \end{pmatrix} \end{pmatrix}$$

and therefore nonsingular.

In the new coordinates we obtain equations in normal form, which, because of the linearity of the system, have the following structure

$$\begin{aligned}
\dot{z} &= Fz + G\xi \\
\dot{\xi}_1 &= \xi_2 \\
\dot{\xi}_2 &= \xi_3 \\
& \cdots \\
\dot{\xi}_{r-1} &= \xi_r \\
\dot{\xi}_r &= Hz + K\xi + bu
\end{aligned}$$

where H and K are row vectors and F and G matrices, of suitable dimensions. The zero dynamics of this system, according to our previous definition, are those of

$$\dot{z} = Fz .$$

The particular choice of the first $n-r$ new coordinates (i.e. of the elements of z) entails a particularly simple structure for the matrices F and G. As a matter of fact, is easily checked that

$$
\begin{aligned}
\frac{dz_1}{dt} &= \frac{dx_1}{dt} = x_2(t) = z_2(t) \\
&\cdots \\
\frac{dz_{n-r-1}}{dt} &= \frac{dx_{n-r-1}}{dt} = x_{n-r}(t) = z_{n-r}(t) \\
\frac{dz_{n-r}}{dt} &= \frac{dx_{n-r}}{dt} = x_{n-r+1}(t) = -b_0x_1(t) - \cdots - b_{n-r-1}x_{n-r}(t) + \xi_1(t) \\
&= -b_0z_1(t) - \cdots - b_{n-r-1}z_{n-r}(t) + \xi_1(t)
\end{aligned}
$$

from which we deduce that

$$
F = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & \cdots & 1 \\ -b_0 & -b_1 & -b_2 & \cdots & -b_{n-r-1} \end{pmatrix}, \qquad G = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \end{pmatrix}.
$$

From the particular form of this matrix, it is clear that the eigenvalues of F coincide with the zeros of the numerator polynomial of $T(s)$, i.e. with the zeros of the transfer function. Thus it is concluded that in a linear system the zero dynamics are linear dynamics with eigenvalues coinciding with the zeros of the transfer function of the system. $\triangleleft$

3 Global Normal Forms

We address, in this section, the problem of deriving the global version of the coordinates transformation and normal form introduced in section 1. Consider a single-input single-output system described by equations of the form

$$
\begin{aligned}
\dot{x} &= f(x) + g(x)u \\
y &= h(x)
\end{aligned} \tag{5}
$$

in which $f(x)$ and $g(x)$ are smooth vector fields, and $h(x)$ is a smooth function, defined on $\mathbb{R}^n$. Assume, as usual, that $f(0) = 0$ and $h(0) = 0$. This system is said to have *uniform* relative degree r if it has relative degree r at each $x^\circ \in \mathbb{R}^n$.

If system (5) has uniform relative degree r, the r differentials

$$
dh(x), dL_f h(x), \ldots, dL_f^{r-1}h(x)
$$

are linearly independent at each $x \in \mathbb{R}^n$ and therefore the set

$$
Z^* = \{x \in \mathbb{R}^n : h(x) = L_f h(x) = \ldots = L_f^{r-1}h(x) = 0\}
$$

(which is nonempty in view of the hypothesis that $f(0) = 0$ and $h(0) = 0$) is a smooth embedded submanifold of $\mathbb{R}^n$, of dimension $n - r$. In particular, each connected component of Z^* is a maximal integral manifold of the (nonsingular and involutive) distribution

$$
\Delta^* = (\mathrm{span}\{dh, dL_f h, \ldots, dL_f^{r-1}h\})^\perp .
$$

The submanifold Z^* is the point of departure for the construction a globally defined version of the coordinates transformation considered in section 1.

Proposition 2 *Suppose (5) has uniform relative degree r. Set*

$$\alpha(x) = \frac{-L_f^r h(x)}{L_g L_f^{r-1} h(x)} \qquad \beta(x) = \frac{1}{L_g L_f^{r-1} h(x)}$$

and consider the (globally defined) vector fields

$$\tilde{f}(x) = f(x) + g(x)\alpha(x), \qquad \tilde{g}(x) = g(x)\beta(x) \, .$$

Suppose the vector fields

$$\tau_i = (-1)^{i-1} ad_{\tilde{f}}^{i-1} \tilde{g}(x), \qquad 1 \le i \le r \tag{6}$$

are complete.

