Introduction to the Analysis and Control of Nonlinear Systems

Scuola di dottorato SIDRA 2024

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Program

Monday July 8th 2024		Tuesday July 9th 2024		Wednesday July 10th 2024	
9:00 – 10:30	Introduction to the course and to nonlinear phenomena	9:00 – 10:30	Relative degree, feedback linearization and zero dynamics	9:00 – 10:30	Steady state for nonlinear systems and output regulation
11:00 – 12:30	Stability notions for nonlinear systems. Lyapunov criteria	11:00 – 12:30	Global, semiglobal and practical stabilizability by state and output feedback	11:00 – 12:30	Principles of internal model-based control
14:30 – 16:00	Nonlinear systems with input – Input-to-State Stability - small gain	14:30 – 16:00	Global, semiglobal and practical stabilizability by state and output feedback		
16:30 – 18:00	Nonlinear systems with input – Input-to-State Stability - small gain	16:30 – 18:00	Nonlinear observers and nonlinear separation principle		

State-space representation

We shall adopt **state-space** representations to describe the dynamic system by means of **O**rdinary **D**ifferential **E**quations (ODEs) for continuous-time systems

Continuous-time systems ($t \in \mathbb{R}$)

$$\begin{array}{rcll} \dot{x}(t) & = & f(x(t), u(t), t) & x \in \mathbb{R}^n, \ u \in \mathbb{R}^m \\ y(t) & = & h(x(t), u(t), t) & y \in \mathbb{R}^p \end{array}$$

$x(t_0) \in \mathbb{R}$ Typically assigned

- **Stationary** systems if the vector fields $f(\cdot)$ and $h(\cdot)$ do not depend on t ($t_0 = 0$ without loss of generality)
- **Linear** systems if the vector fields $f(\cdot)$ and $h(\cdot)$ are linear
- **Autonomous** systems if the vector fields $f(\cdot)$ and $h(\cdot)$ do not depend on independent variables $(u(t), t)$

Only ODEs?

ODEs could be not the appropriate math description tool to model certain physical systems:

Discrete-time systems ($t \in \mathbb{Z}$)

$$\begin{aligned} x(t+1) &= f(x(t), u(t), t) \qquad x \in \mathbb{R}^n,\ u \in \mathbb{R}^m \\ y(t) &= h(x(t), u(t), t) \qquad y \in \mathbb{R}^p \end{aligned}$$

(difference equations)

Partial Differential Equations (PDEs)

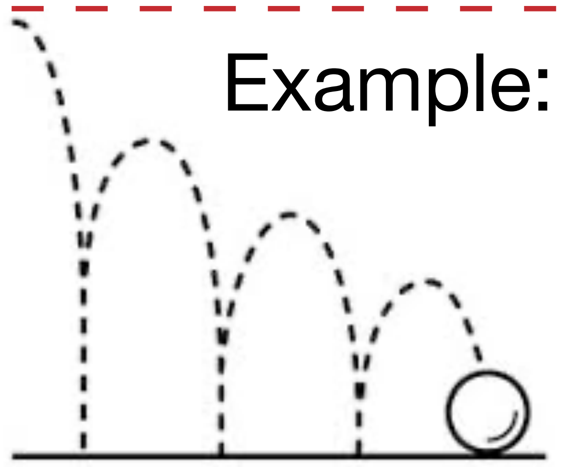
$$\frac{\partial T(t,x)}{\partial x} = c\frac{\partial^2 T(t,x)}{\partial x^2}$$

Example: heat transport dynamics

Hybrid systems (differential and difference equations)

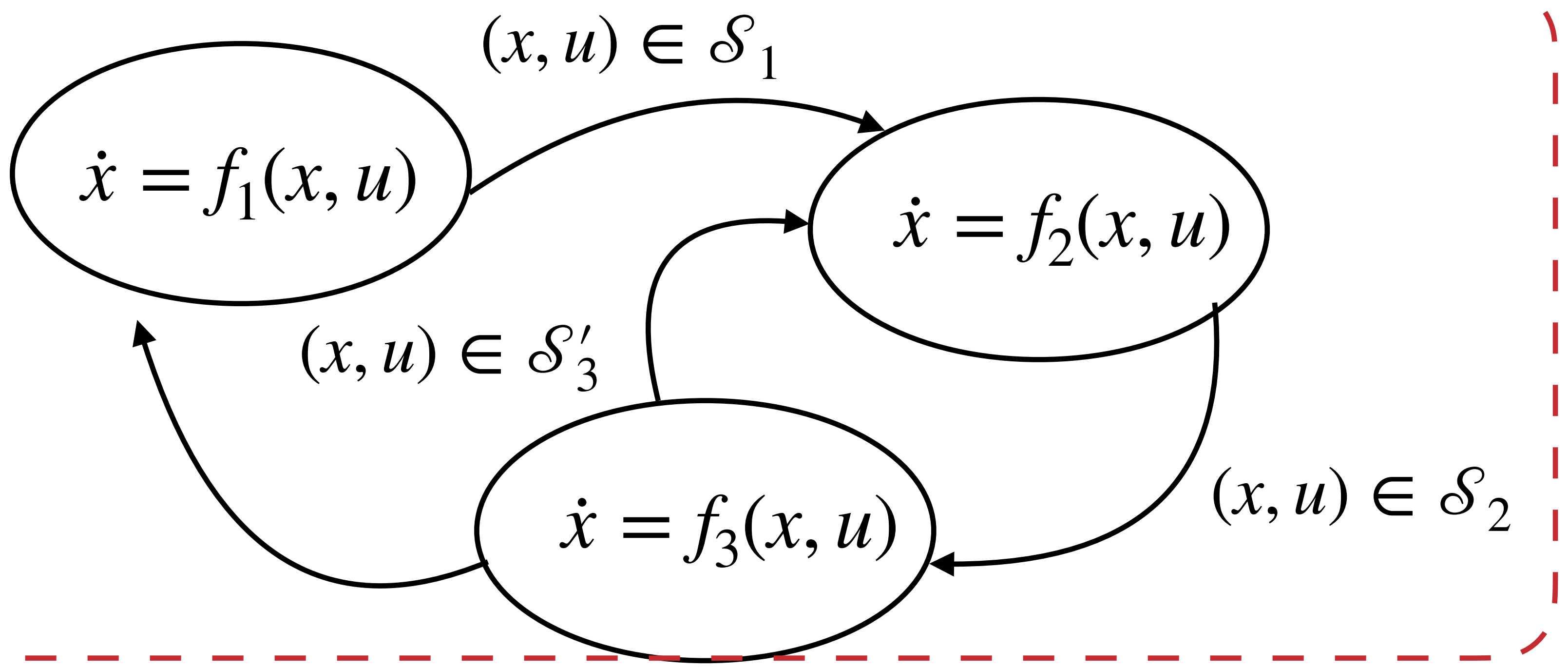
$$\begin{cases} \dot{x} = F(x,u) & (x,u) \in \mathcal{F} \\ x^+ = J(x,u) & (x,u) \in \mathcal{J} \end{cases}$$

Example: bouncing ball



Hybrid Automata

Example: drone airborne and in physical contact



Differential-Algebraic Equations (Descriptor systems)

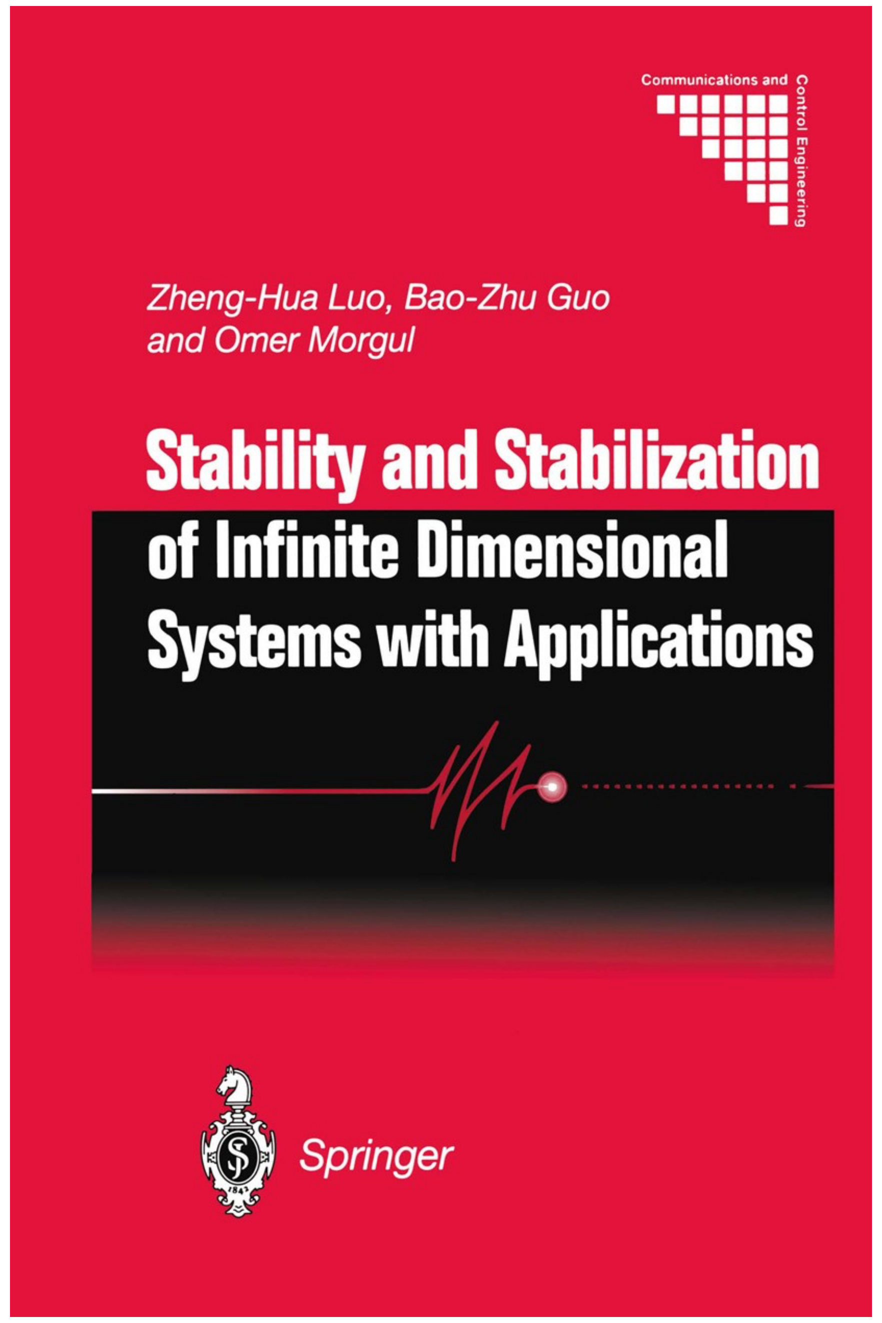
Example: constrained mechanical systems

$$\begin{cases} \dot{x} = F(x,u) \\ 0 = C(x,u) \end{cases}$$

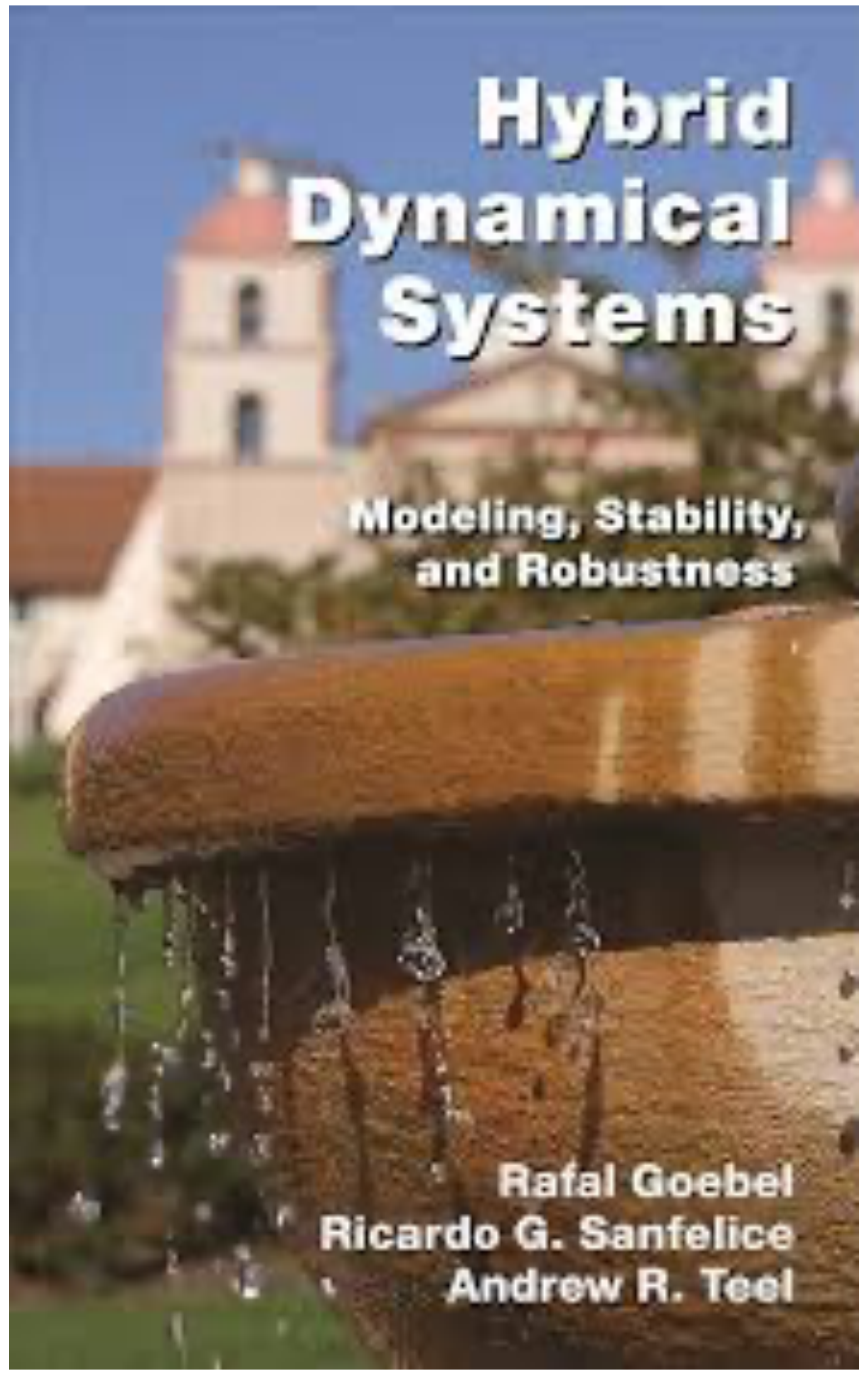
Non treated in the course!

References (topics not treated in the course)

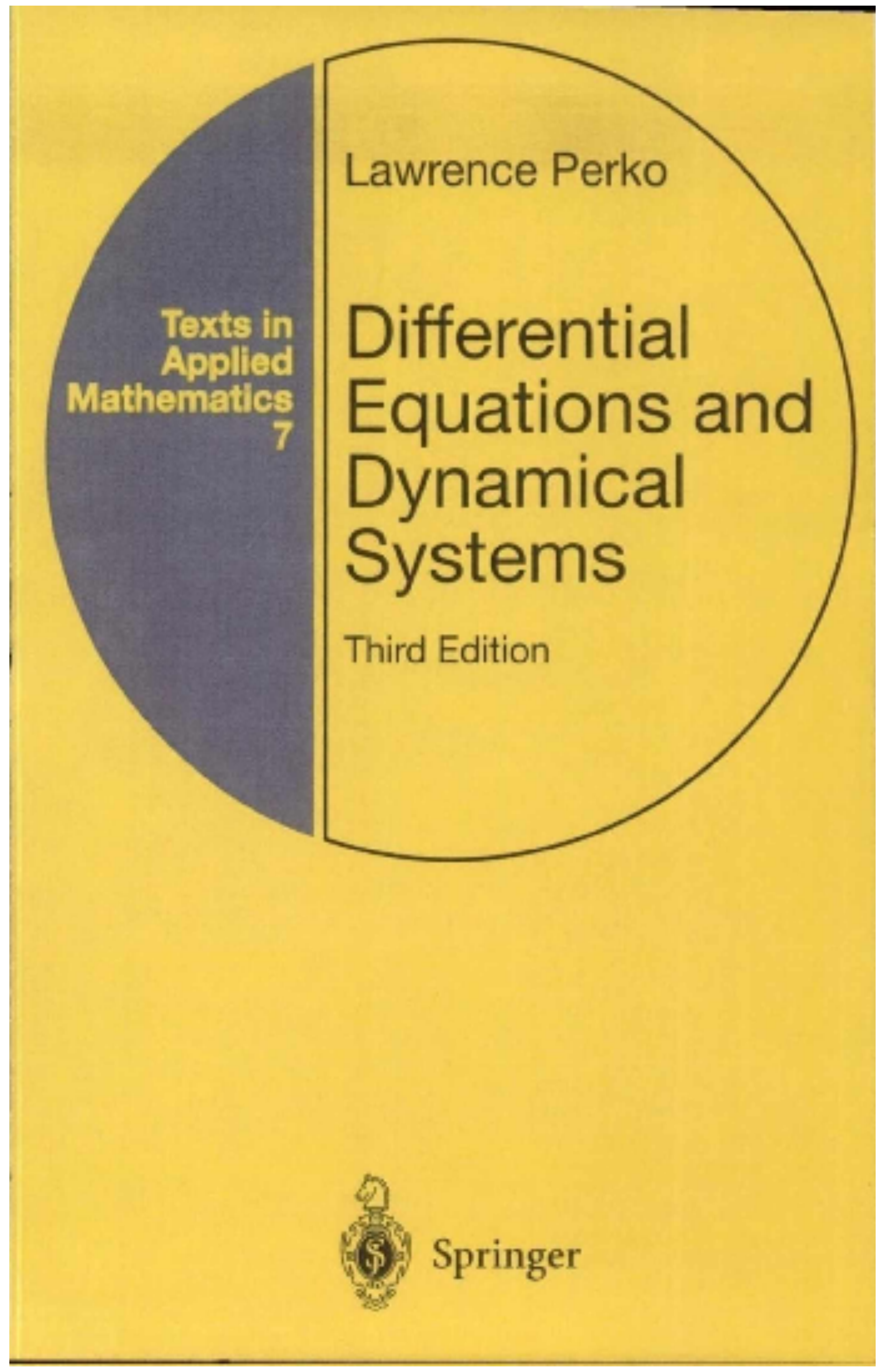
Partial Differential Equations (PDEs)



Hybrid systems (differential and difference equations)



Time-varying systems

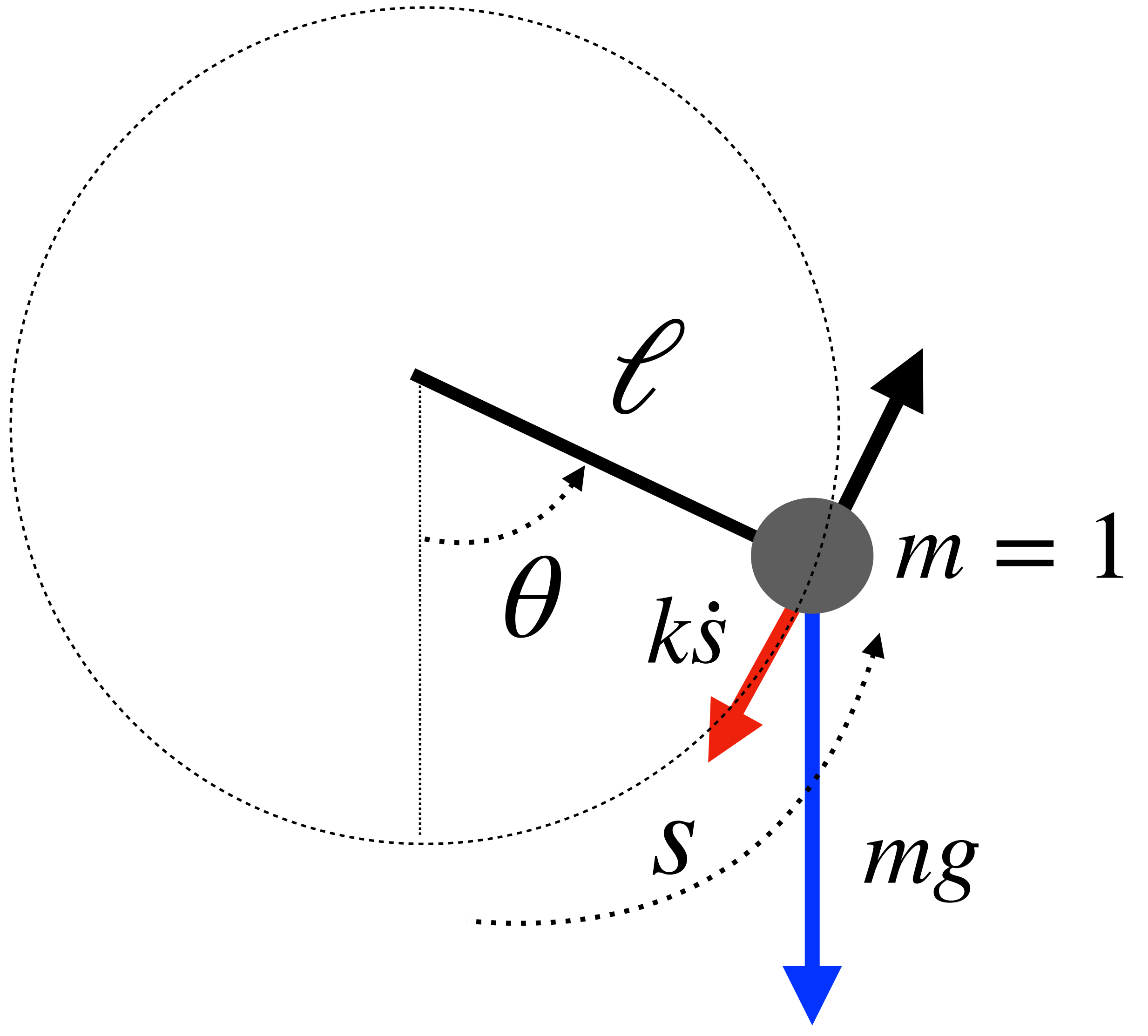


Where the system “lives”?

