PhD School SIDRA 2024
Introduction to the analysis and control of nonlinear systems
Homeworks

Problem 1. Let

$$H(x,y) = \frac{x^4}{4} + \frac{y^4}{4} - \frac{x^2}{2} - \frac{y^2}{2}$$

and consider the system

$$\begin{cases} \dot{x} = \dfrac{\partial H}{\partial y} \\ \\ \dot{y} = -\dfrac{\partial H}{\partial x} \end{cases}$$

1. Find the equilibrium points of the system
2. For each equilibrium point, compute the linear part of the system and discuss the stability properties
3. Write a MATLAB program in order to simulate the phase portrait of the system.
4. Repeat the simulation for the "perturbed" system

$$\begin{cases} \dot{x} = \dfrac{\partial H}{\partial y} - \dfrac{1}{10}x \\ \\ \dot{y} = -\dfrac{\partial H}{\partial x} - \dfrac{1}{10}y \end{cases}$$

and compare with the "unperturbed" system.

Problem 2. Consider the system

$$\dot{x} = -\frac{x^3}{1+x^2} + \tan^{-1}(u)$$

Show that the system is input-to-state stable and find a "gain function".

Problem 3. Consider the system

$$\begin{aligned} \dot{x}_1 &= -x_1 + \frac{x_1}{1+x_1^2}x_2 \\ \dot{x}_2 &= ax_1^2 - x_2 \end{aligned}$$

Using the small gain theorem for input-to-state stable systems, find values of a for which the equilibrium $(x_1, x_2) = (0, 0)$ is globally asymptotically stable.

Problem 4. Consider the system

$$\begin{aligned} \dot{x}_1 &= x_2 + 2x_1^2 \\ \dot{x}_2 &= x_3 + u \\ \dot{x}_3 &= x_1 - x_3 \\ y &= x_1 \end{aligned}$$

Show that the system has relative degree 2 at each x. Determine all pairs (initial state, input) yielding an identically zero output. Show that, for any of such pairs, the corresponding state trajectory $x(t)$ decays to zero as $t \to \infty$, i.e. that the zero dynamics are globally asymptotically stable.

Problem 5. Consider the system

$$\begin{aligned} \dot{x}_1 &= -x_1 + x_2 u \\ \dot{x}_2 &= x_2 + x_3 \\ \dot{x}_3 &= x_1 + (x_2 + x_3^3)^2 + u \\ y &= x_2\,. \end{aligned}$$

Check that the system is transformable into a *uniform observability* canonical form and find its uniform observability canonical form (hint: check the diffeomorphism $\varphi_1(x) = h(x)$, $\varphi_i(x,u) = [\frac{\partial \varphi_{i-1}}{\partial x}] f(x,u)$, $i = 2, 3$, with $u = 0$).