Then Z^* is connected. Moreover, the smooth mapping

$$\begin{array}{rcccl} \Phi & : & Z^* \times \mathbb{R}^r & \to & \mathbb{R}^n \\ & & (z, (\xi_1, \ldots, \xi_r)) & \mapsto & \Phi^{\tau_1}_{\xi_r} \circ \Phi^{\tau_2}_{\xi_{r-1}} \circ \cdots \circ \Phi^{\tau_r}_{\xi_1}(z) \, , \end{array} \tag{7}$$

in which – as usual – $\Phi^\tau_t(x)$ denotes the flow of the vector field τ, has a globally defined smooth inverse

$$(z, (\xi_1, \ldots, \xi_r)) = \Phi^{-1}(x) \tag{8}$$

in which

$$\begin{array}{rcl} z & = & \Phi^{\tau_r}_{-h(x)} \circ \cdots \circ \Phi^{\tau_2}_{-L_{\tilde{f}}^{r-2} h(x)} \circ \Phi^{\tau_1}_{-L_{\tilde{f}}^{r-1} h(x)}(x) \\ \xi_i & = & L_{\tilde{f}}^{i-1} h(x) \qquad 1 \le i \le r \, . \end{array}$$

The globally defined diffeomorphism (8) changes system (5) into a system described by equations of the form

$$\begin{array}{rcl} \dot{z} & = & f_0(z, \xi_1, \ldots, \xi_r) \\ \dot{\xi}_1 & = & \xi_2 \\ & & \cdots \\ \dot{\xi}_{r-1} & = & \xi_r \\ \dot{\xi}_r & = & q(z, \xi_1, \ldots, \xi_r) + b(z, \xi_1, \ldots, \xi_r) u \\ y & = & \xi_1 \end{array} \tag{9}$$

where

$$\begin{array}{rcl} q(z, \xi_1, \ldots, \xi_r) & = & L_f^r h \circ \Phi(z, (\xi_1, \ldots, \xi_r)) \\ b(z, \xi_1, \ldots, \xi_r) & = & L_g L_f^{r-1} h \circ \Phi(z, (\xi_1, \ldots, \xi_r)) \, . \end{array}$$

If, and only if, the vector fields (6) are such that [3]

$$[\tau_i, \tau_j] = 0 \qquad \text{for all } \ 1 \le i, j \le r \, ,$$

[3] Let f and g be vector fields on $\mathbb{R}^n$. Their *Lie bracket*, denoted $[f, g]$, is the vector field of $\mathbb{R}^n$ defined as

$$[f, g](x) = \frac{\partial g}{\partial x} f(x) - \frac{\partial f}{\partial x} g(x) \, .$$

then the globally defined diffeomorphism (8) changes system (5) into a system described by equations of the form

$$
\begin{array}{rcl}
\dot{z} & = & f_0(z, \xi_1) \\
\dot{\xi}_1 & = & \xi_2 \\
 & & \cdots \\
\dot{\xi}_{r-1} & = & \xi_r \\
\dot{\xi}_r & = & q(z, \xi_1, \ldots, \xi_r) + b(z, \xi_1, \ldots, \xi_r)u \\
y & = & \xi_1 \ .
\end{array}
\tag{10}
$$

Note also that, if $r < n$, the submanifold Z^* is the largest (with respect to inclusion) smooth submanifold of $h^{-1}(0)$ with the property that, at each $x \in Z^*$, there is $u^*(x)$ such that $f^*(x) = f(x) + g(x)u^*(x)$ is tangent to Z^*. Actually, for each $x \in Z^*$ there is only one $u^*(x)$ rendering this condition satisfied, namely,

$$u^*(x) = \frac{-L_f^r h(x)}{L_g L_f^{r-1} h(x)} \ .$$

The submanifold Z^* is an *invariant* manifold of the (autonomous) system

$$\dot{x} = f^*(x) \tag{11}$$

and the restriction of this system to Z^*, which can be identified with $(n-r)$-dimensional system

$$\dot{z} = f_0(z, 0, \ldots, 0)$$

of Z^*, is the system which characterizes the *zero dynamics* of (5).