In the course we shall treat only systems whose state evolves on Euclidean spaces (distance -> Euclidean norm)

However, often the physics of the system implies that the state $x(t)$ does not evolve on an Euclidean space but rather on a “**manifold**” (topological space that locally resembles a an Euclidean space)

Example: Pendulum



Tangential force (gravity and friction)

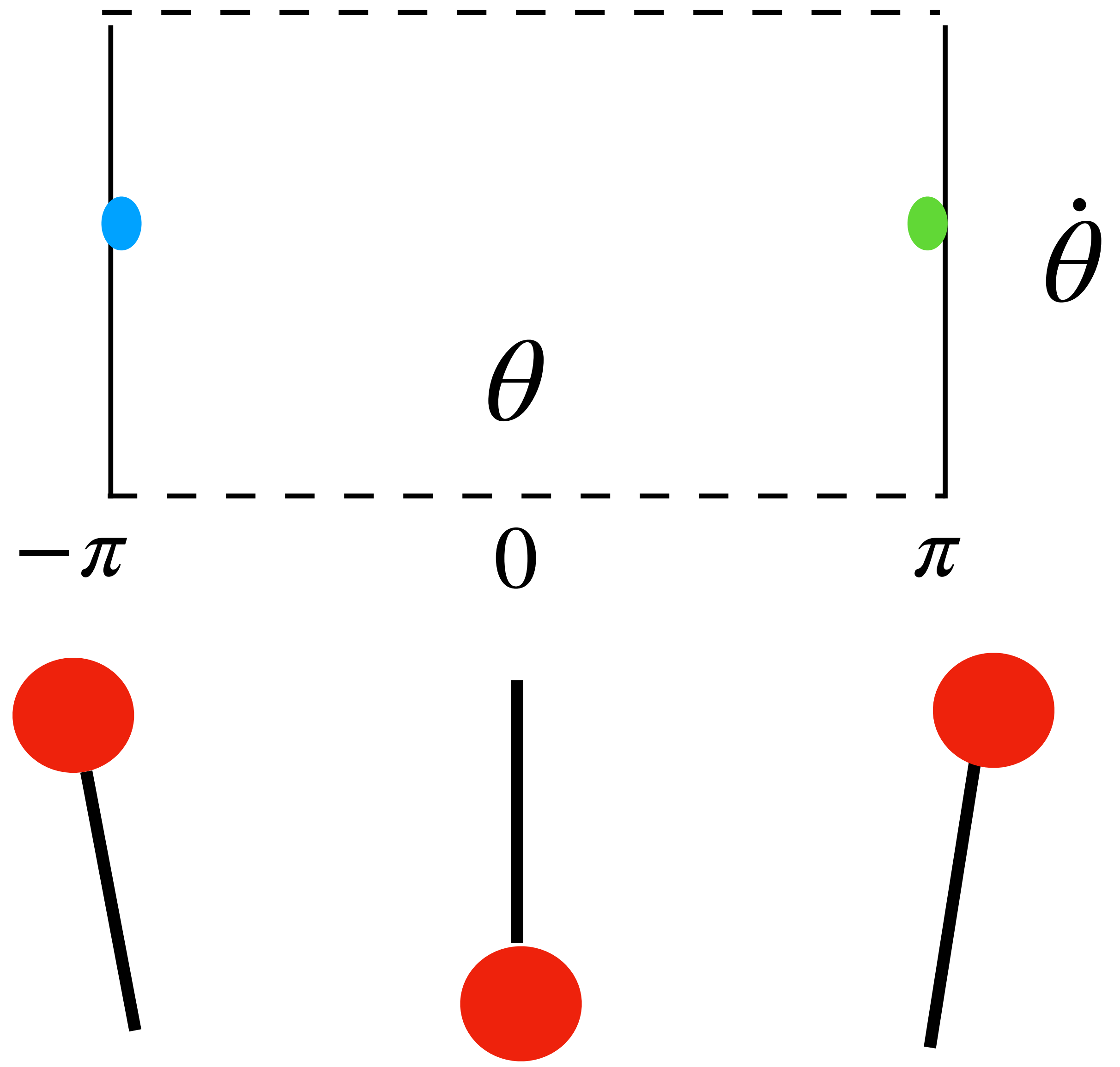
$$F = ma$$

Tangential acceleration ($\ell\ddot{\theta}$)

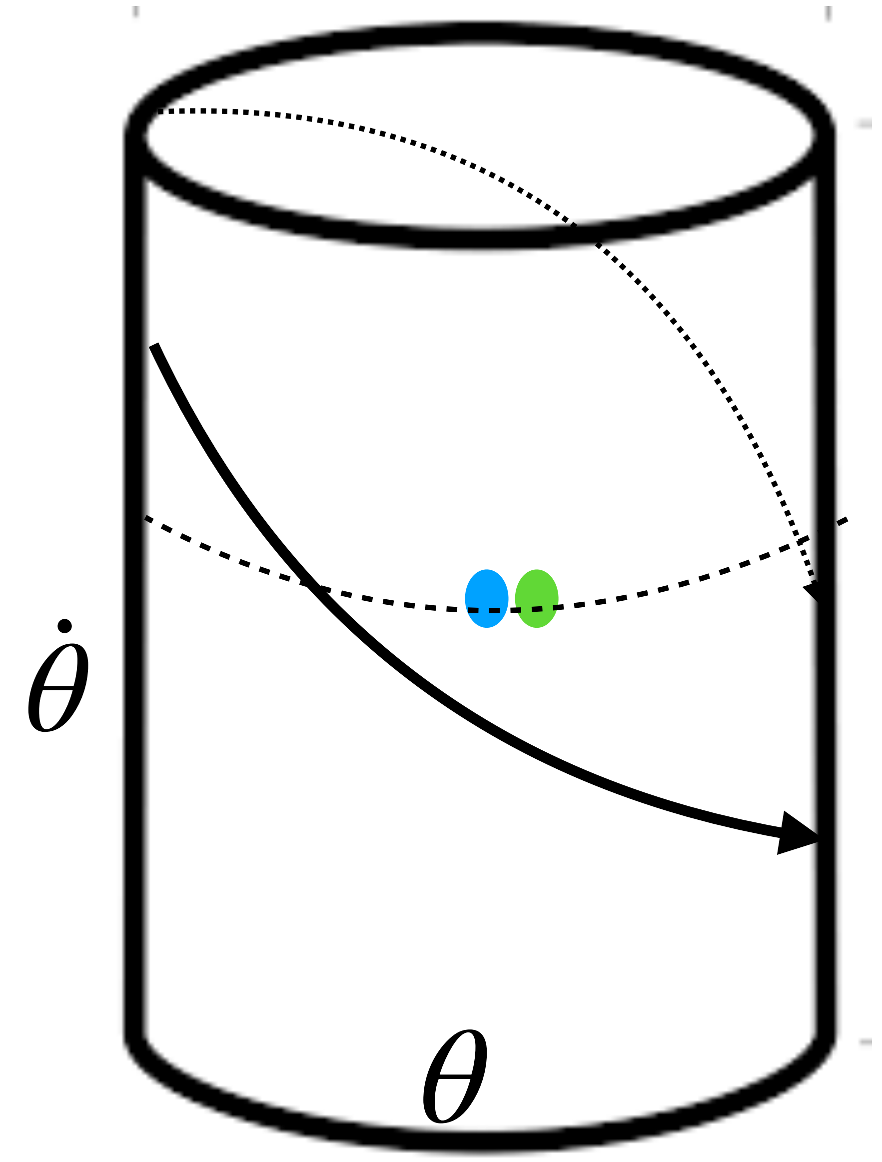
$$x_1 := \theta,\ x_2 := \dot{\theta}$$

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -\frac{g}{\ell}\sin x_1 - k\ell x_2$$

θ is not evolving on a euclidean space (distance between $\theta_1 = \pi - \epsilon$ and $\theta_2 = -\pi + \epsilon$ is not $2(\pi - \epsilon)$!)



The “supporting space” of the pendulum is the “cylinder” manifold $S^1 \times \mathbb{R}$



Where the system "lives"?

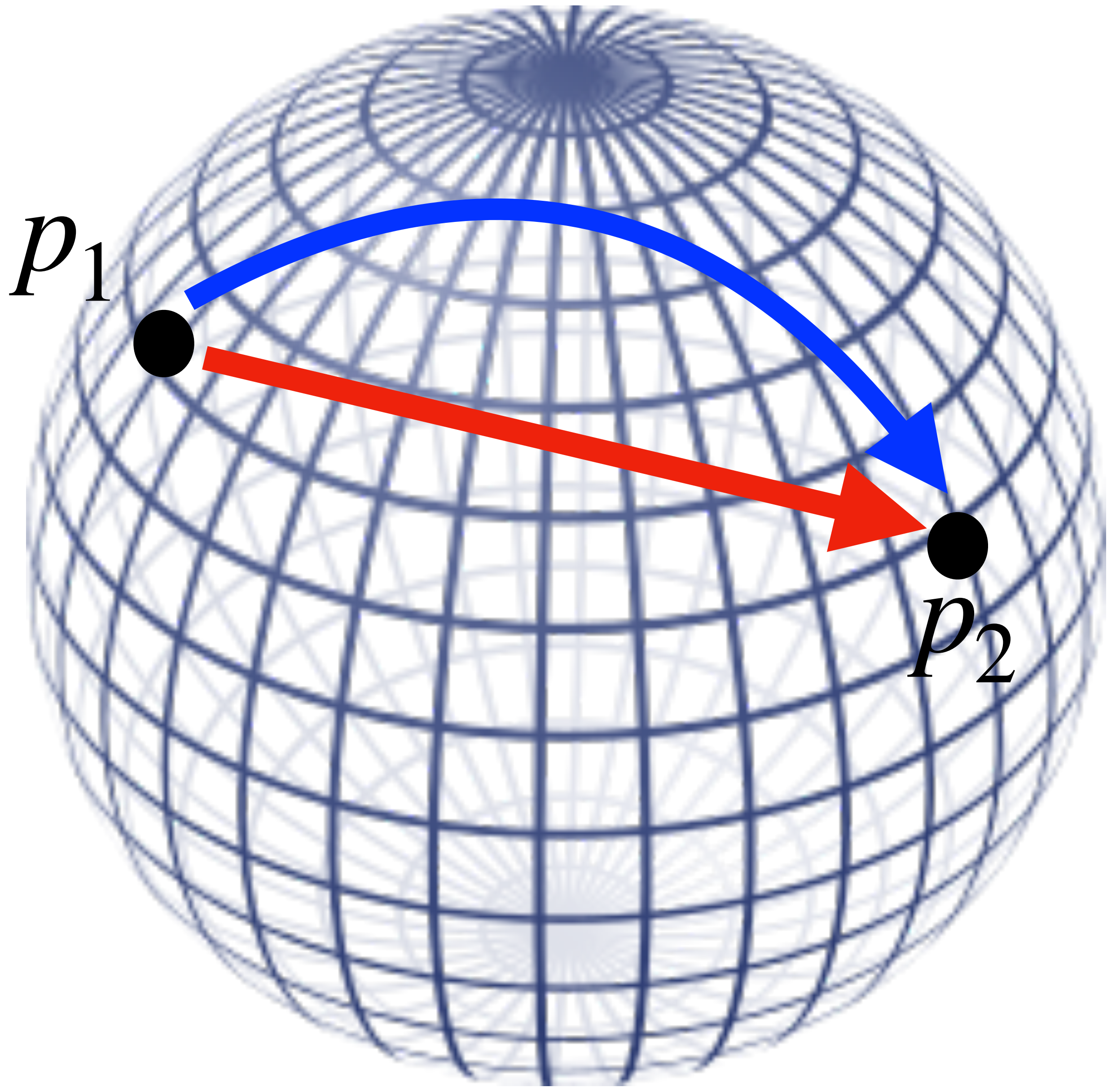
Example: aircraft

Tangent Euclidean space

Sphere manifold

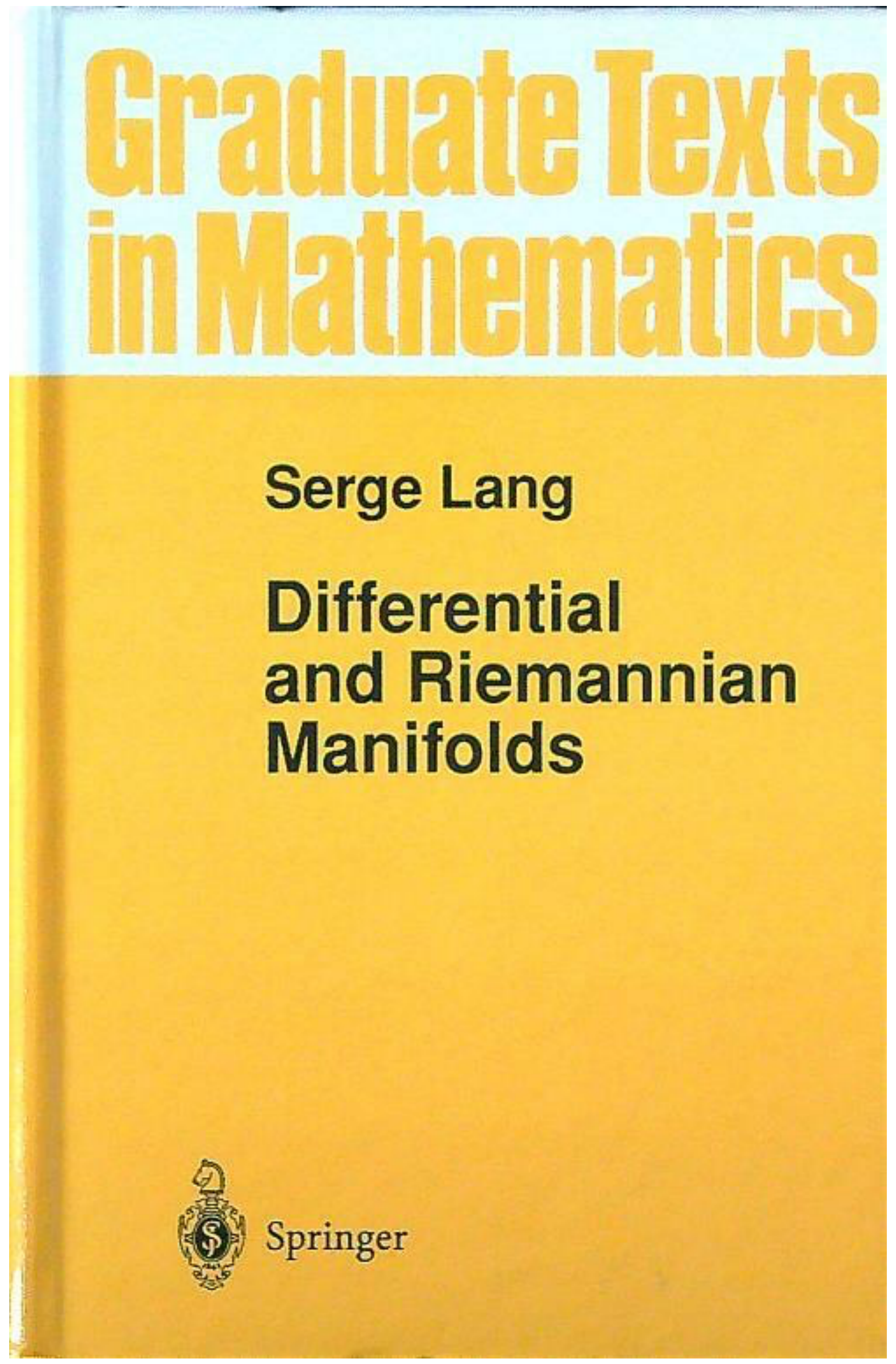
Euclidean distance (unfeasible trajectory)

Sphere distance (feasible trajectory)



S^2 (2-D sphere)

In the course we shall treat only systems whose state evolves on Euclidean spaces (possibly local study)



Trajectories

Lagrange Formula (linear systems)

The state and output trajectory $(x(t), y(t))$ depends on the initial state $x(t_0)$ and on the specific input applied to the system in the time interval $[t_0, t]$. For linear systems (A, B, C, D) the trajectory can be given an explicit form in which the effect of the initial state and of the input are decoupled.

Continuous-time ($t \in \mathbb{R}$)

$$x(t) = \underbrace{e^{A(t-t_0)}\, x(t_0)}_{\text{Free state evolution}} + \underbrace{\int_{t_0}^{t} e^{A(t-s)}\, B\, u(s)\, ds}_{\text{Forced state evolution}}$$

$$y(t) = \underbrace{Ce^{A(t-t_0)}\, x(t_0)}_{\text{Free output evolution}} + \underbrace{\int_{t_0}^{t} Ce^{A(t-s)}\, B\, u(s)\, ds + Du(t)}_{\text{Forced output evolution}}$$

$$\boxed{e^M := I + M + \frac{1}{2!}M^2 + \ldots + \frac{1}{k!}M^k \ldots \; k \to \infty}$$

Lagrange Formula (linear systems)

Nice properties of linear (stationary) systems immediately visible from the Lagrange formula:

- Effects of (sum of) initial state(s) and (sum of) input(s) add up (**Additivity**)
- "Large"/"small" experiments differ only by scaling factors (**Homogeneity**)

Principle of superimposition

- The effects of a stimulus (initial state/input) applied at different times are the same, simply shifted in time (**shift-invariance**)
- Properties of the A affects the free state evolution, properties of (A, B) affects the state forced evolution
- C plays a role in the output free and forced evolution

Principle of superimposition: property of linear systems (not necessarily stationary)

Shift-invariance: property of stationary systems

What about nonlinear systems?

Unfortunately, Lagrange-type formulas do not exist, in general, for nonlinear systems. The effect of the initial state and of the input are not "decoupled" but "mix-up".

Let's consider **autonomous systems** of the form $\dot{x} = f(x)$ (continuous-time without too much loss of generality). Formally, we say that $x(t)$ is a solution for this ODE (trajectory for the system) with initial state x_0 at $t_0 = 0$ if

$x(0) = x_0$ and $\dot{x}(t) = f(x(t))$ for all $t > 0$

For general nonlinear systems trajectories could be "pathological":

- **Finite Escape time**. Example: $\dot{x} = x^2 \quad x(0) = 1$

$$x(t) = \frac{1}{1-t}$$

The trajectory does not exist for $t \geq 1$

- **Uniqueness**. Example: $\dot{x} = 2\sqrt{x} \quad x(0) = 0$

$$x(t) = \begin{cases} 0 & \text{if} \quad 0 \leq t \leq a \\ (t-a)^2 & \text{if} \quad t > a \end{cases}$$

$a \geq 0$ any

The trajectory is not unique

What about nonlinear systems?

Existence (for all t) and uniqueness are guaranteed if certain regularity properties on the vector field $f(\cdot)$ are assumed.

Property: The function $f(\cdot)$ is said to be **Locally Lipschitz at $\bar{x}$** if there exist an $r_{\bar{x}} > 0$ and $c_{\bar{x}} > 0$ such that $\|f(x) - f(y)\| \leq c_{\bar{x}}\|x - y\|$ for all $(x, y) \in \mathscr{B}_{r_{\bar{x}}}(\bar{x}) := \{ x \in \mathbb{R}^n \ : \ \|x - \bar{x}\| < r_{\bar{x}} \}$.

Property: The function $f(\cdot)$ is said to be **Locally Lipschitz on $\mathscr{D}$** (open set of $\mathbb{R}^n$) if it is locally Lipschitz at each $\bar{x} \in \mathscr{D}$.

Property: The function $f(\cdot)$ is said to be **Lipschitz on $\mathscr{D}$** if it is locally Lipschitz on $\mathscr{D}$ with the same Lipschitz constant (c not dependent on $\bar{x}$).

Property: The function $f(\cdot)$ is said to be **Globally Lipschitz** if it is Lipschitz on $\mathbb{R}^n$.

Remark: locally Lipschitz on a open set $\mathscr{D}$ implies Lipschitz on any compact subset of $\mathscr{D}$.

What about nonlinear systems?

Continuous differentiability

Result: if $\boxed{df(x)/dx \text{ is continuous}}$ on $\mathscr{D}$ then $f(\cdot)$ is locally Lipschitz on $\mathscr{D}$.

Result: if $df(x)/dx$ is continuous on $\mathbb{R}^n$ then $f(\cdot)$ is globally Lipschitz if the Jacobian $\partial f(x)/\partial x$ is **uniformly bounded** (there exists an $L > 0$ such that $\left\| \partial f(x)/\partial x \right\| \leq L$ for all $x \in \mathbb{R}^n$).