4 Robust stabilization via partial-state feedback

An typical setting in which normal forms are useful is a systematic method for stabilization *in the large* of certain classes of nonlinear system, even in the presence of parameter uncertainties. We discuss first the case of systems having relative degree 1, i.e. we consider systems modelled by equations of the form

$$
\begin{array}{rcl}
\dot{z} & = & f_0(z, \xi) \\
\dot{\xi} & = & q(z, \xi) + b(z, \xi)u \\
y & = & \xi_1
\end{array}
\tag{12}
$$

in which $z \in \mathbb{R}^{n-1}$ and $\xi \in \mathbb{R}$. All functions are smooth functions of their arguments.

About this system we assume the following. The coefficient $b(z, \xi)$ satisfies

$$b(z, \xi) \geq b_0 > 0$$

for some b_0. Moreover, there is a positive definite and proper smooth real-valued function $V(z)$ that satisfies

$$\frac{\partial V}{\partial z} f_0(z, 0) \leq -\alpha(|z|) \qquad \forall z \in \mathbb{R}^{n-1}$$

in which $\alpha(|z|)$ is a class $\mathcal{K}_\infty$ function. This is equivalent to assume that the equilibrium $z=0$ of

$$\dot{z} = f_0(z,0)$$

is globally asymptotically stable.

Consider now for (12) a control law

$$u = -ky$$

in which $k>0$, which yields the closed loop system

$$\begin{array}{rcl} \dot{z} & = & f_0(z,\xi) \\ \dot{\xi} & = & q(z,\xi) - b(z,\xi)k\xi \,. \end{array} \tag{13}$$

For convenience, we set

$$x = \begin{pmatrix} z \\ \xi \end{pmatrix}$$

and rewrite the latter as

$$\dot{x} = F_k(x) \tag{14}$$

in which

$$F_k(x) = \begin{pmatrix} f_0(z,\xi) \\ q(z,\xi) - b(z,\xi)k\xi \end{pmatrix} .$$

Consider, for this system, the candidate Lyapunov function

$$W(x) = V(z) + \frac{1}{2}\xi^2$$

which is positive definite and proper. For any real number $a \geq 0$, let

$$\Omega_a = \{x \in \mathbb{R}^n : W(x) \leq a\}$$

denote the sublevel set consisting of all points of $\mathbb{R}^n$ at which the value of $W(x)$ is less than or equal to a and let

$$B_a = \{x \in \mathbb{R}^n : |x| \leq a\}$$

denotes the closed ball consisting of all points of $\mathbb{R}^n$ whose norm does not exceed a. Since $W(x)$ is proper, the set Ω_a is a compact set for any a. Pick any arbitrarily large number R, any arbitrarily small number r. Since $W(x)$ is positive definite and proper, there exist numbers numbers $0 < d < c$ such that

$$\Omega_d \subset B_r \subset B_R \subset \Omega_c \,.$$

Consider also the compact "annular" region

$$S_d^c = \{x \in \mathbb{R}^n : d \leq V(x) \leq c\} \,.$$

Our goal is to show that – if the gain coefficient k is large enough – the function

$$\dot{W}(x) := \frac{\partial W}{\partial x} F_k(x)$$

is negative at each point of S_d^c. Observe, in this respect, that

$$\dot{W}(x) = \frac{\partial V}{\partial x} f_0(z,\xi) + \xi q(z,\xi) - b(z,\xi)k\xi^2 \,.$$

To this end, we proceed as follows. Consider the compact set

$$Z = S_d^c \bigcap \{x \in \mathbb{R}^n : \xi = 0\} \,.$$

At each point of Z

$$\dot{W}(x) = \frac{\partial V}{\partial x} f_0(z,0) \le -\alpha(|z|)$$

Since $z \neq 0$ at each point of Z, there is a number $a > 0$ such that

$$\dot{W}(x) \le -a \qquad \forall x \in Z \,.$$

Hence, by continuity, there is an open set Z_ε containing Z such

$$\dot{W}(x) \le -a/2 \qquad \forall x \in Z_\varepsilon \,. \tag{15}$$

Consider now the set

$$\tilde{S} = \{x \in S_d^c : x \notin Z_\varepsilon\} \,.$$

which is a compact set, let

$$M = \max_{x \in \tilde{S}} \{\frac{\partial V}{\partial x} f_0(z,\xi) + \xi q(z,\xi)\}$$

$$m = \min_{x \in \tilde{S}} \{b(z,\xi)\xi^2\}$$

and observe that $m > 0$ because $b(z,\xi) \ge b_0 > 0$ and ξ cannot vanish at any point of $\tilde{S}$. Thus, since $k > 0$, we obtain

$$\dot{W}(x) \le M - km \qquad \forall x \in \tilde{S} \,.$$

Let k_1 be such that $M - k_1 m = -a/2$. Then, if $k \ge k_1$,

$$\dot{W}(x) \le -a/2 \qquad \forall x \in \tilde{S} \,. \tag{16}$$

This, together with (15) shows that

$$k \ge k_1 \qquad \Rightarrow \qquad \dot{W}(x) \le -a/2 \qquad \forall x \in S_d^c \,,$$

as requested. This being the case, we see that for any trajectory with initial condition in S_d^c, so long as $z(t) \in S_d^c$ we have

$$W(x(t)) \le W(x(0)) - (a/2)t \,.$$

As a consequence, the trajectory in finite time enters the set Ω_d, and remains there (because $\dot{W}(x)$ is negative on the boundary of this set). We summarize the result as follows.

Proposition 3 *Consider system (12). Suppose that $b(z,\xi) \geq b_0 > 0$ for some b_0 and suppose that the equilibrium $z = 0$ of $\dot{z} = f_0(z,0)$ is globally asymptotically stable. Then, for every choice of two positive numbers R, r, with $R \gg r$, there is a number k_1 such that, for all $k \geq k_1$ there is a finite time T such that all trajectories of (13) with initial condition $|x(0)| \leq R$ remain bounded and satisfy $|x(t)| \leq r$ for all $t \geq T$.*

This property is commonly known as property of *semi-global practical* stabilizability, of the equilibrium $(z,\xi) = (0,0)$ of (12).

To obtain asymptotic stability, an extra assumption is required. The simplest case in which this can be achieved is when the equilibrium $z = 0$ of $\dot{z} = f_0(z,0)$ is not just asymptotically, but also *locally exponentially*, stable, i.e. when the eigenvalues of the matrix

$$F_0 := \frac{\partial f_0}{\partial z}(0,0) \tag{17}$$

have negative real part. Linear arguments can be invoked to prove the existence of a number k_2 such that, if $k \geq k_2$, the equilibrium $x = 0$ of (12) is locally asymptotically stable. It is also possible to show that there is a number r' such that, for all $k \geq k_2$, the closed ball $B_{r'}$ is always contained in the domain of attraction of $x = 0$. To check that this is the case, let P be the positive definite solution of

$$PF_0 + F_0^{\mathrm{T}} P = -2I\,,$$

which exists because all eigenvalues of F_0 have negative real part and consider, for the system the candidate quadratic Lyapunov function

$$U(x) = z^{\mathrm{T}} \tilde{P} z + \frac{1}{2}\xi^2$$

yielding

$$\dot{U}(x) = 2z^{\mathrm{T}} P f_0(z,\xi) + \xi q(z,\xi) - b(z,\xi) k \xi^2\,.$$

Expand $f_0(z,\xi)$ as

$$f_0(z,\xi) = F_0 z + g(z) + [f_0(z,\xi) - f_0(z,0)]$$

in which

$$\lim_{z \to 0} \frac{|g(z)|}{|z|} = 0\,.$$

Then, there is a number δ such that,

$$|z| \leq \delta \qquad \Rightarrow \qquad |g(z)| \leq \frac{1}{2|P|}|z|$$

and this yields

$$2z^{\mathrm{T}} P[F_0 z + g(z)] = -2|z|^2 + 2z^{\mathrm{T}} P g(z) \leq -|z|^2 \qquad \forall z \in B_\delta\,.$$