Continuity $\Leftarrow$ **Lipschitz property** $\Leftarrow$ Continuous differentiability

What about nonlinear systems?

Lipschitz property $\Rightarrow$ Existence and Uniqueness

Theorem: If the function $f(\cdot)$ is **Locally Lipschitz at x_0** then there exist an $\delta > 0$ such that the solution of the system $\dot{x} = f(x)$ with initial condition x_0 exists and it is unique over $[0,\delta]$.

Remark: The property $f(\cdot)$ locally Lipschtz on $\mathbb{R}^n$ is not sufficient for the existence of the solution for all $t \geq 0$

Example: $\dot{x} = -x^2 \quad x(0) = -1$

Theorem: If the function $f(\cdot)$ is **Locally Lipschitz at x_0** then there exist an $T > 0$ (**maximal interval of definition**) such that the solution of the system $\dot{x} = f(x)$ with initial condition x_0 exists and it is unique over $[0,T)$.

Continuity of the right-hand side of an ODE is enough for existence of solutions, but not enough for their uniqueness (see [Khalil, p. 88] for a reference).

What about nonlinear systems?

Lemma. Let $f(x)$ locally Lipschitz for some domani $\mathscr{D} \subseteq \mathbb{R}^n$. Let $x(t)$ be a solution of $\dot{x} = f(x)$ on a maximal open interval $[0,T)$ with $T < \infty$. Let W be any compact subset of $\mathscr{D}$. Then there is some $t \in (0, T)$ with $x(t) \notin W$.

Theorem: Let $f(\cdot)$ be **Locally Lipschitz on $\mathscr{D}$** (open set of $\mathbb{R}^n$). Let $W \subset \mathscr{D}$ be compact set and suppose that it is known that every solution originating from x_0 is such that $x(t, x_0) \in W$ for all $t \geq 0$. Then $x(t, x_0)$ exists and is unique for all $t \geq 0$.

Theorem: If the function $f(\cdot)$ is s **Globally Lipschitz** then the system $\dot{x} = f(x)$ with initial condition $x_0 \in \mathbb{R}^n$ has a unique solution $x(t)$ that exists for all $t \geq 0$.

What about nonlinear systems?

From now $f(x)$ and others real-valued functions that will be used in the analysis are assumed to be continuously differentiable (existence and uniqueness of the solutions are guaranteed). But not necessary….

Dini Derivatives

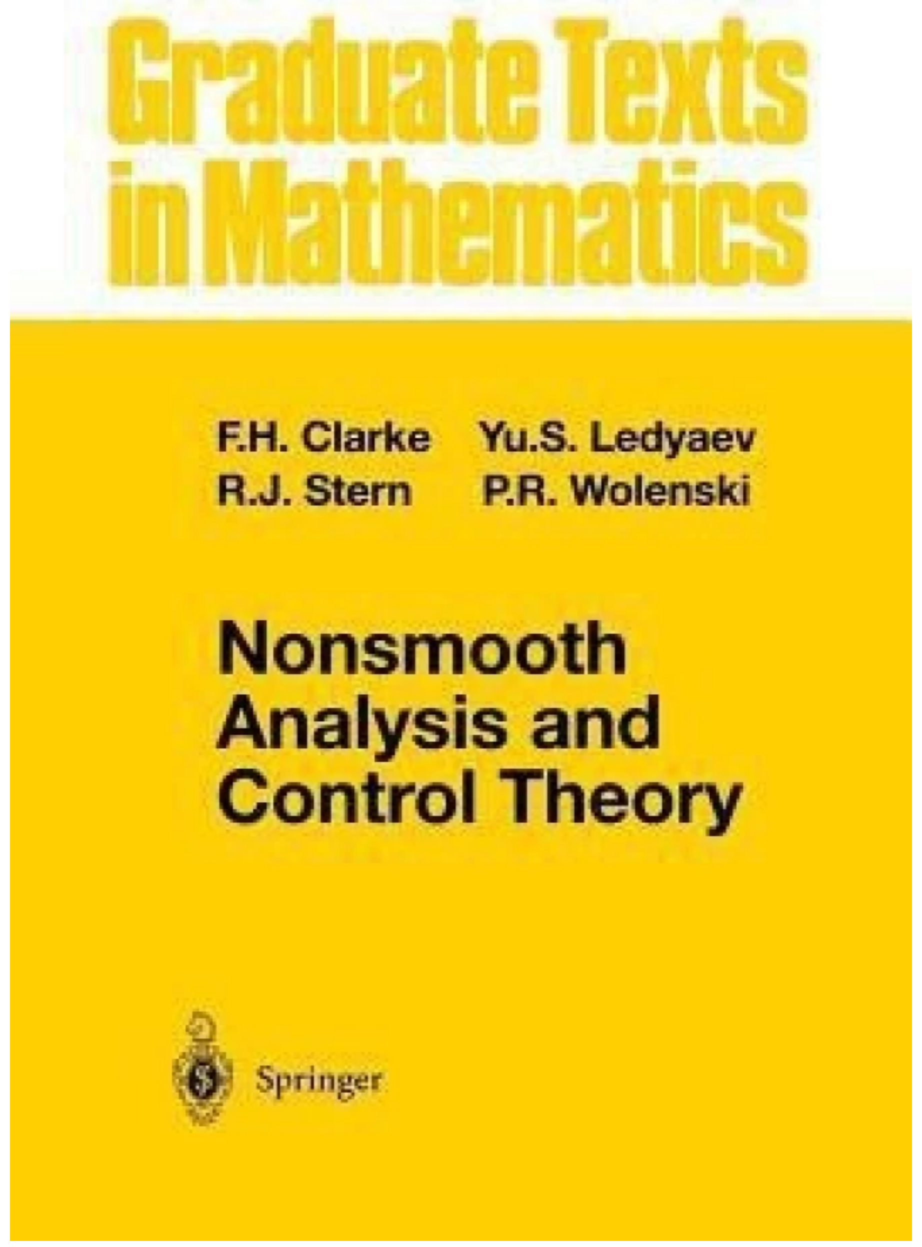
If $V : \mathbb{R} \to \mathbb{R}$ is locally Lipschitz

$$D^+V(t) = \lim_{h \to 0^+} \sup \frac{1}{h}[V(t+h) - V(t)]$$

If $V : \mathbb{R}^n \to \mathbb{R}$ is locally Lipschitz and evaluated along $x(t, x_0)$

$$D^+V(x_0) = \lim_{h \to 0^+} \sup \frac{1}{h}[V(x(h, x_0) - V(x_0)]$$

Note: extremely rich and elegant literature on nonsmooth analysis



Continuity wrt the initial conditions and parameters

Let $x(t, \mu, x_0)$ solution of $\dot{x} = f_\mu(x)$, $\mu \in \mathbb{R}^n$ a set of parameters, with initial condition $x(0) = x_0$. Is $x(t, \mu, x_0)$ continuous with respect to x_0 and μ ?

Closeness of solutions

Let $f(x)$ be Lipschitz on an open connected set $\mathscr{D} \subset \mathbb{R}^n$ with Lipschitz constant L and let $x(t, x_0)$ solution of $\dot{x} = f(x)$ with initial condition $x(0) = x_0$. Let $g(\cdot) : \mathbb{R}^n \to \mathbb{R}^n$ be such that $\|g(y)\| \leq \mu$ for some $\mu > 0$ and for all $y \in \mathscr{D}$ and let $y(t, y_0)$ be the solution of $\dot{y} = f(y) + g(y)$ with initial condition $y(0) = y_0$. Assume that $x(t, x_0) \in \mathscr{D}$ and $y(t, y_0) \in \mathscr{D}$ for all $t \in [0, \bar{t}]$ for some $\bar{t}$. Then

$$\|x(t, x_0) - y(t, y_0)\| \leq \|x_0 - y_0\| e^{Lt} + \frac{\mu}{L}\left(e^{Lt} - 1\right) \qquad \forall t \in [0, \bar{t}]$$

Continuity comes as a corollary

What about nonlinear systems?

The effect of the input in nonlinear systems could be catastrophic. Systems with "nice" properties when $u = 0$ could have "deteriorated" behaviour when an input is applied. A clear distinction between free and forced evolution doesn't exist in general.

$$\dot{x} = -x + xu$$

$\dot{x} = -x$ when $u = 0$ (exponential convergence to zero without input)

$\dot{x} = x$ when $u = 2$ (exponential explosion with (sufficiently large) input)

$$\dot{x} = -x + (1+x)xu$$

$\dot{x} = -x$ when $u = 0$ (exponential convergence to zero without input)

$\dot{x} = x^2$ when $u = 1$ (finite escape time with bounded input)

$$\dot{x} = -x + (x^2+1)u$$

$\dot{x} = -x$ when $u = 0$ (exponential convergence to zero without input)

$u(t) = 1/\sqrt{2t+2}$ and $x(0) = \sqrt{2}$ leads to $x(t) = \sqrt{2t+2}$ (explosion with vanishing input)

Superimposition properties are far to be true

Qualitative Behaviour of Trajectories

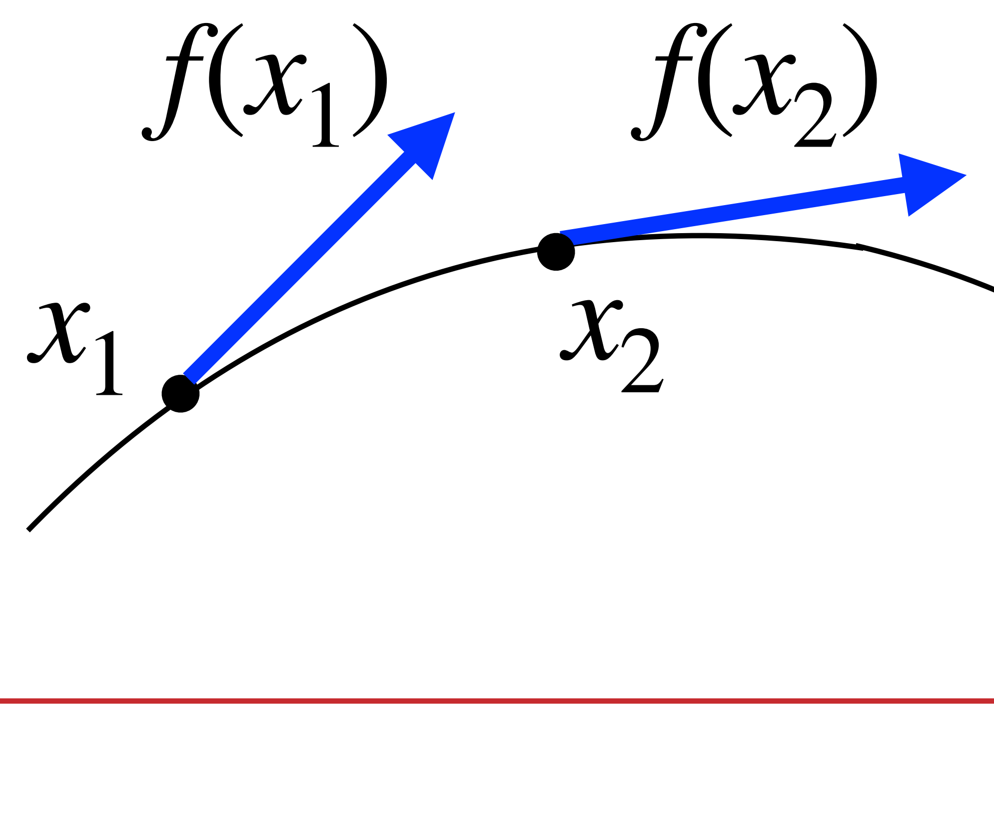
Let $u = 0$

For dynamics of dimension $2, 3$ ($n = 2, 3$), a qualitative behaviour of the trajectory can be obtained by plotting the “vector field” and the associated “phase portrait” (“phase plane” method)

A vector field is an assignment of a vector (which is $f(x)$) to each point x in the space/manifold

A phase portrait is a geometric representation of the trajectories of a dynamical system in the phase plane. Each set of initial conditions is represented by a different curve, or point

For **continuous-time systems**, at each $x \in \mathbb{R}^n$ $f(x)$ is a vector of $\mathbb{R}^n$, which is tangent to the trajectory (flow) passing through x. The geometry of the vector yields local information about the direction of the trajectory. The amplitude about the local “speed” $f(x)$ is the velocity vector at x.

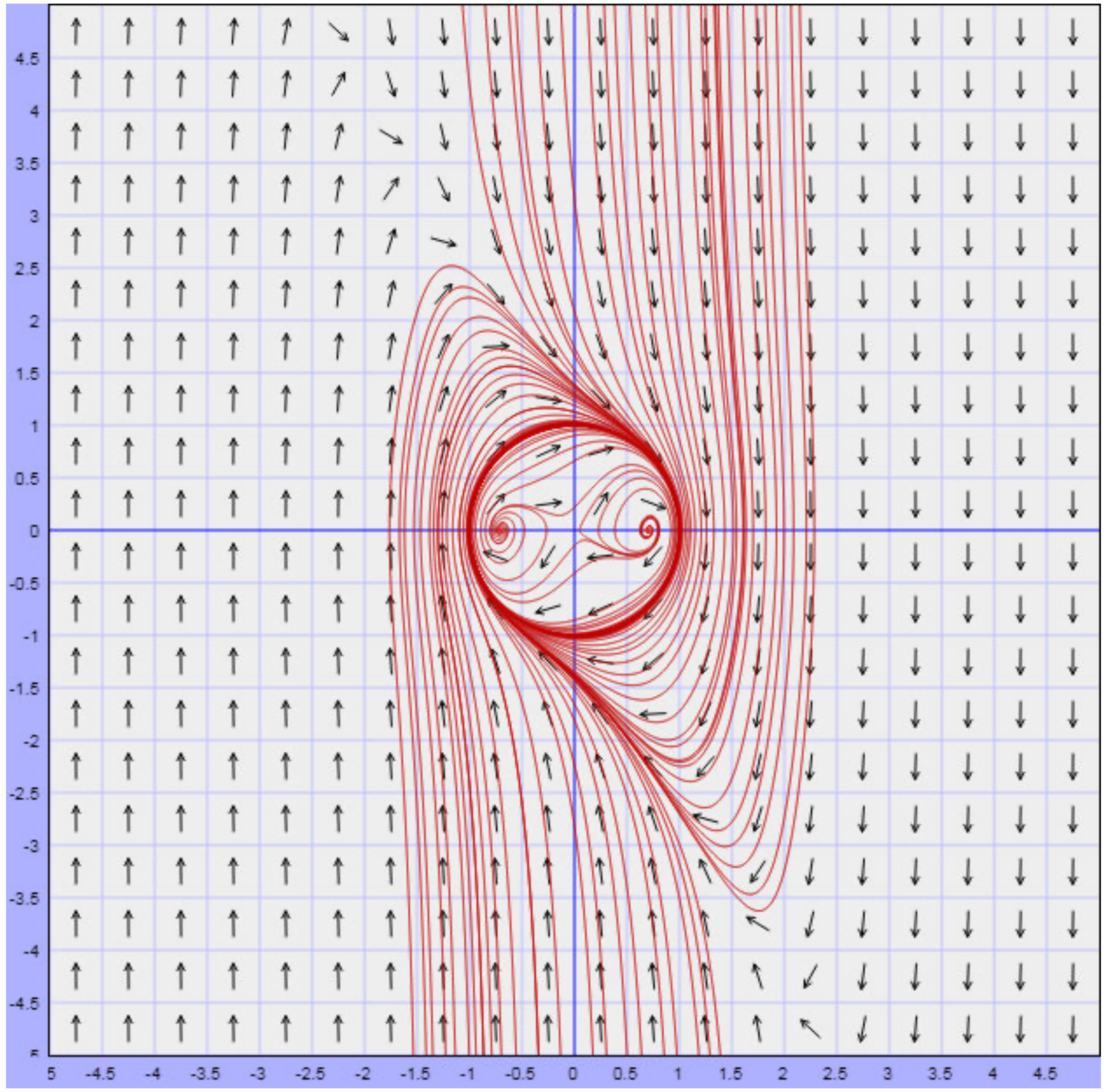


A vector field is easy to plot. Covering $\mathbb{R}^2$ with vectors yield qualitative insight about the whole set of trajectories originating from different initial conditions. Up to $n = 3$ the tool is valuable.

Qualitative Behaviour of Trajectories

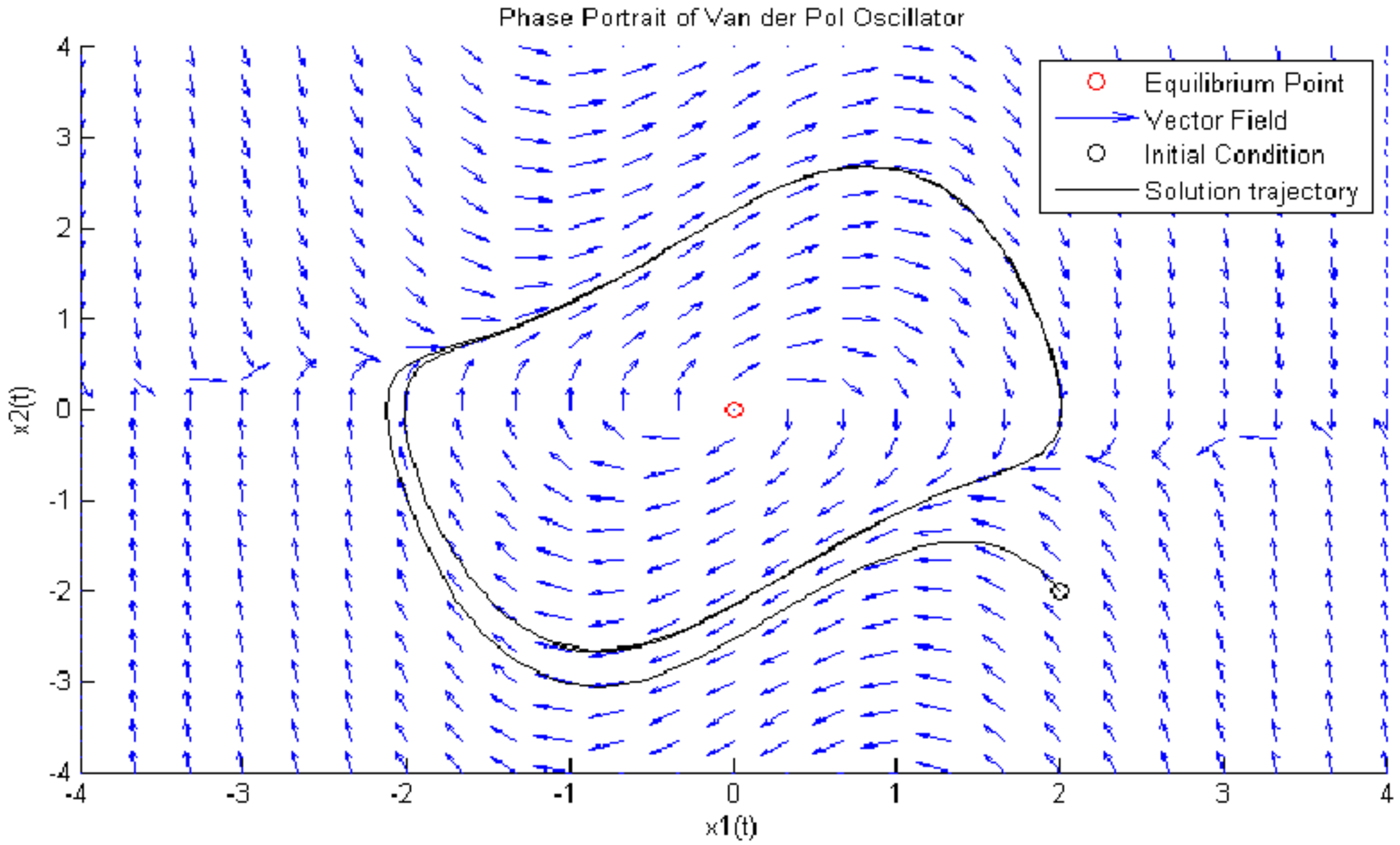
$n = 2$

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -x_2^3 - 2x_1^3 - x_1^2 x_2 - 2x_2^2 x_1 + x_1 + x_2$$