Since the function $[f_0(z,\xi) - f_0(z,0)]$ is a continuously differentiable function that vanish at $\xi = 0$, there is a number M_1 such that

$$|2z^{\mathrm{T}} P[f_0(z,\xi) - f_0(z,0)]| \leq M_1 |z||\xi| \qquad \text{for all } (z,\xi) \text{ such that } z \in B_\delta \text{ and } |\xi| \leq \delta\,.$$

Likewise, since $q(z,\xi)$ is a continuously differentiable function that vanish at $(z,\xi)=(0,0)$, there are numbers N_1 and N_2 such that

$$|\xi q(z,\xi)| \le N_1|z||\xi| + N_2|\xi|^2 \qquad \text{for all } (z,\xi) \text{ such that } z\in B_\delta \text{ and } |\xi|\le\delta\,.$$

Finally, since $b(z,\xi)$ is positive and nowhere zero, there is a number b_0 such that (recall that $k>0$)

$$-kb(z,\xi) \le -kb_0 \qquad \text{for all } (z,\xi) \text{ such that } z\in B_\delta \text{ and } |\xi|\le\delta\,.$$

Putting all these inequalities together, one finds that, for all for all (z,ξ) such that $z\in B_\delta$ and $|\xi|\le\delta$

$$\dot U(x) \le -|z|^2 + (M_1+N_1)|z||\xi| - (kb_0 - N_2)|\xi|^2\,.$$

It is easy to check that if k is such that

$$(2kb_0 - 2N_2 - 1) \ge (M_1+N_1)^2\,,$$

$$\dot U(x) \le -\frac{1}{2}|x|^2\,.$$

This shows that there is a number k_2 such that, if $k\ge k_2$, the function $\dot U(x)$ is negative definite for all x satisfying $z\in B_\delta$ and $|\xi|\le\delta$. Pick now any (nontrivial) sublevel set $\tilde\Omega_c$ of $U(x)$ entirely contained in the set of all x satisfying $z\in B_\delta$ and $|\xi|\le\delta$ and let r' be such that $B_{r'}\subset\tilde\Omega_c$. Then, the argument above shows that, for all $k\ge k_2$, the equilibrium $x=0$ is asymptotically stable with a domain of attraction that contains $B_{r'}$, which is a set independent of the choice of k.

Pick now $r\le r'$, pick any $R>0$ and use the result of the Proposition above. There is a number k_1 such that, if $k\ge k_1$ all trajectories with initial condition in B_R in finite time enter the region B_r, and hence enter the region of attraction of $x=0$. It can be concluded, setting $k^*=\max\{k_1,k_2\}$, that for all $k\ge k^*$ the equilibrium $x=0$ of (13) is asymptotically stable, with a domain of attraction that contains B_R.

The case of higher relative degree

$$\begin{aligned}
\dot z &= f_0(z,\xi_1)\\
\dot\xi_1 &= \xi_2\\
&\cdots\\
\dot\xi_{r-1} &= \xi_r\\
\dot\xi_r &= q(z,\xi_1,\ldots,\xi_r) + b(z,\xi_1,\ldots,\xi_r)u
\end{aligned}$$

can be reduced to the case already studied. To this end, it suffices to consider the "dummy" output variable

$$\theta = \xi_r + g^{r-1}a_0\xi_1 + g^{r-2}a_1\xi_2 + \cdots ga_{r-2}\xi_{r-1}$$

with $a_0,\ldots,a_{r-2}$ such that the polynomial

$$d(\lambda) = \lambda^{r-1} + a_{r-2}\lambda^{r-1} + \cdots + a_1\lambda + a_0$$

is Hurwitz. The system thus obtained has relative degree 1 and a "normal form"

$$\begin{aligned}
\dot{\tilde z} &= \tilde f_0(\tilde z,\theta)\\
\dot\theta &= \tilde q(\tilde z,\theta) + \tilde b(\tilde z,\theta)u
\end{aligned}$$

in which

$$\tilde{z} = \mathrm{col}(z, \xi_1, \ldots, \xi_{r-1})$$

$$\tilde{f}_0(\tilde{z}, \theta) = \begin{pmatrix} f_0(z, \xi_1) \\ \xi_2 \\ \cdots \\ \xi_{r-2} \\ -[g^{r-1}a_0\xi_1 + g^{r-2}a_1\xi_2 + \cdots g a_{r-2}\xi_{r-1}] + \theta \end{pmatrix}$$

and $\tilde{q}(\tilde{z}, \theta), \tilde{b}(\tilde{z}, \theta)$ suitable functions.