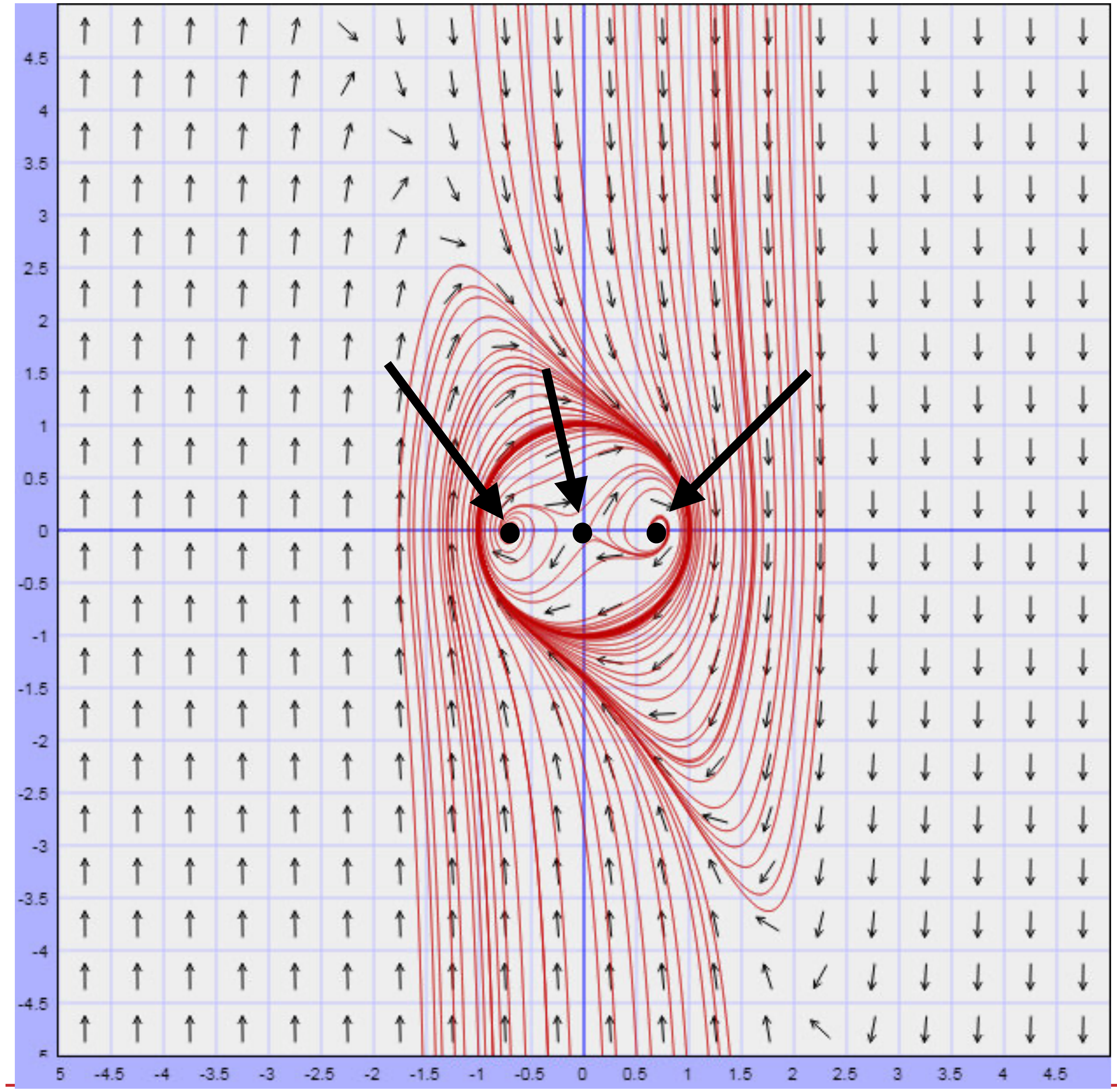
Van der Pol oscillator

$$\begin{aligned} \dot{x}_1 &= \kappa x_2 \\ \dot{x}_2 &= \mu(1 - x_1^2)x_2 - x_1 \end{aligned}$$



Qualitative Behaviour of Trajectories

Fixed Points (Equilibrium Points): points $x^\star \in \mathbb{R}^n$ such that $f(x^\star) = 0$



Isoclines: locus where $f_1(x) = 0, \; f_2(x) = 0$

$$\dot{x}_1 = \boxed{x_2}$$
$$\dot{x}_2 = \boxed{-x_2^3 - 2x_1^3 - x_1^2 x_2 - 2x_2^2 x_1 + x_1 + x_2}$$
$$= 0$$

$$x_2^\star = 0, \;\; x_1^\star = 0, \;\; x_1^\star = \pm \frac{1}{\sqrt{2}}$$

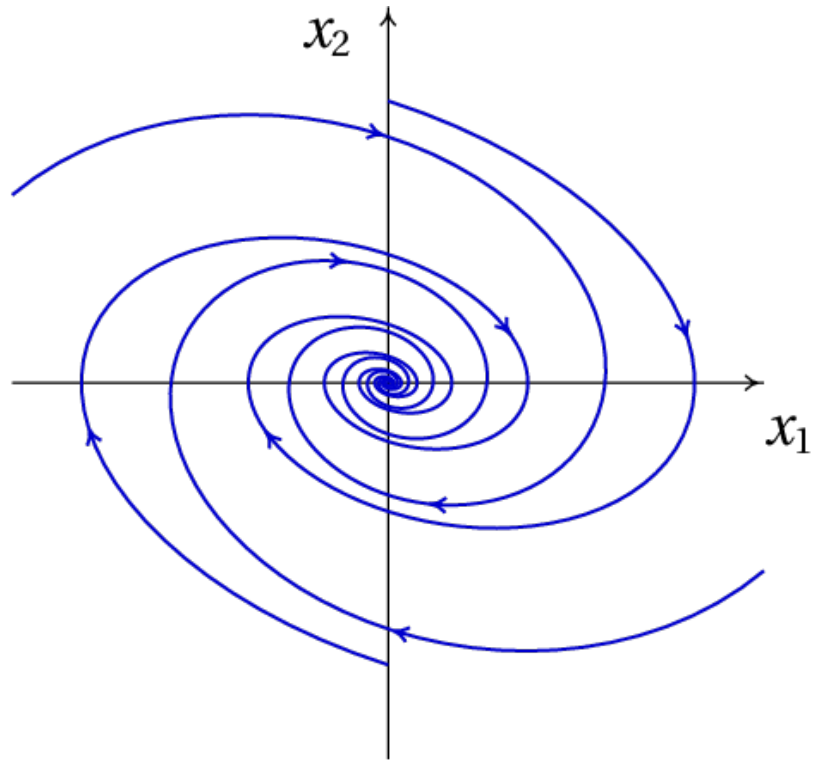
Three isolated equilibria

Nonlinear systems could have **isolated** equilibrium points

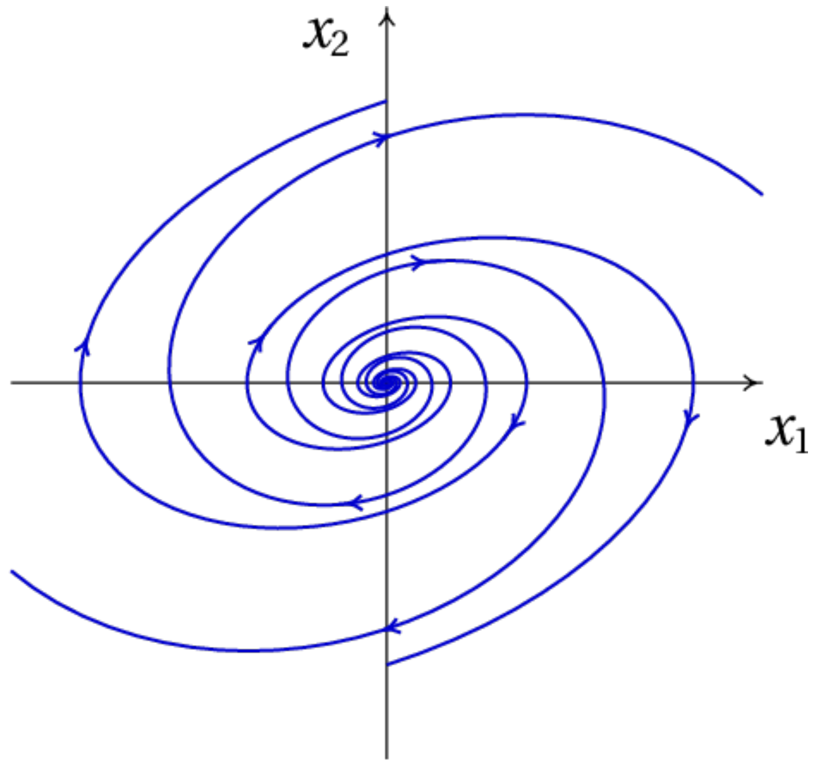
Linear systems have a single equilibrium or a subspace of equilibria ($\{x^\star \; : \; x^\star \in \mathrm{Ker} A\}$)

Qualitative Behaviour of Trajectories

Equilibrium points can be **attractive** (all nearby trajectories converge to the point as $t \to \infty$), **repulsive** (all nearby trajectories converge to the point as $t \to -\infty$), "**saddles**" (points with some directions attractive and others repulsive), they can be "anchor points" of "complex" orbits, such as **homoclinic orbits** (a trajectory hanged at the equilibrium), **heteroclinic orbits** (a trajectory connecting two isolated equilibria), or "**centers**" around which closed curves take place….

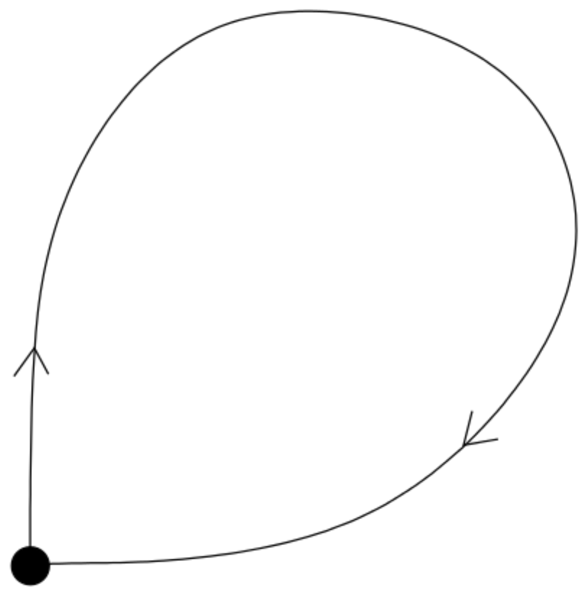


Attractive

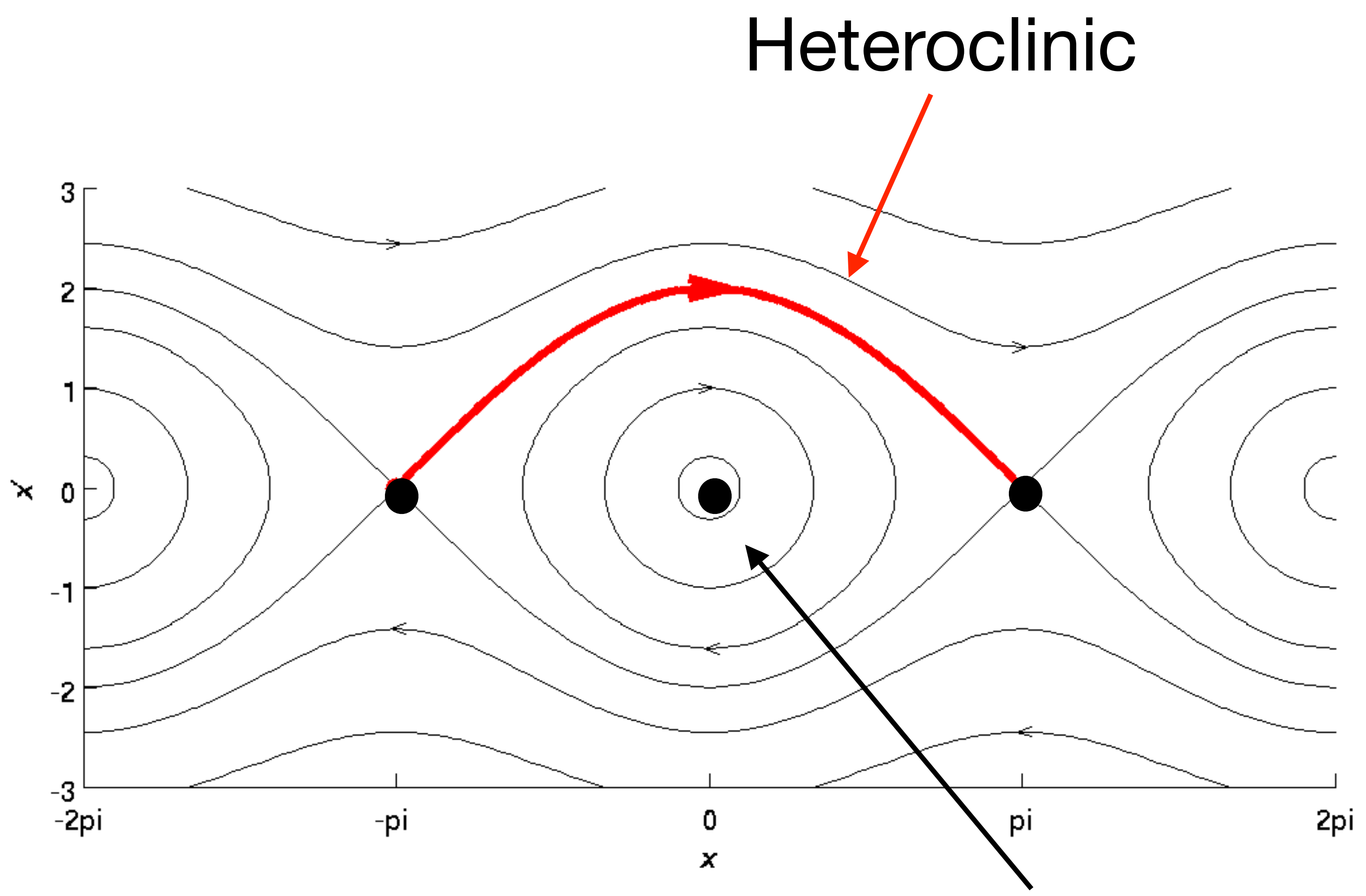


Repulsive

Saddle



Homoclinic



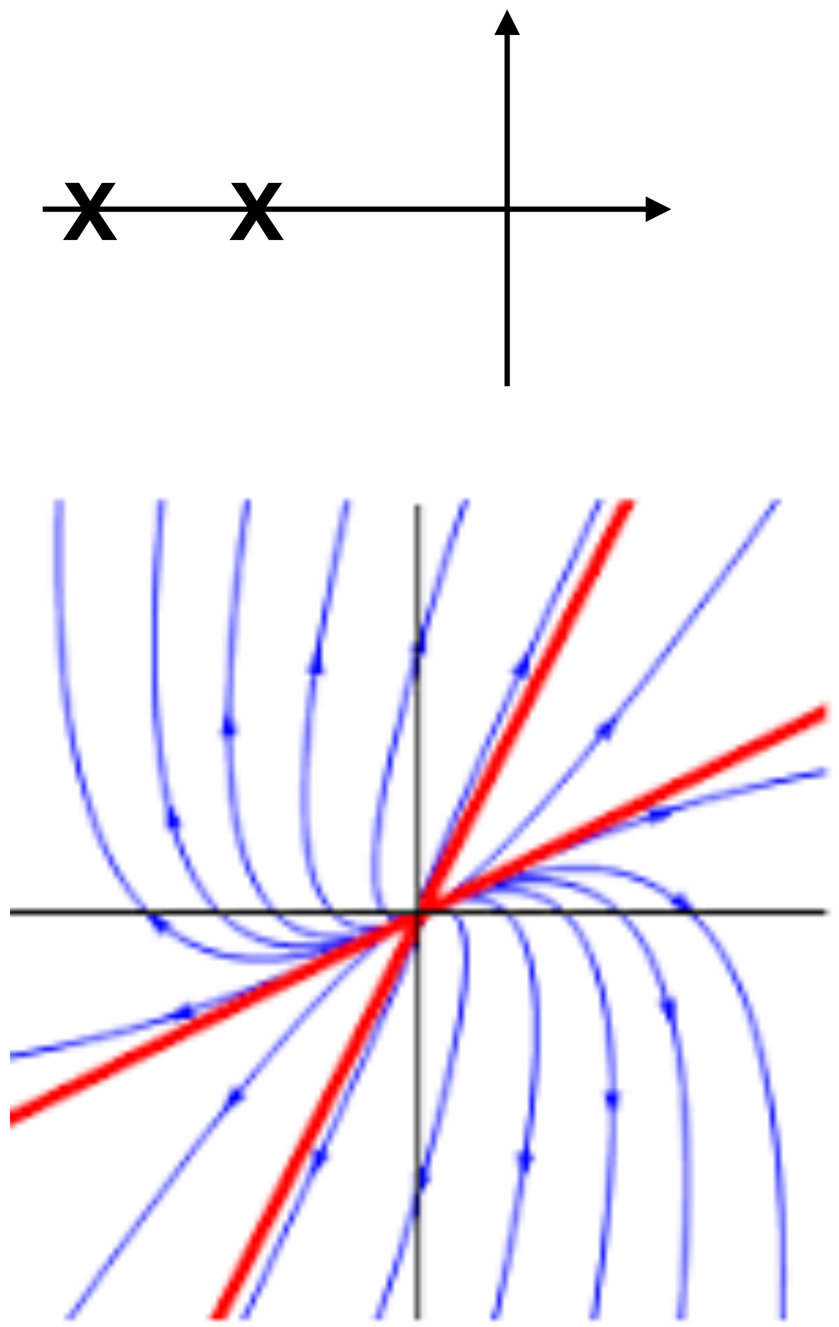
center of closed curves

Rich geometry!

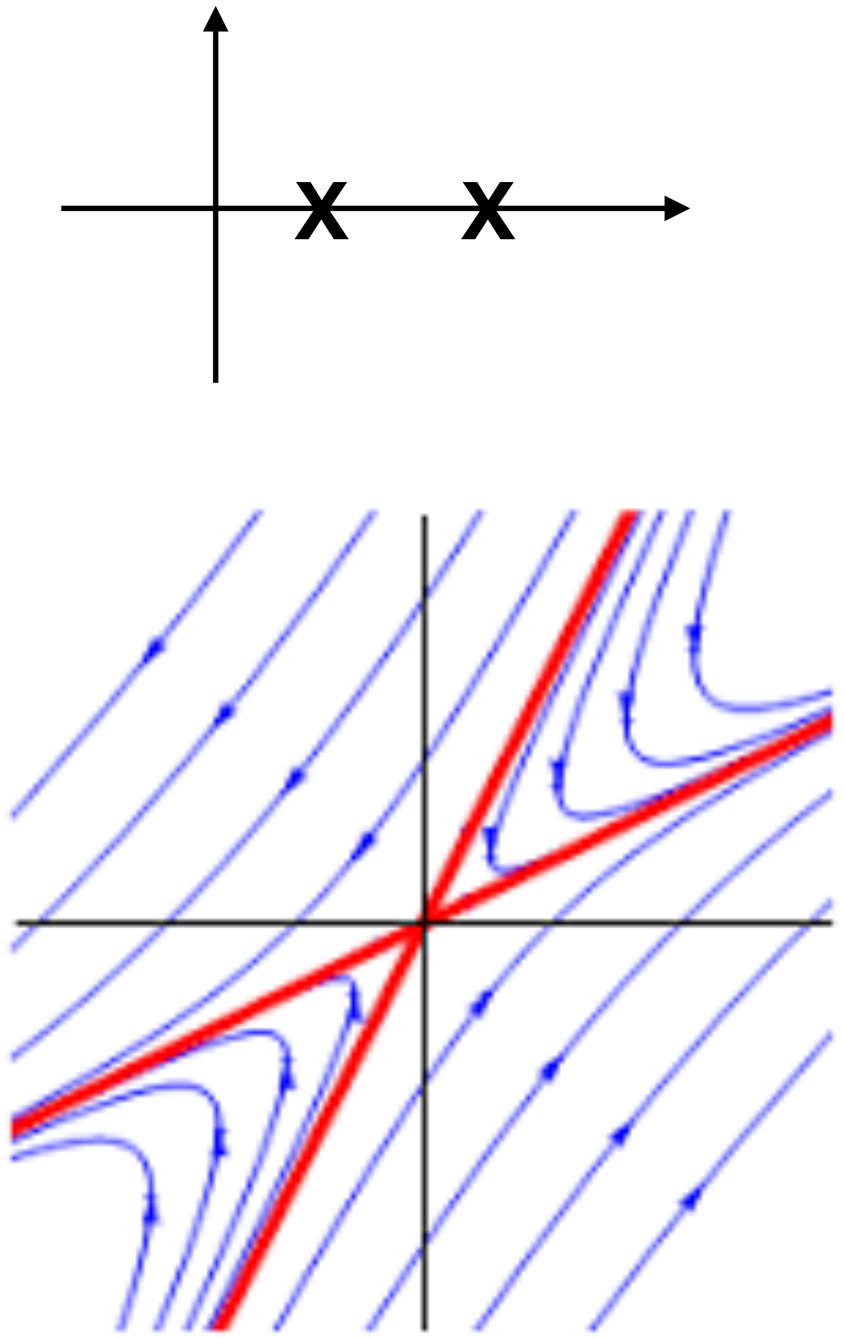
> **Locally** around the equilibrium point the "behaviour" of the trajectory can be studied by using the linearised dynamics

Qualitative Behaviour of Trajectories

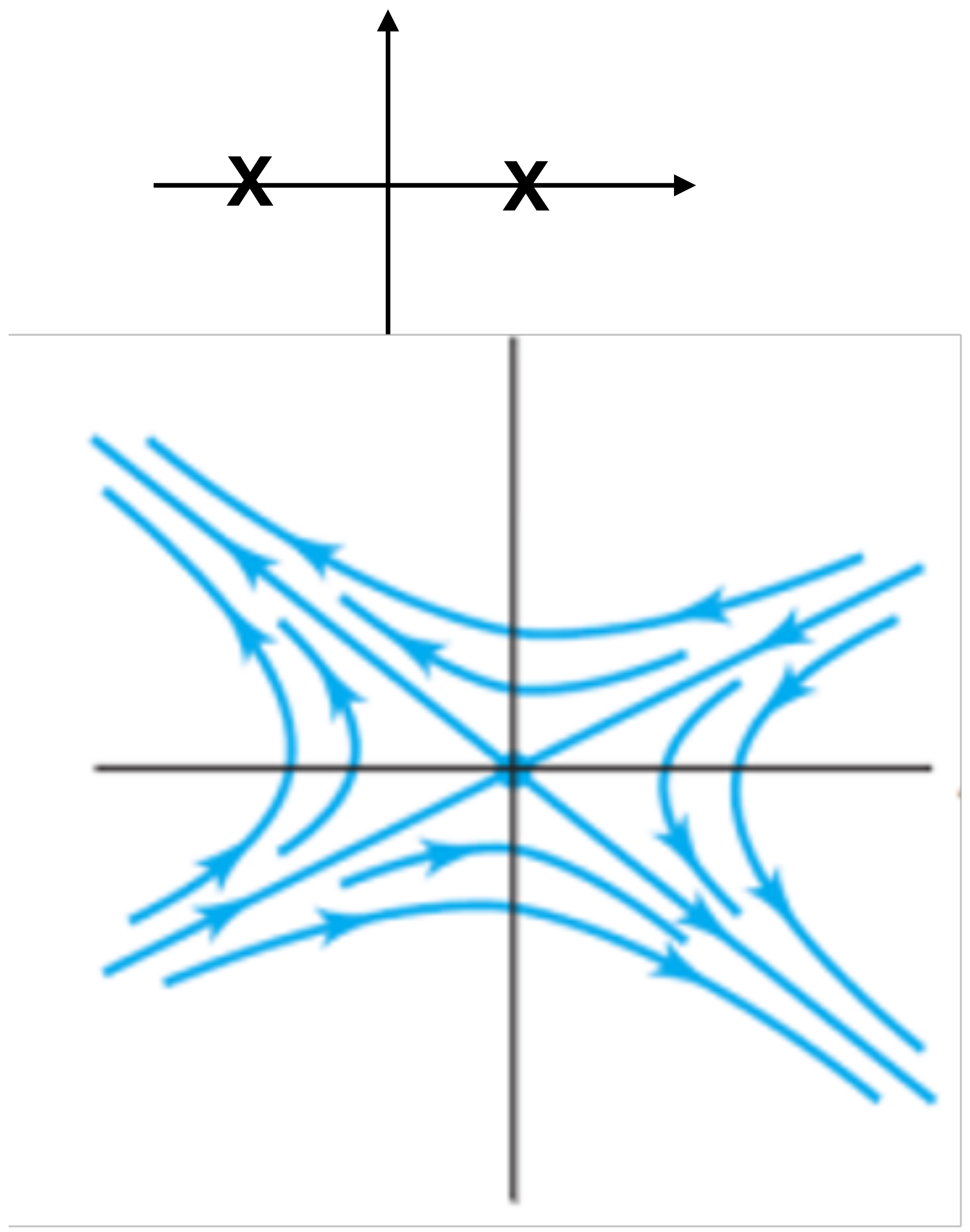
$$\dot{x} = f(x) \qquad A = \left.\frac{\partial f}{\partial x}\right|_{x^\star}$$



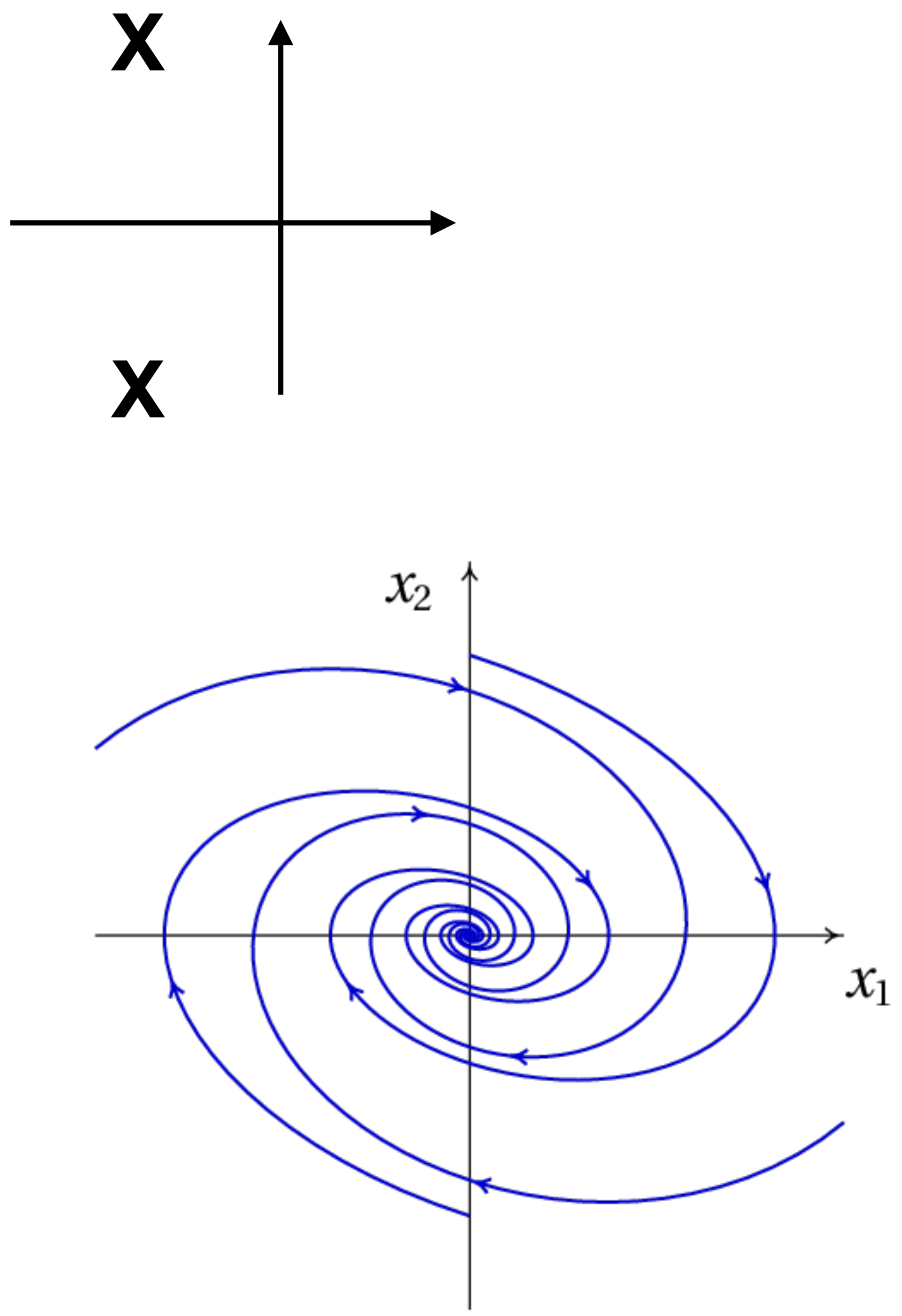
stable node



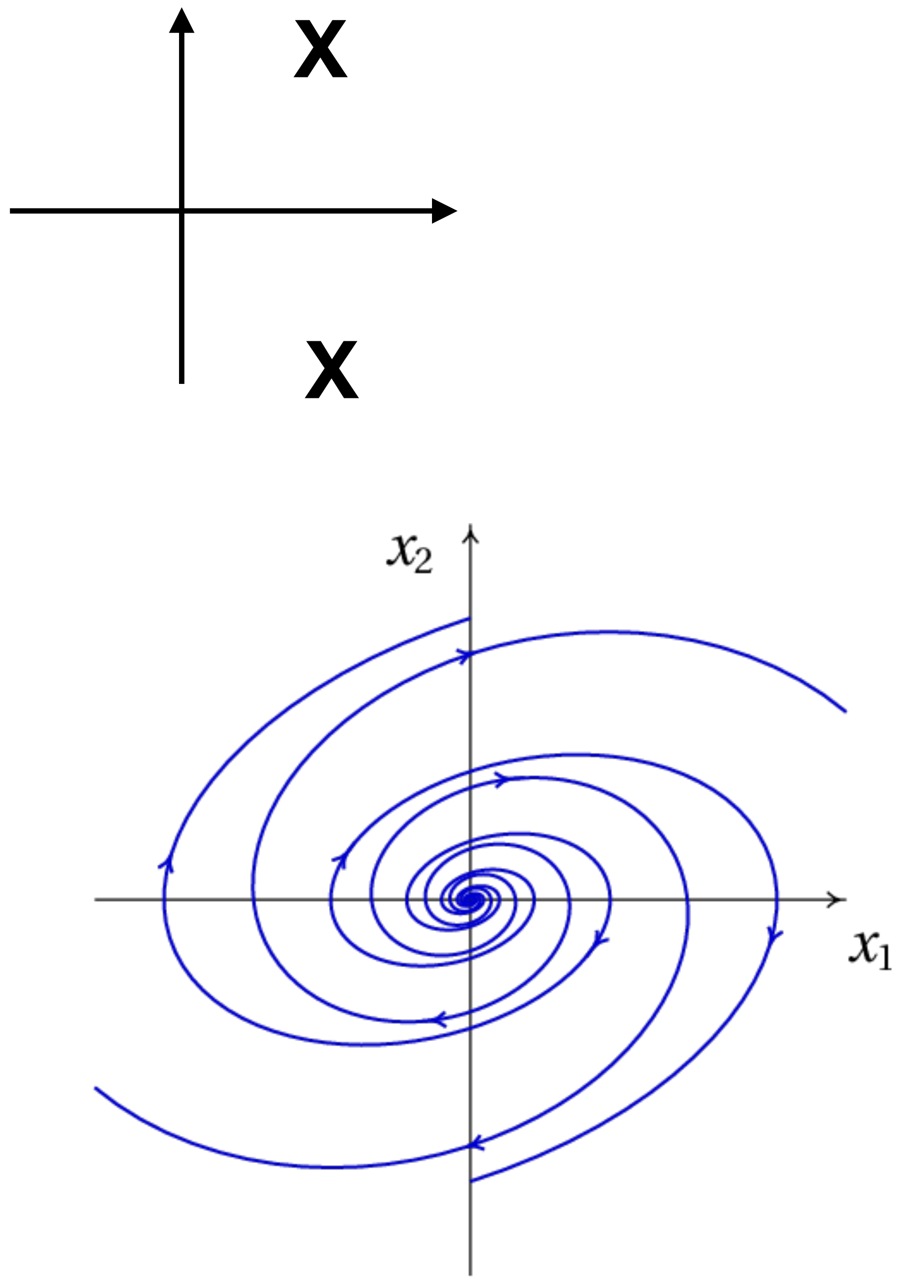
Unstable node



Saddle



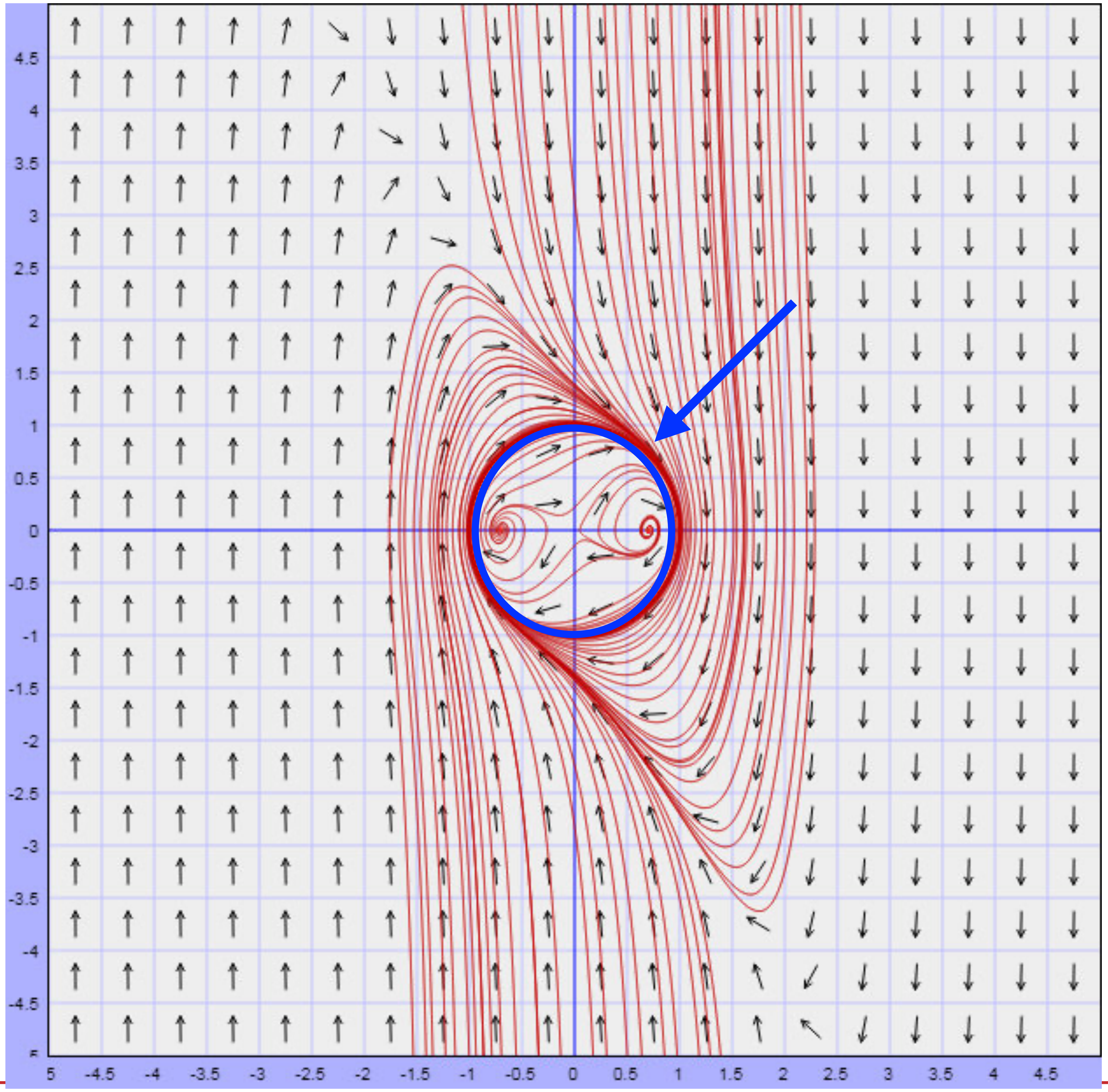
stable focus



unstable focus

Qualitative Behaviour of Trajectories

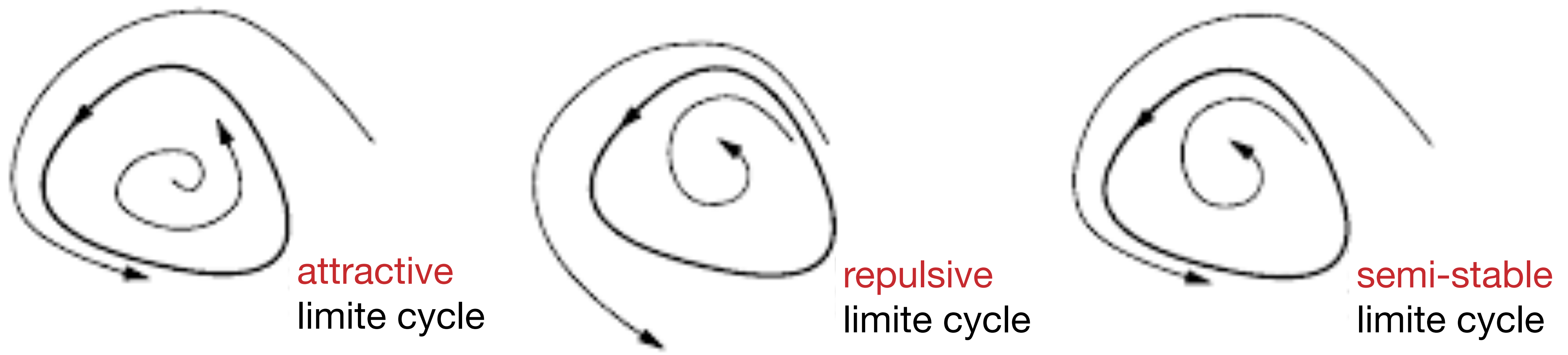
Limit cycle: **isolated** closed trajectory having the property that at least one other trajectory spirals into it either as time approaches infinity or as time approaches negative infinity



A limit cycle is a locus of **periodic trajectories** of the system

Period orbit with period T: $x(t) = x(t+T)\ \forall t \geq 0$

Limit cycles can be categorised according to their attractive/repulsive properties:



All neighbourhood trajectories converge to the limit cycle as $t \to \infty$

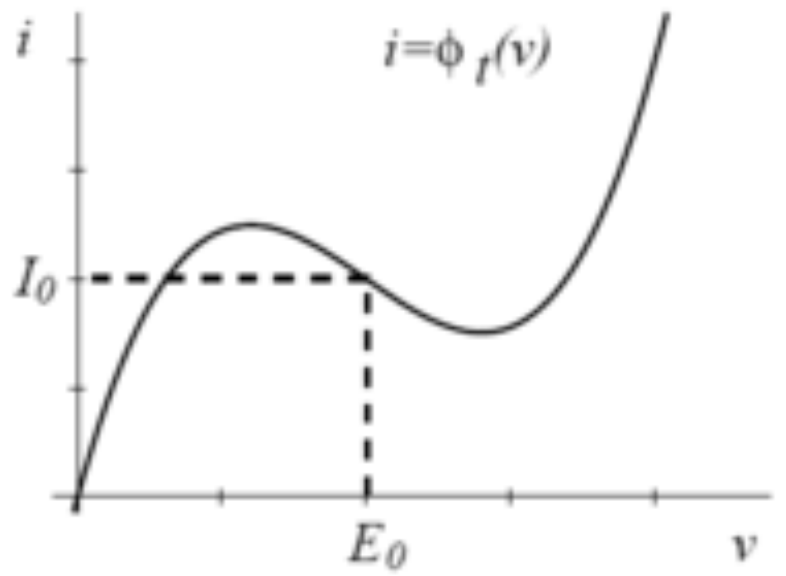
All neighbourhood trajectories converge to the limit cycle as $t \to -\infty$

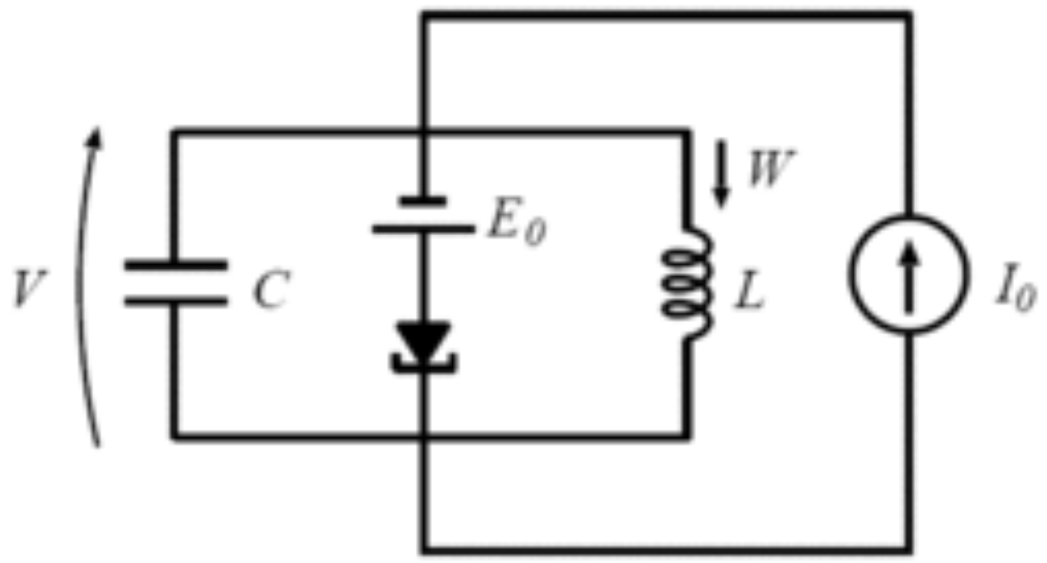
Convergence form one side, repulsive from the other

Example: Van der Pol oscillator

Van der Pol oscillator ($\mu > 0$, $\kappa > 0$)

$$\begin{aligned} \dot{x}_1 &= \kappa x_2 \\ \dot{x}_2 &= \mu(1 - x_1^2)x_2 - x_1 \end{aligned}$$