The zero dynamics of this system are characterized by

$$\begin{aligned} \dot{z} &= f_0(z, \xi_1) \\ \dot{\xi}_1 &= \xi_2 \\ &\cdots \\ \dot{\xi}_{r-1} &= -[g^{r-1}a_0\xi_1 + g^{r-2}a_1\xi_2 + \cdots g a_{r-2}\xi_{r-1}] \end{aligned}$$

Changing variables as

$$\zeta_i = \frac{1}{g^{i-1}}\xi_i, \qquad i = 1, 2, \ldots, i-1\,,$$

yields a system of the form

$$\begin{aligned} \dot{z} &= f_0(z, \zeta_1) \\ \dot{\zeta} &= gA\zeta \end{aligned}$$

in which

$$A = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & \cdot & 1 \\ -a_0 & -a_1 & -a_2 & \cdot & -a_{r-2} \end{pmatrix}$$

is a Hurwitz matrix.

Arguments similar to those used above can be invoked to show that – if the equilibrium $z = 0$ of $\dot{z} = f_0(z, 0)$ is globally asymptotically and locally exponentially stable – the equilibrium $\tilde{z} = 0$ of

$$\dot{\tilde{z}} = \tilde{f}_0(\tilde{z}, 0) \tag{18}$$

is locally exponentially stable, with a domain of attraction that can be made arbitrarily large by increasing the design parameter g. In fact, letting Q to be the positive definite solution of

$$QA + A^{\mathrm{T}}Q = -I$$

and considering the candidate Lyapunov function

$$W(\tilde{z}) = V(z) + \zeta^{\mathrm{T}}Q\zeta$$

one proves, in exactly the same way, that for every choice of two positive numbers R, r, with $R \gg r$, there is a number g_1 such that, for all $g \geq g_1$ there is a finite time T such that all trajectories of (18) with initial condition $|\tilde{z}(0)| \leq R$ remain bounded and satisfy $|\tilde{(}t)| \leq r$ for all $t \geq T$.

Similarly, using instead the Lyapunov function

$$U(\tilde{z}) = z^{\mathrm{T}} P z + \zeta^{\mathrm{T}} Q \zeta \,,$$

one proves, in exactly the same way, that there exists a number g_2 and a number r' such that, for all $g \geq g_2$, the equilibrium $\tilde{z} = 0$ of (18) is locally asymptotically stable, with a domain of attraction that contains $B_{r'}$.

In this way, the problem is reduced to the problem considered at the beginning, namely the problem of controlling a system having relative degree 1. Thus a feedback law of the form

$$u = -k\theta \,,$$

with large k, yields asymptotic stability of the equilibrium $(\tilde{z}, \theta) = (0, 0)$, with a domain of attraction which include a ball of arbitrarily fixed radius R.

Remark. Note that, in the construction thus described, it is *essential* that $f_0(z\xi_1)$ only depends on ξ_1 and not on the other components $\xi_2, \ldots, \xi_r$. $\triangleleft$

In the actual coordinates, the feedback law thus found is a linear law in the ξ_i's, namely

$$u = -k[\xi_r + g^{r-1} a_0 \xi_1 + g^{r-2} a_1 \xi_2 + \cdots g a_{r-2} \xi_{r-1} \,.$$

This sometimes called a *partial–state* feedback.

Finally we observe that the result discussed so far can be extended to the case in which the functions characterizing the system also depend, in a smooth fashion, on a vector μ of uncertain parameters, so long as the latter ranges on a compact set. The proof proceeds essentially in the same way.