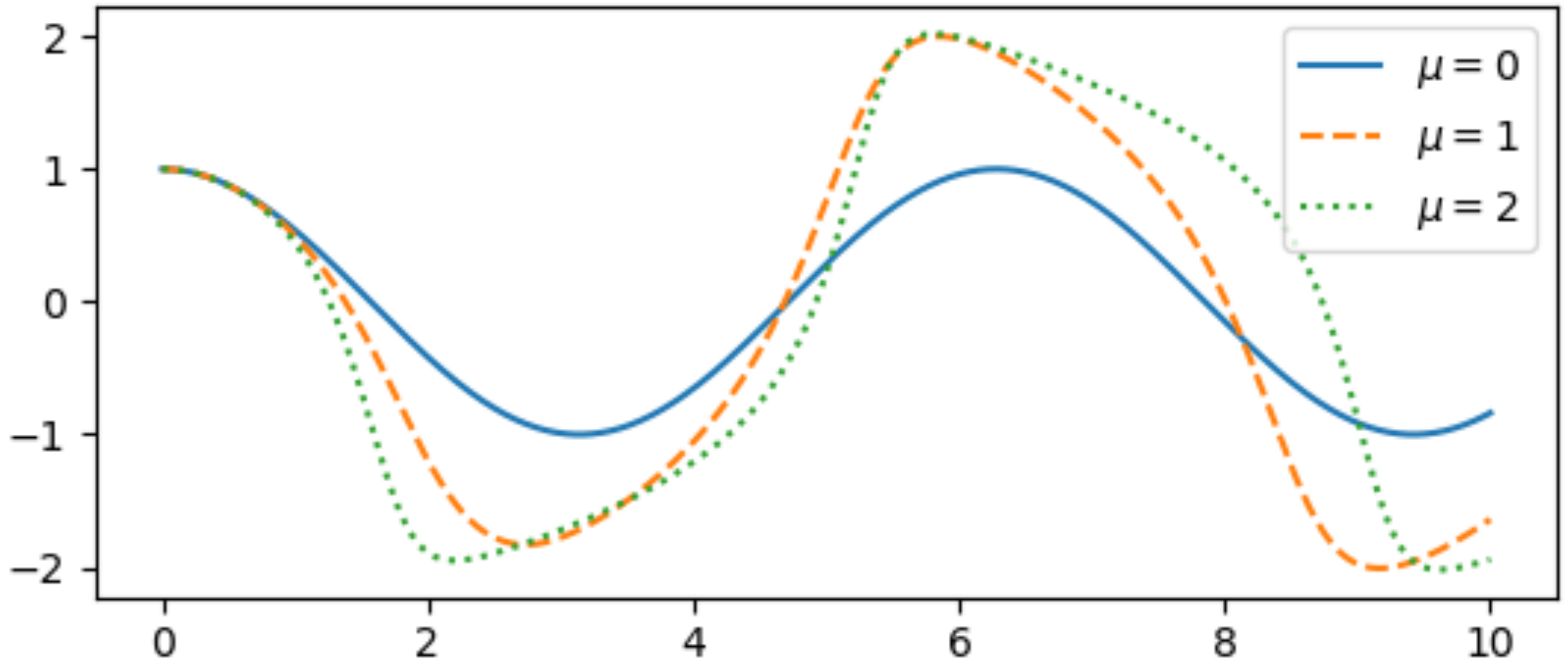
Balthasar van der Pol
(27 January 1889 – 6 October 1959)

Linear oscillator ($\mu = 0$)

$$\begin{aligned} \dot{x}_1 &= \kappa x_2 \\ \dot{x}_2 &= -x_1 \end{aligned}$$

Basic model for oscillatory processes in physics, electronics, biology, neurology, sociology and economics

Self-sustained oscillations whose amplitude is not dependent on initial conditions



Qualitative Behaviour of Trajectories

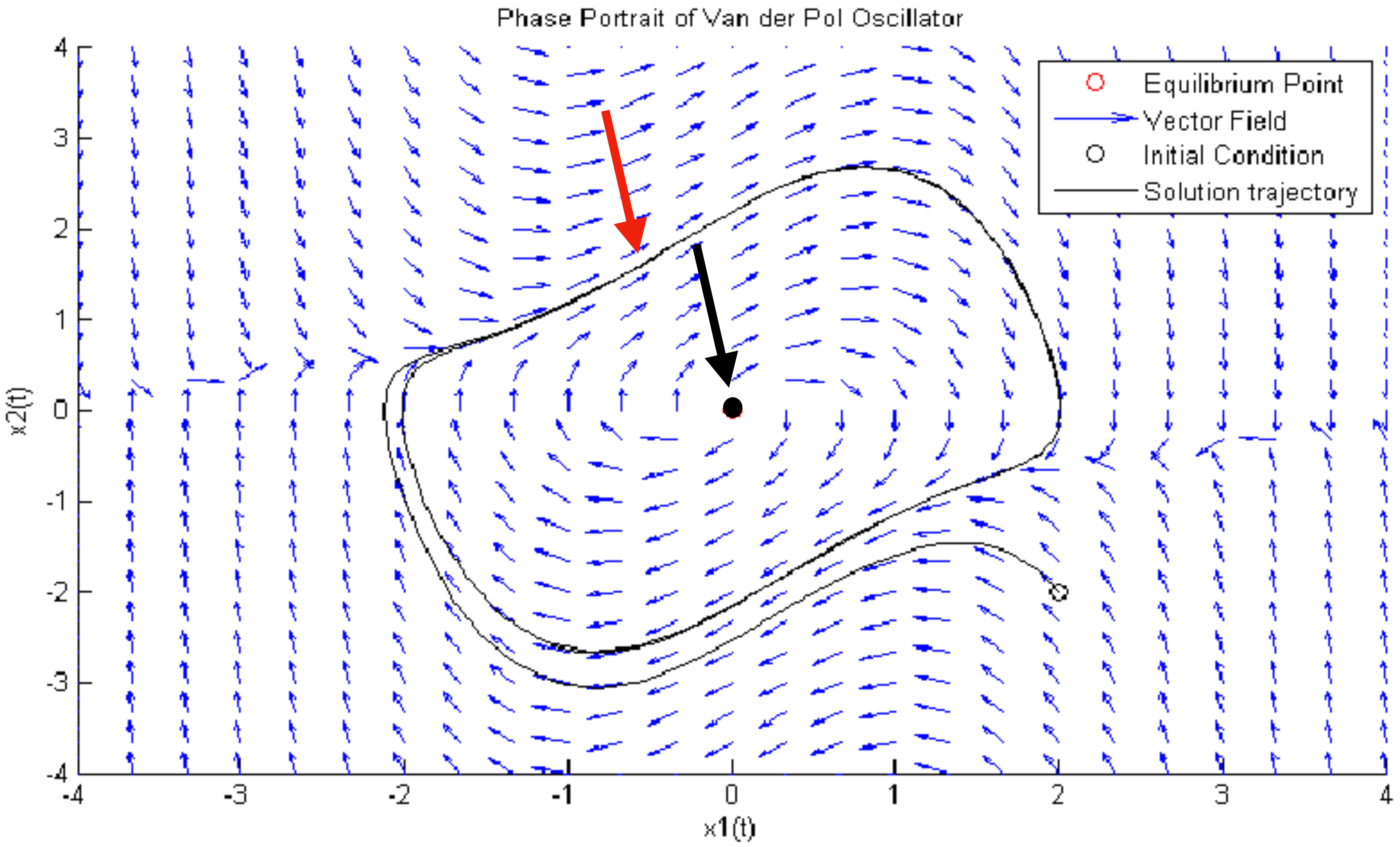
Van der Pol oscillator

$$\begin{aligned}\dot{x}_1 &= \kappa x_2\\ \dot{x}_2 &= \mu(1-x_1^2)x_2 - x_1\end{aligned}$$

$= 0$

$$x_1 = x_2 = 0$$

The origin is the only equilibrium point, which is repulsive. There exists a limit cycle that is attractive



Homework: plot how the geometry of the limit cycle and the features of the periodic trajectory change when (μ, κ) change

Homework: plot how the phase portrait changes when $\mu = 0$ (linear oscillator)

Qualitative Behaviour of Trajectories

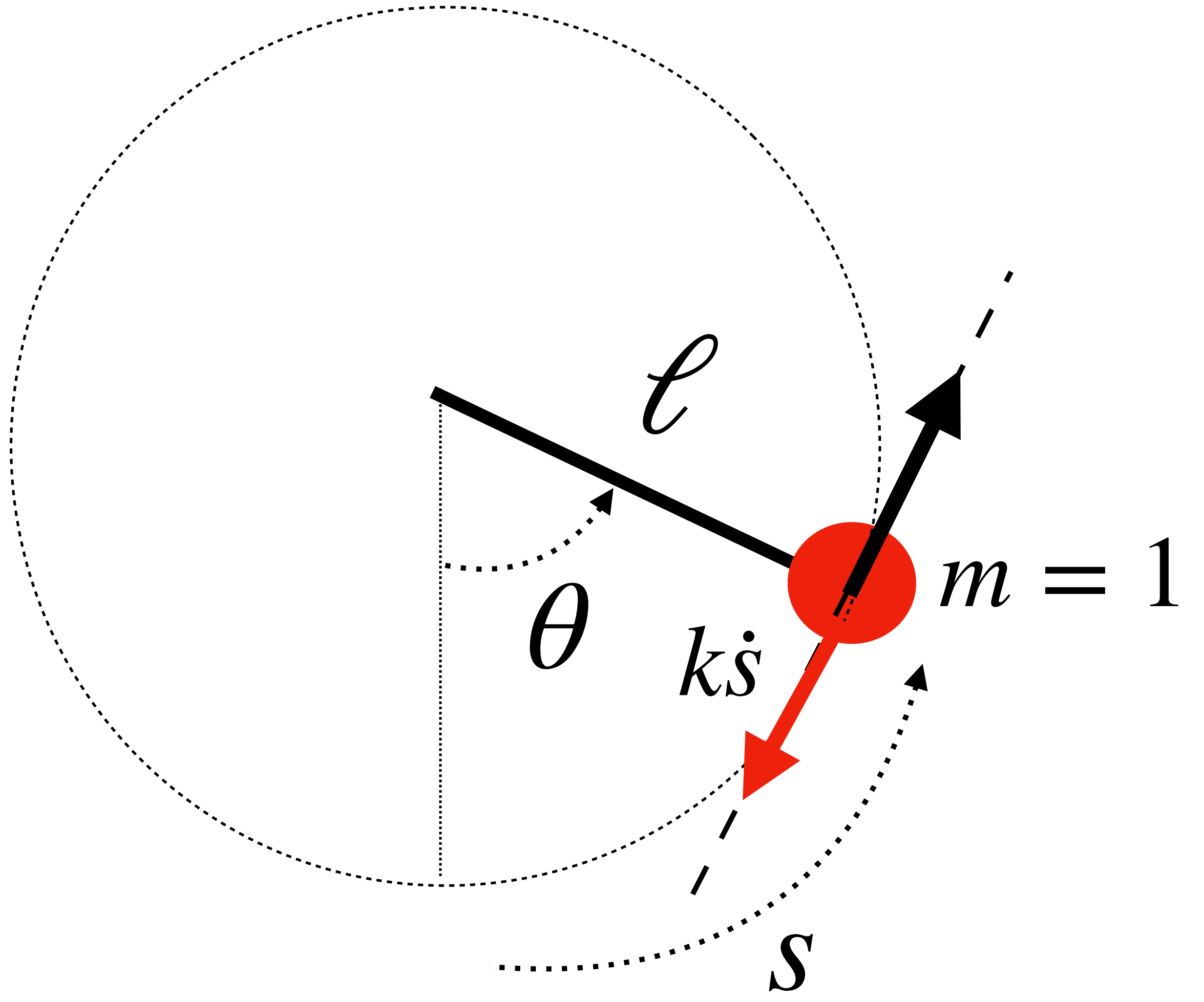
Pendulum

No friction

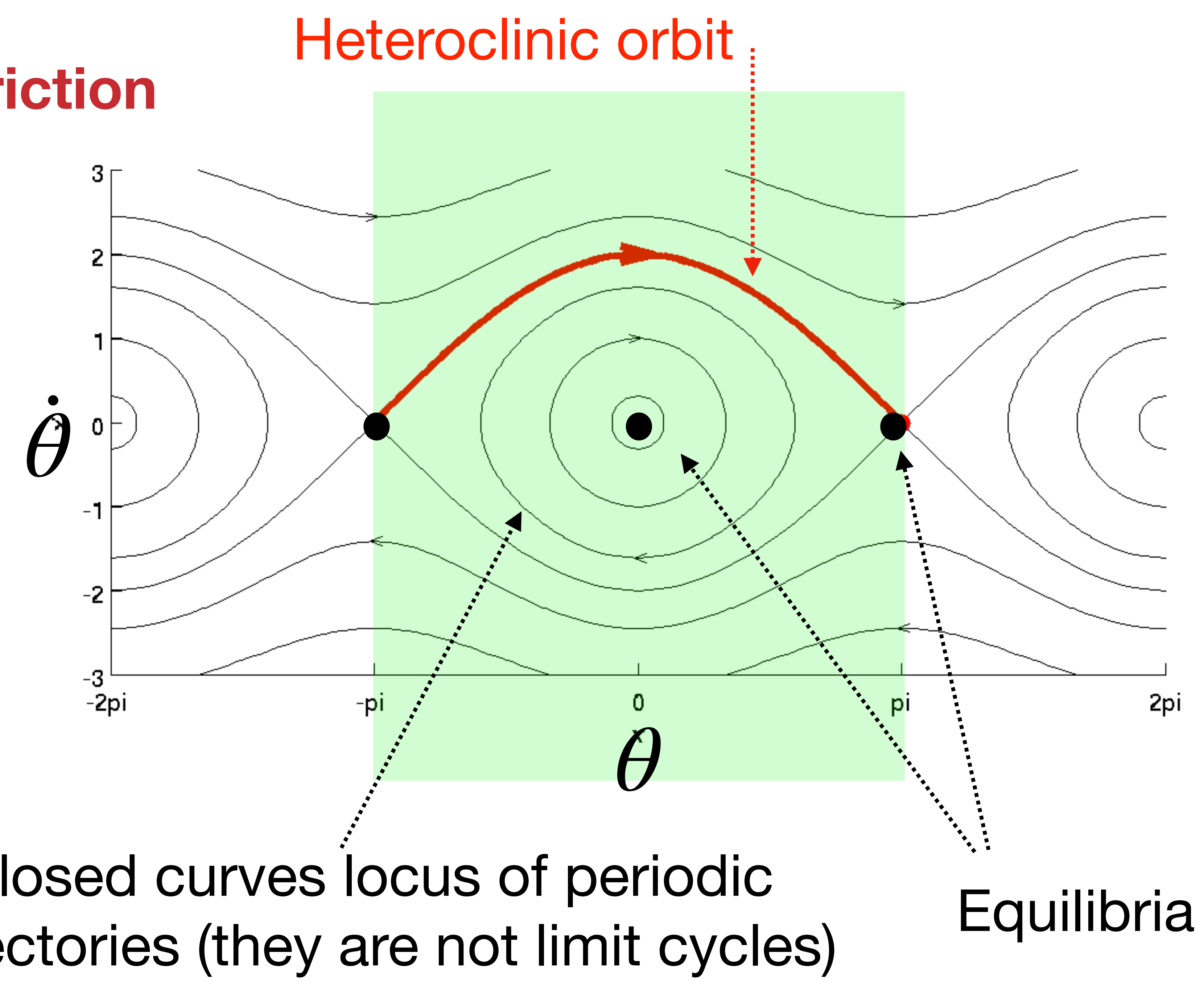
$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -\frac{g}{\ell}\sin x_1$$

With friction

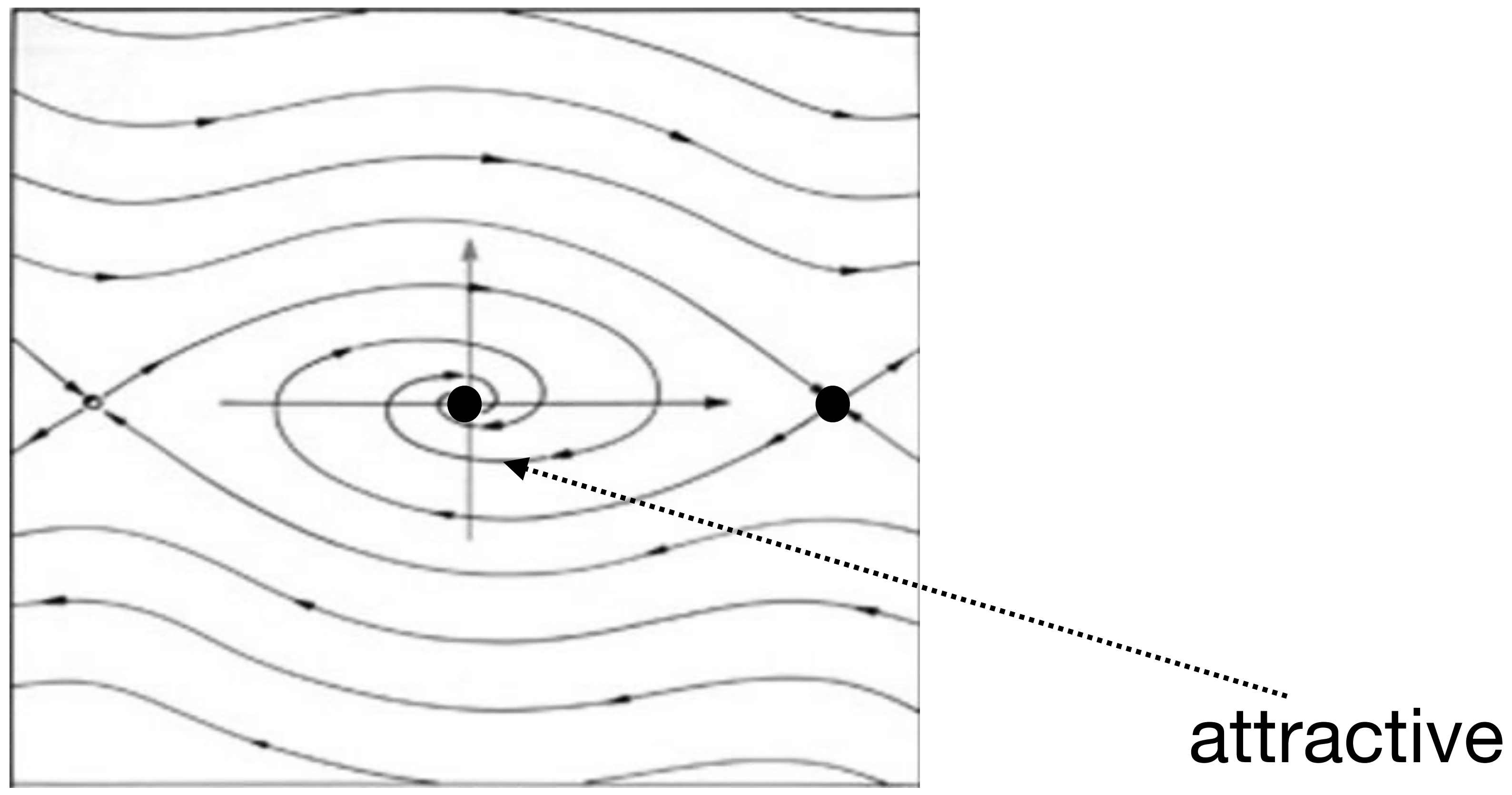
$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -\frac{g}{\ell}\sin x_1 - k\ell x_2$$



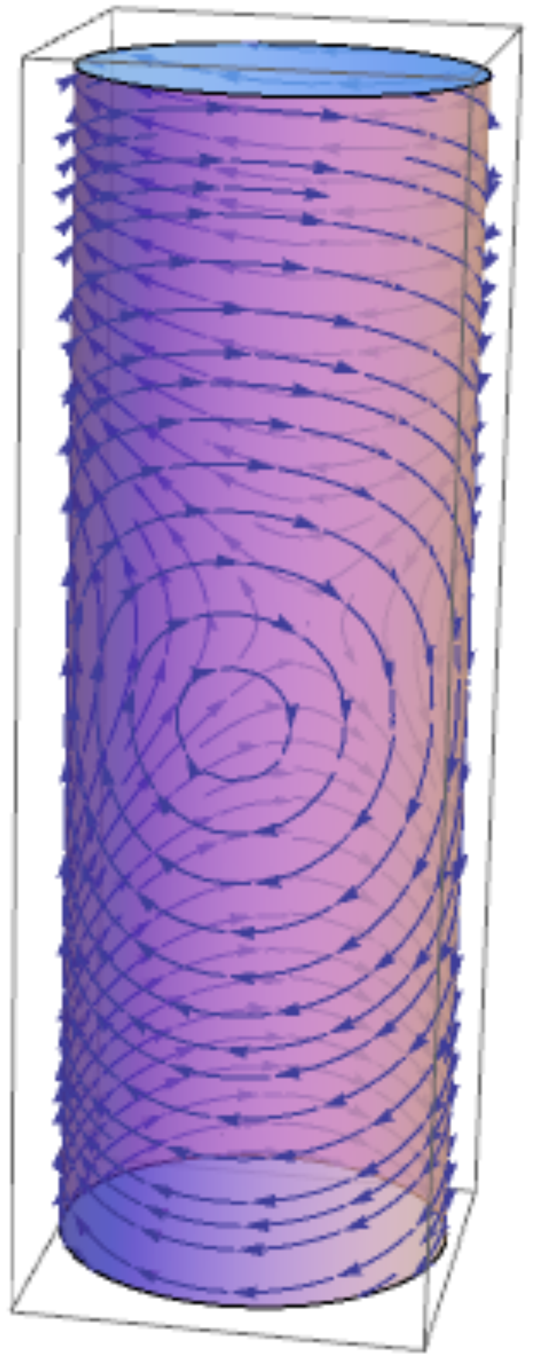
No friction



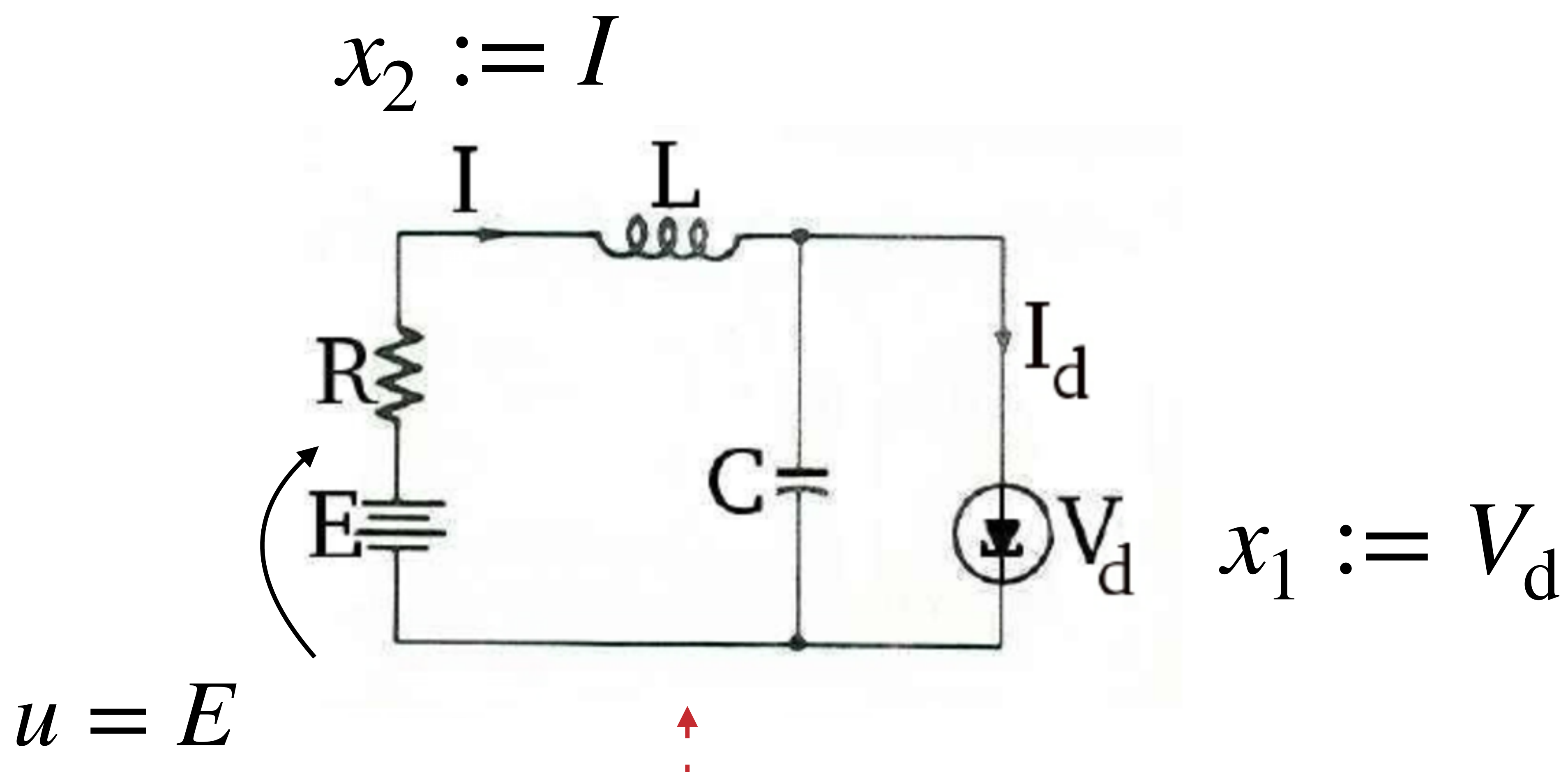
With friction



Phase portrait on the cylinder manifold

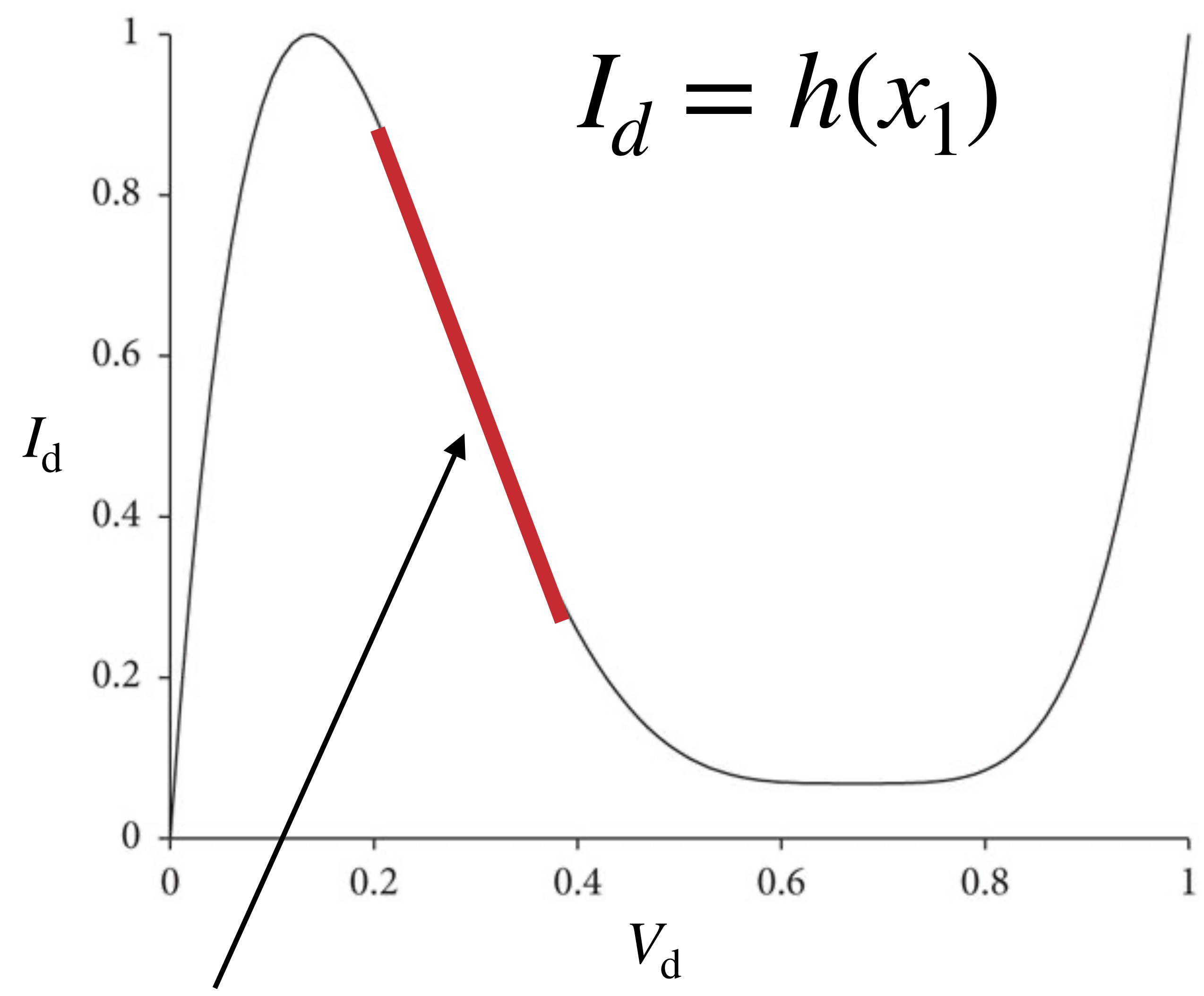


Example: Tunnel-Diode

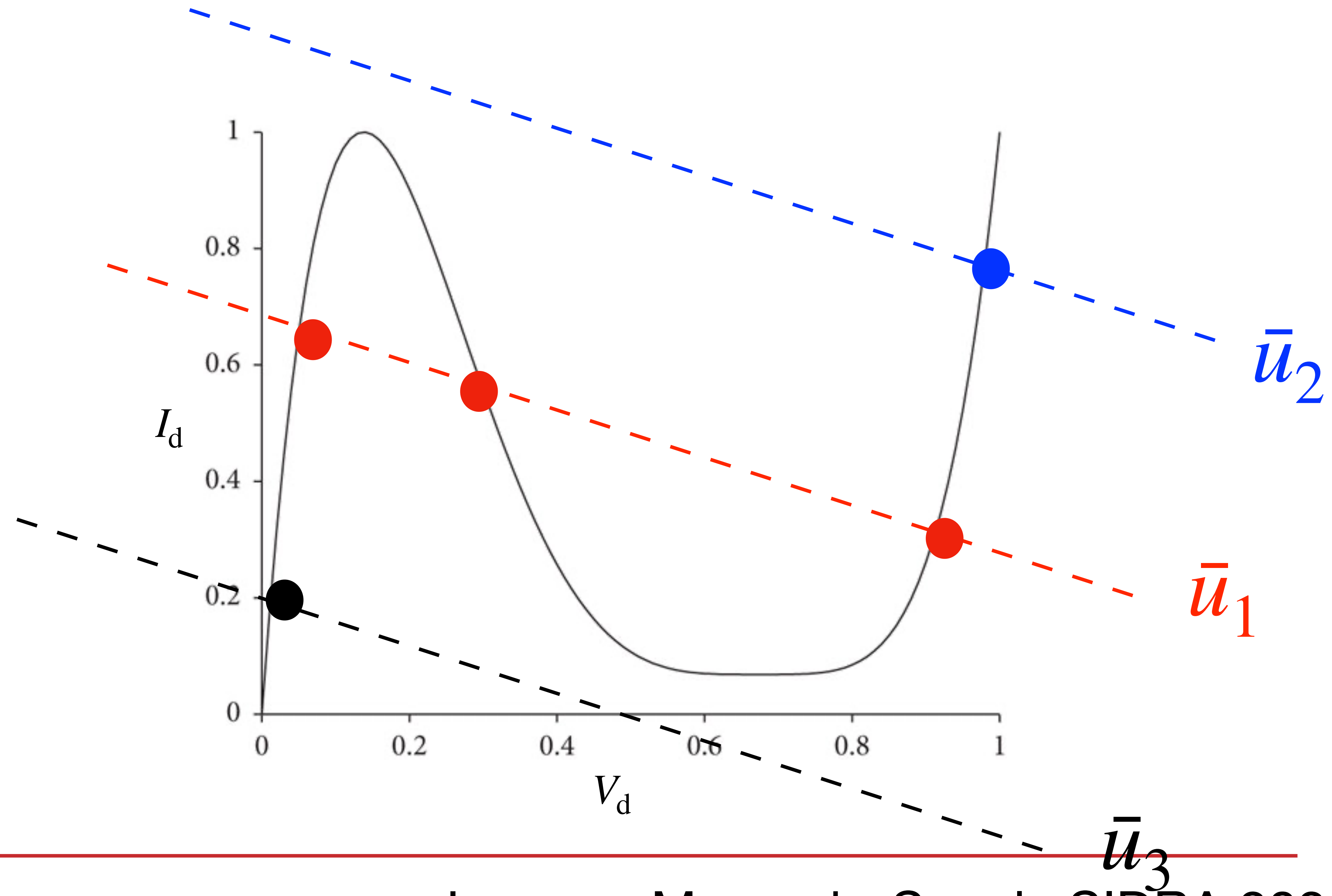


Switching circuit

$$
\begin{aligned}
C\dot{x}_1 &= x_2 - h(x_1) \\
L\dot{x}_2 &= -x_1 - Rx_2 + u
\end{aligned}
$$

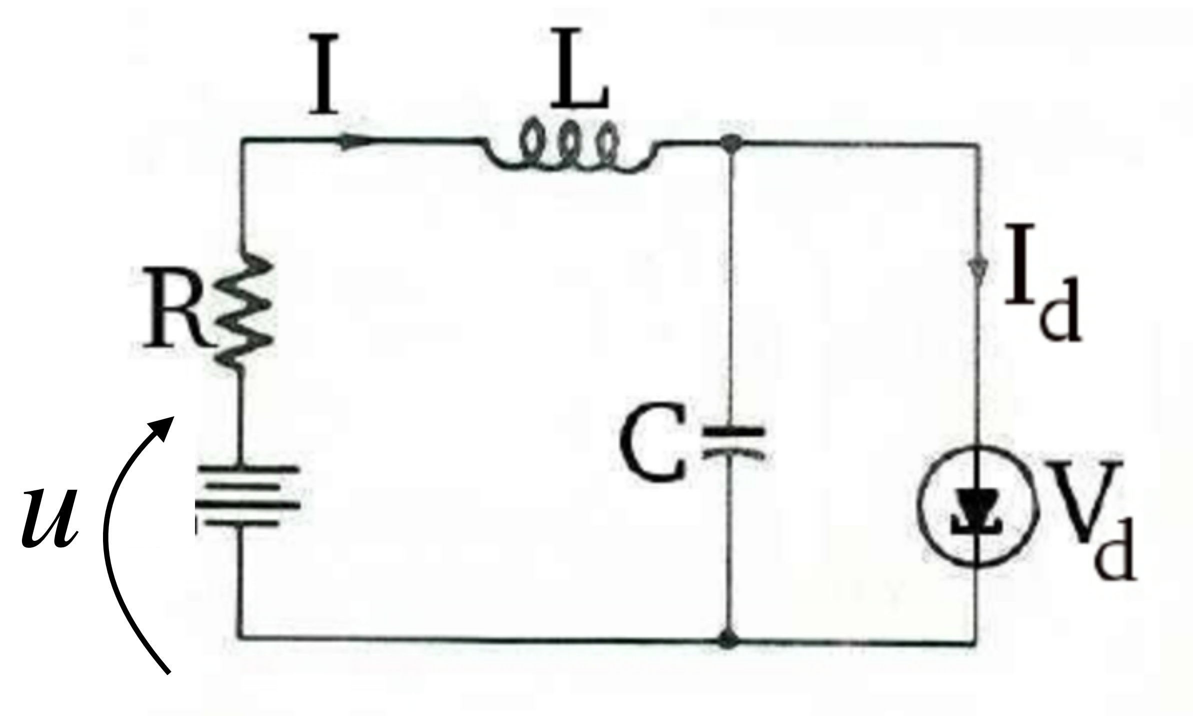


The "negative" differential resistance makes tunnel diodes key elements in circuits like oscillators and switching circuits.

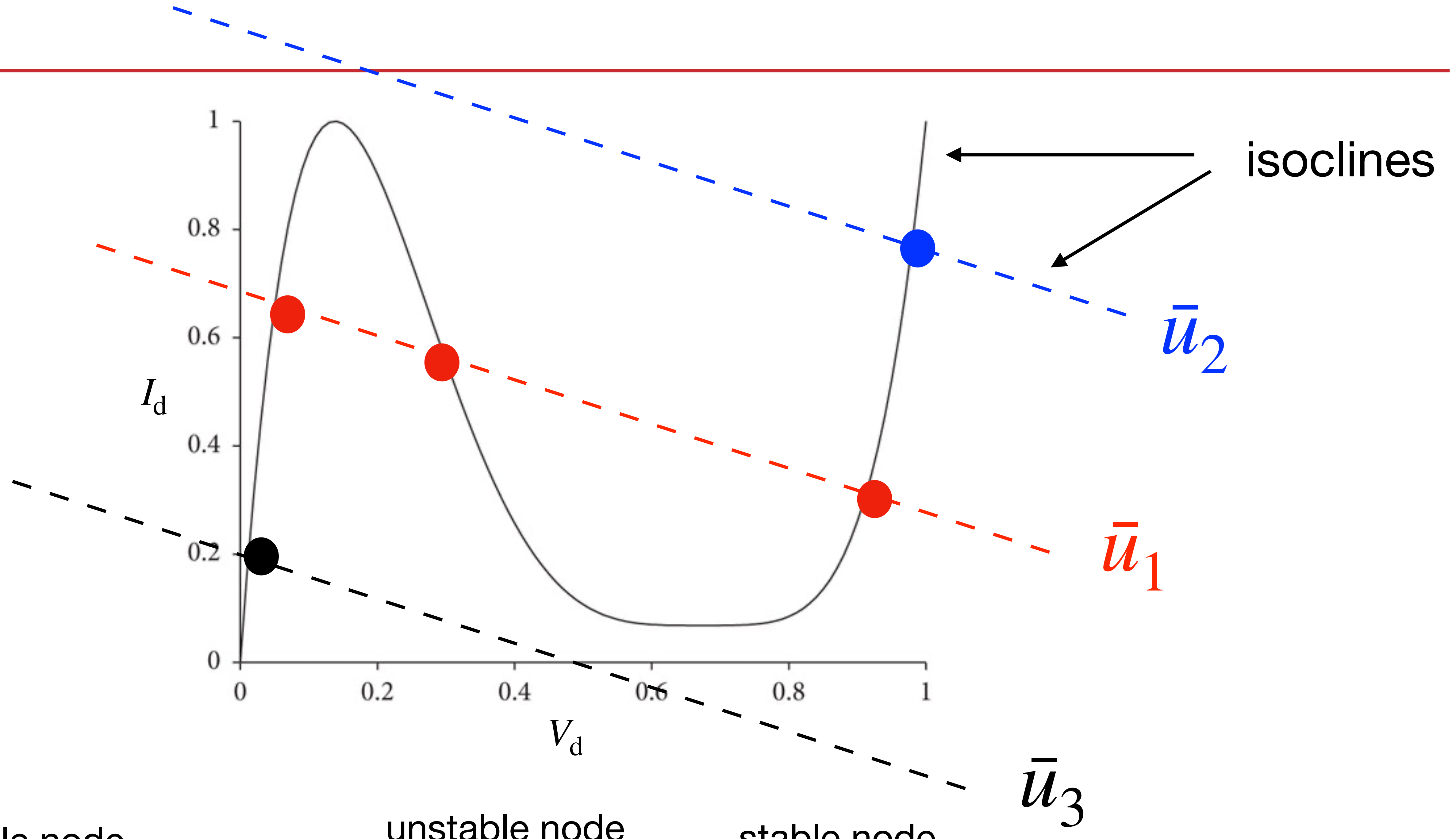


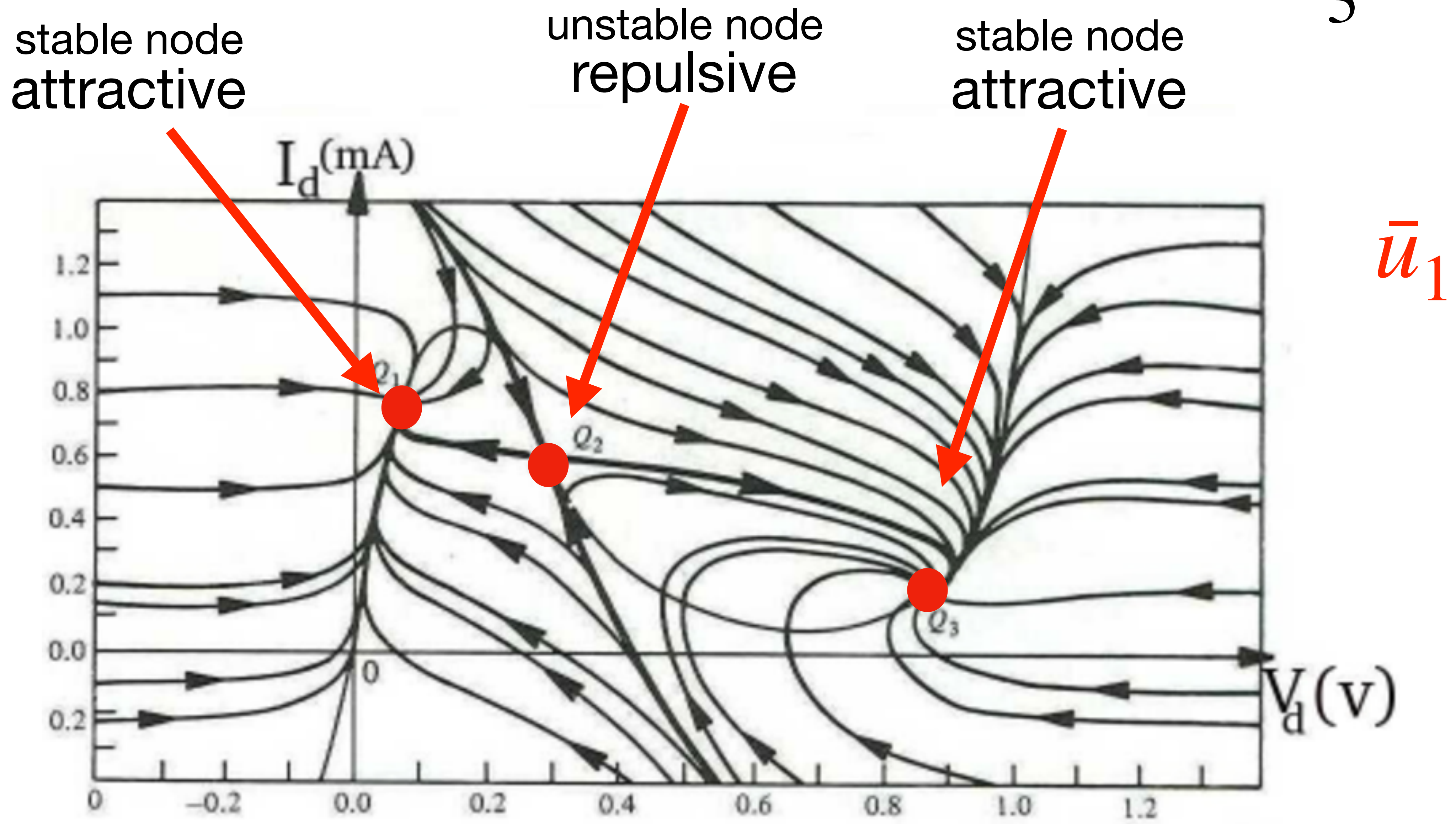
Qualitative Behaviour of Trajectories

Tunnel Diode example



$$C\dot{x}_1 = x_2 - h(x_1)$$
$$L\dot{x}_2 = -x_1 - Rx_2 + u$$



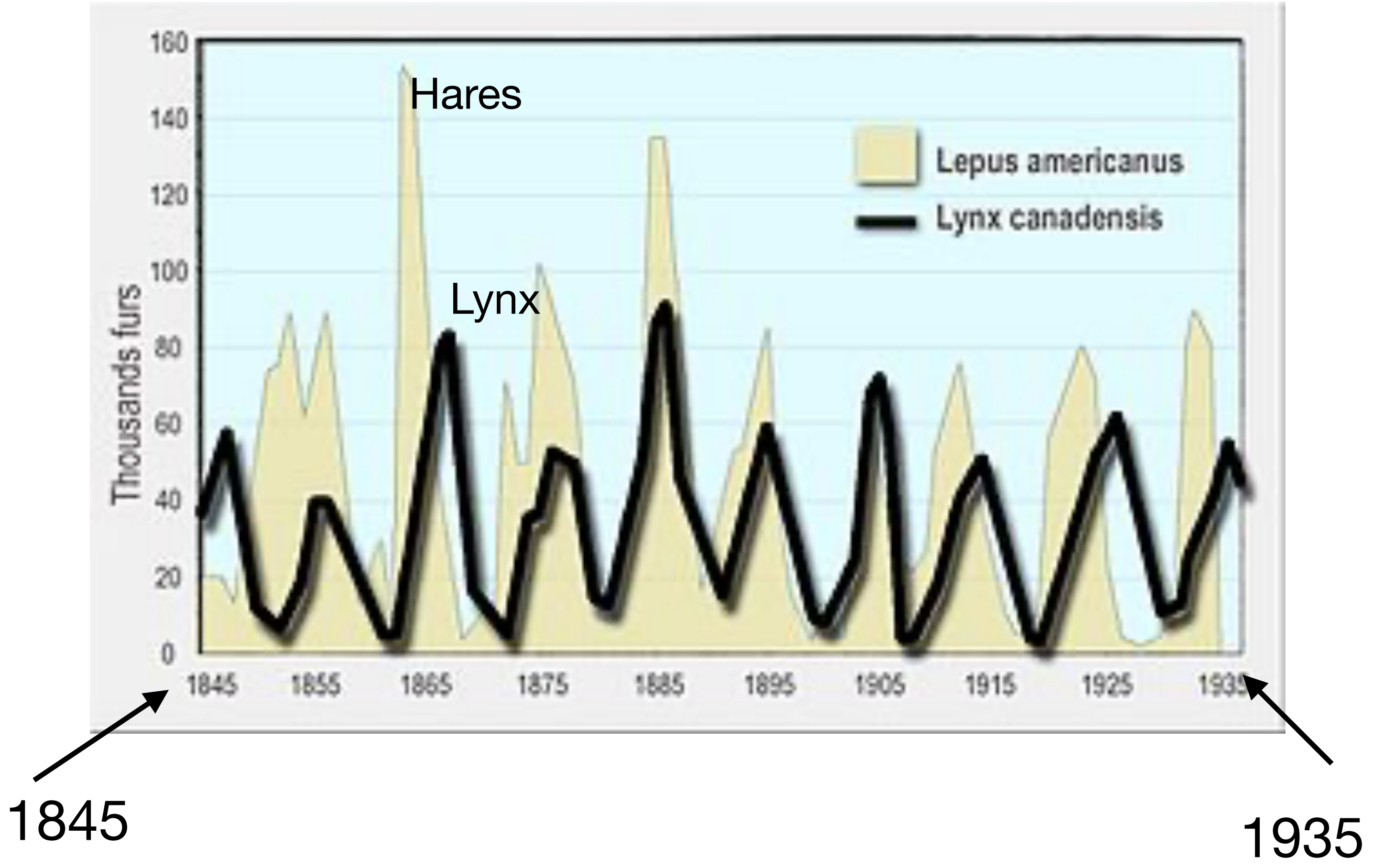


Homework: plot the phase portrait in the other two cases of $\bar{u}_2, \bar{u}_3$

Example: Predator-Pray

Experimental data

Source: Canadian Rockies



Lotka–Volterra equations (1910)

$$\dot{x} = \alpha x - \beta x y$$
$$\dot{y} = \delta x y - \gamma y$$

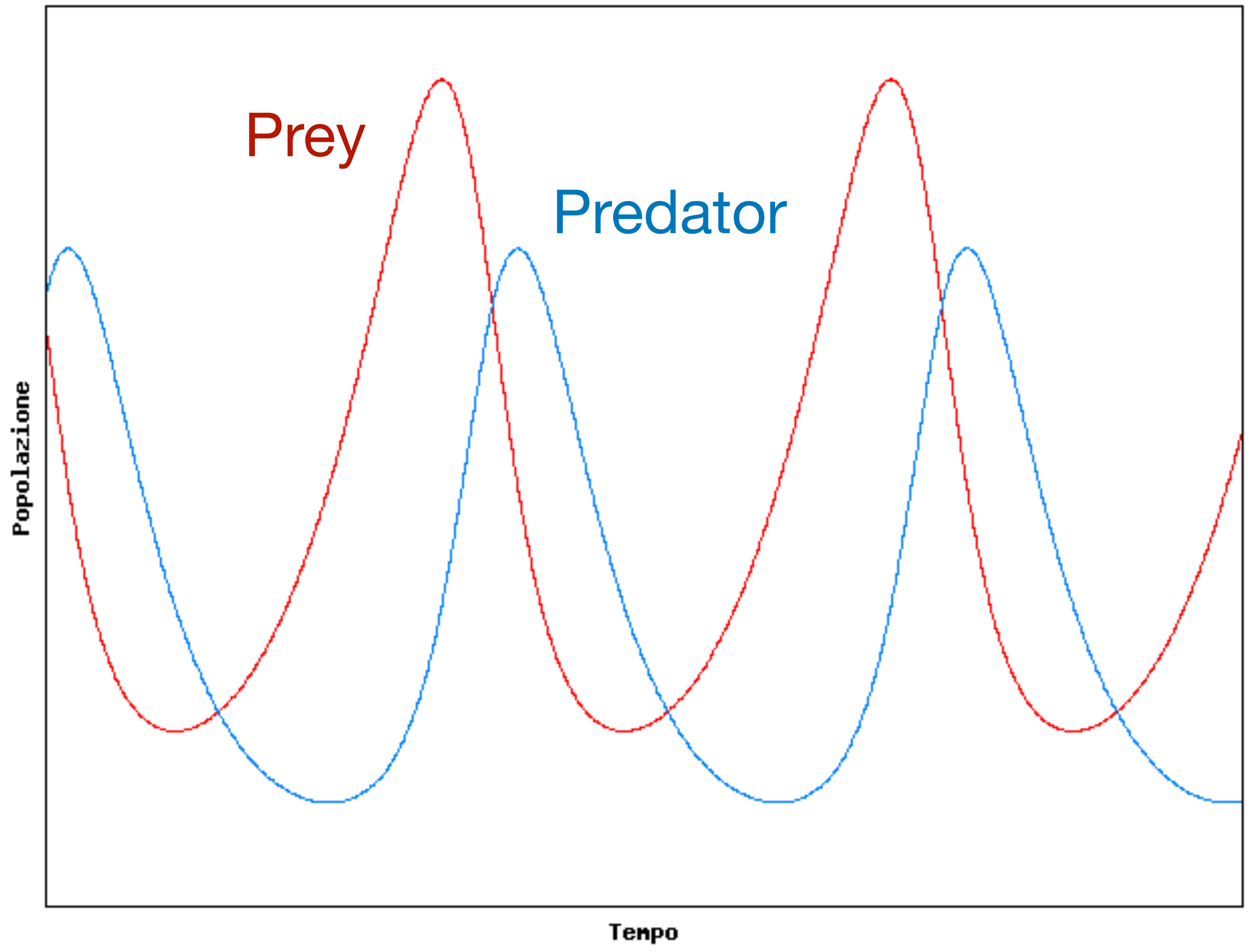
x is the number of prey

y is the number of predator

- Equilibria:
  - $(\bar{x}, \bar{y}) = (0,0)$
  - $(\bar{x}, \bar{y}) = (\frac{\gamma}{\delta}, \frac{\alpha}{\beta})$
- $x(0) = 0, y(0) \neq 0$?
- $x(0) \neq 0, y(0) = 0$?
- $x(0) \neq 0, y(0) \neq 0$?

Amplitude of oscillations depends on the initial state

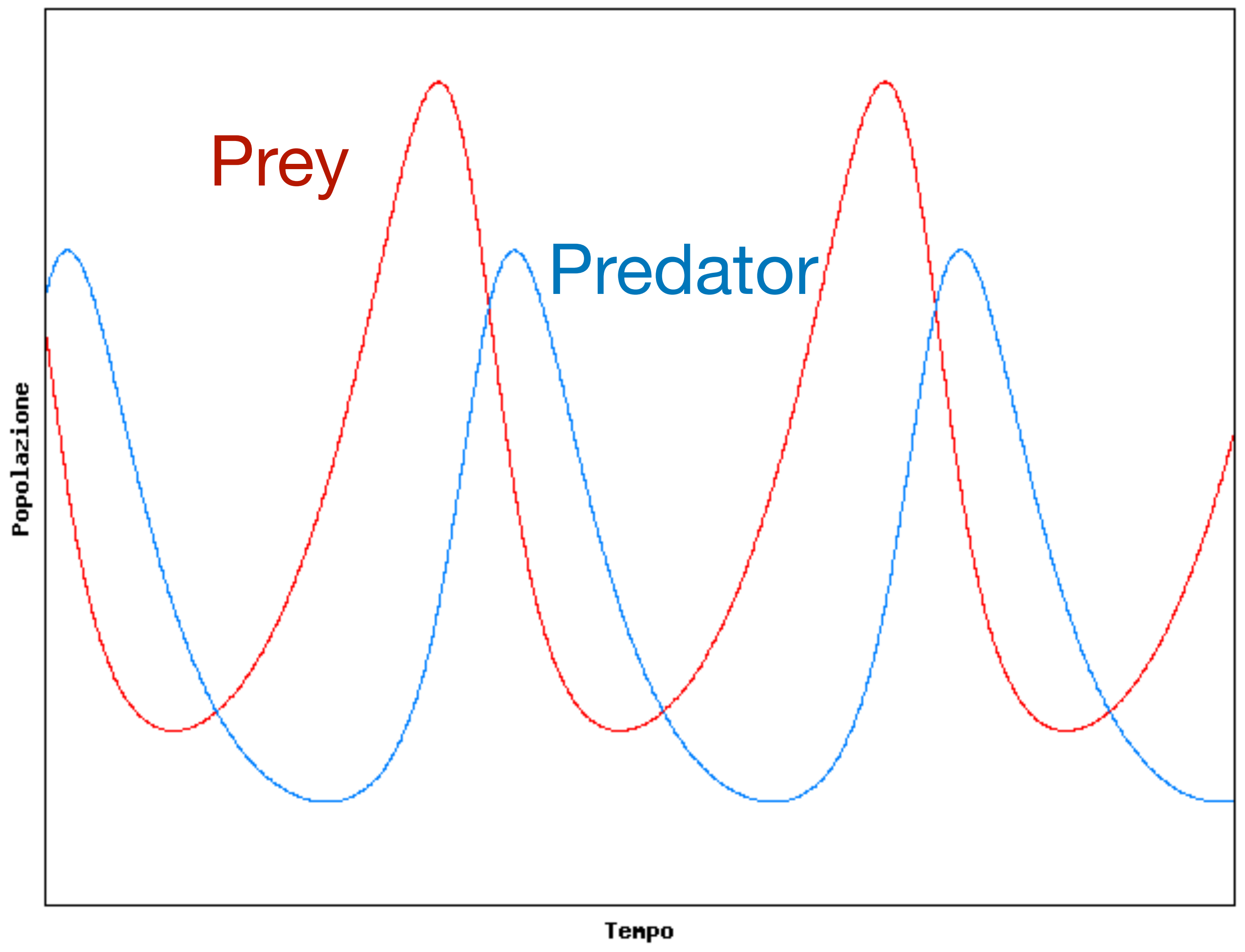
Simulation



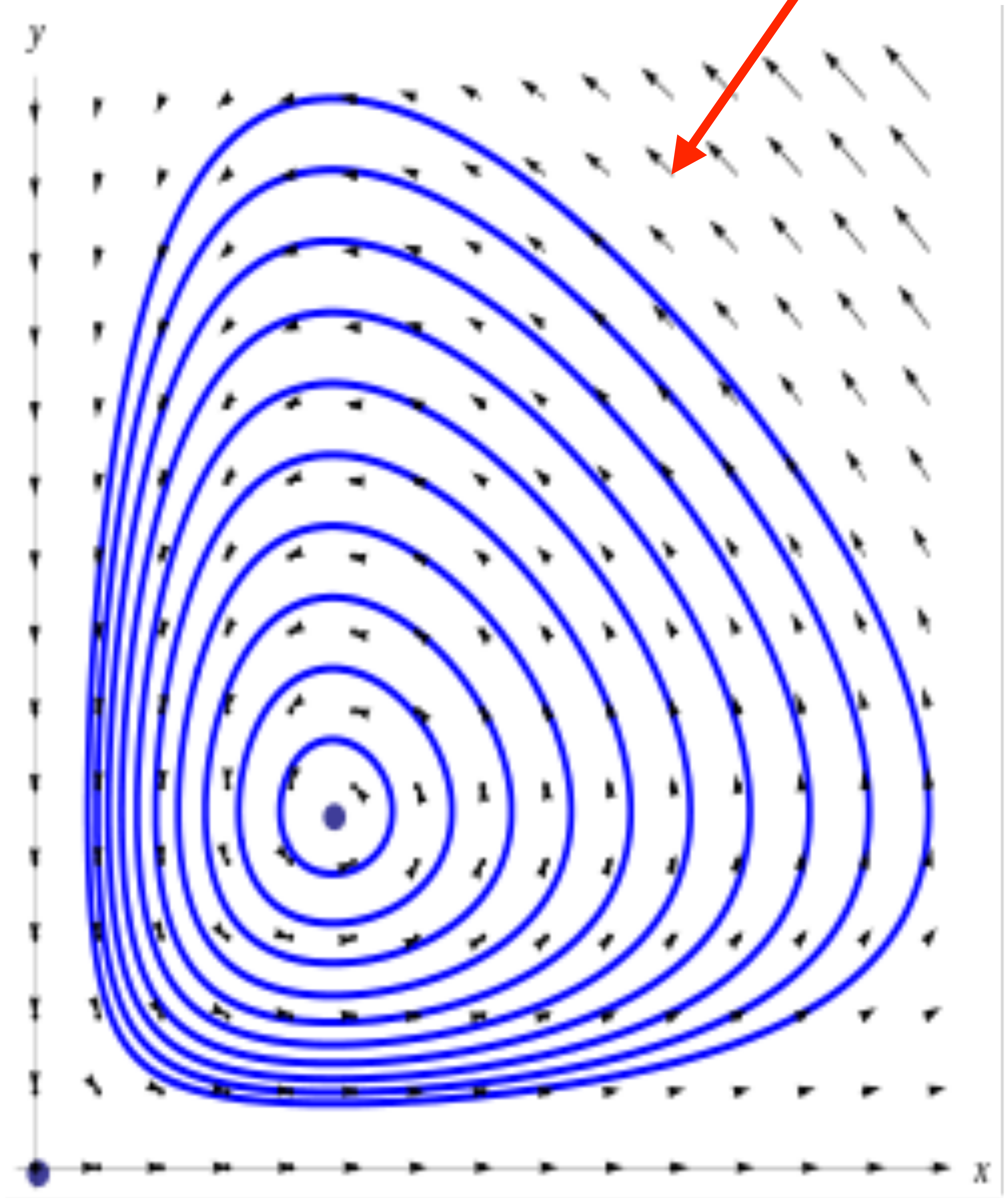
In non-equilibrium situations, predators thrive when there is abundance of prey but, in the long run, there's not enough food and they die out. As the predator population decreases the prey population increases again; this dynamic continues in a cycle of growth and decline.

Predator-Prey system

$$\dot{x} = \alpha x - \beta x y$$
$$\dot{y} = \delta x y - \gamma y$$



Dense periodic trajectories
(they are not limit cycles)



Example: Predator-Prey

More sophisticated/realistic models consider α, β, δ as function of x according to the following rules:

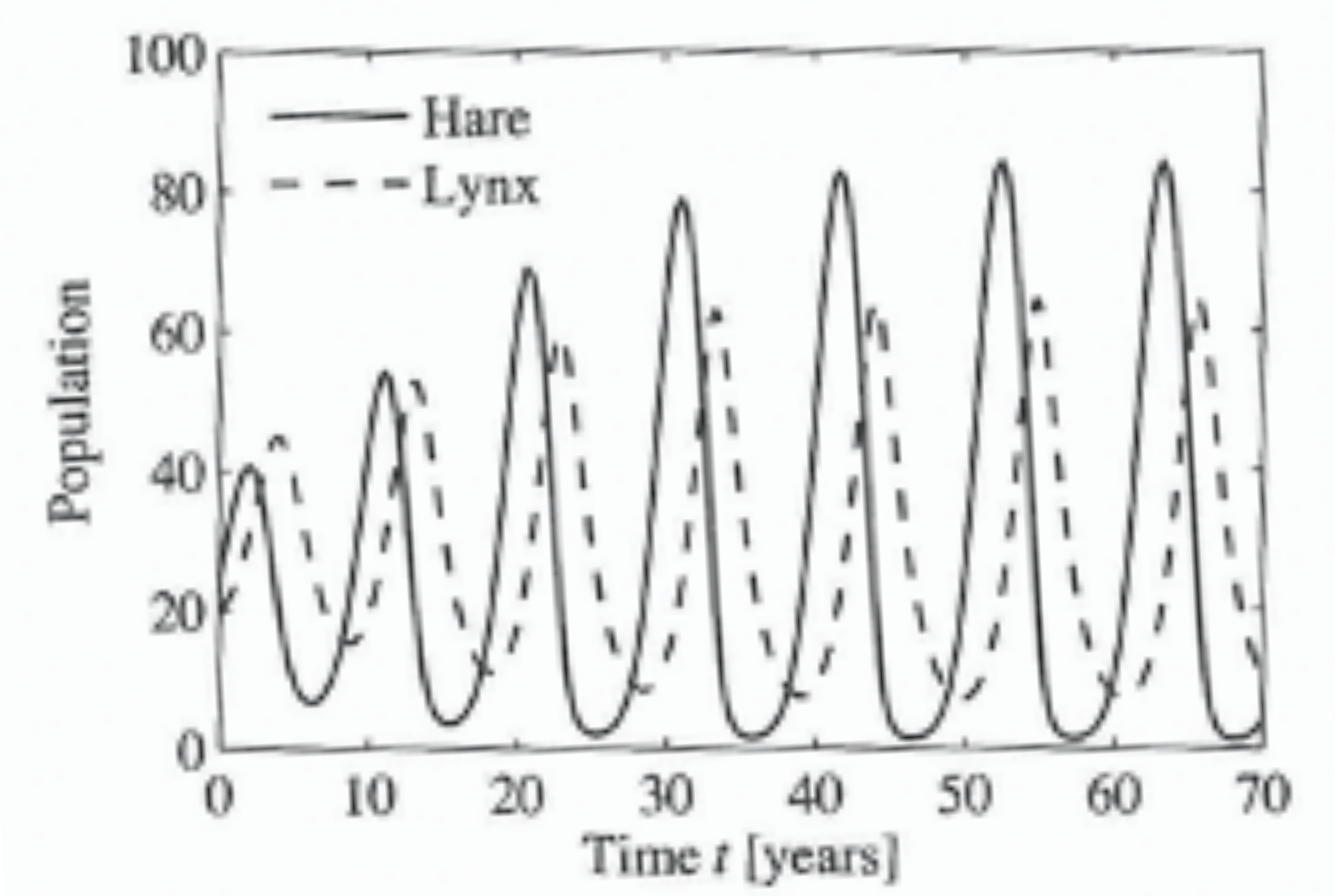
$\alpha(x) = r\,(1 - \dfrac{x}{k})$ namely, the prey growth rate decreases (possibly negative) when x increases ($r, k > 0$)

$\beta(x) = \dfrac{a}{c+x}$ namely, the predation rate decreases when x increases (effect of a population dominating on the other regardless the mutual strength) ($a, c > 0$)

$\delta(x) = b\,\beta(x)$ similarly as above ($b > 0$)

$$\dot{x} = r\,x\,(1 - \frac{x}{k}) - \frac{a\,x\,y}{c+x}$$
$$\dot{y} = b\frac{a\,x\,y}{c+x} - d\,y$$

Amplitude of oscillations **does not** depend on the initial state

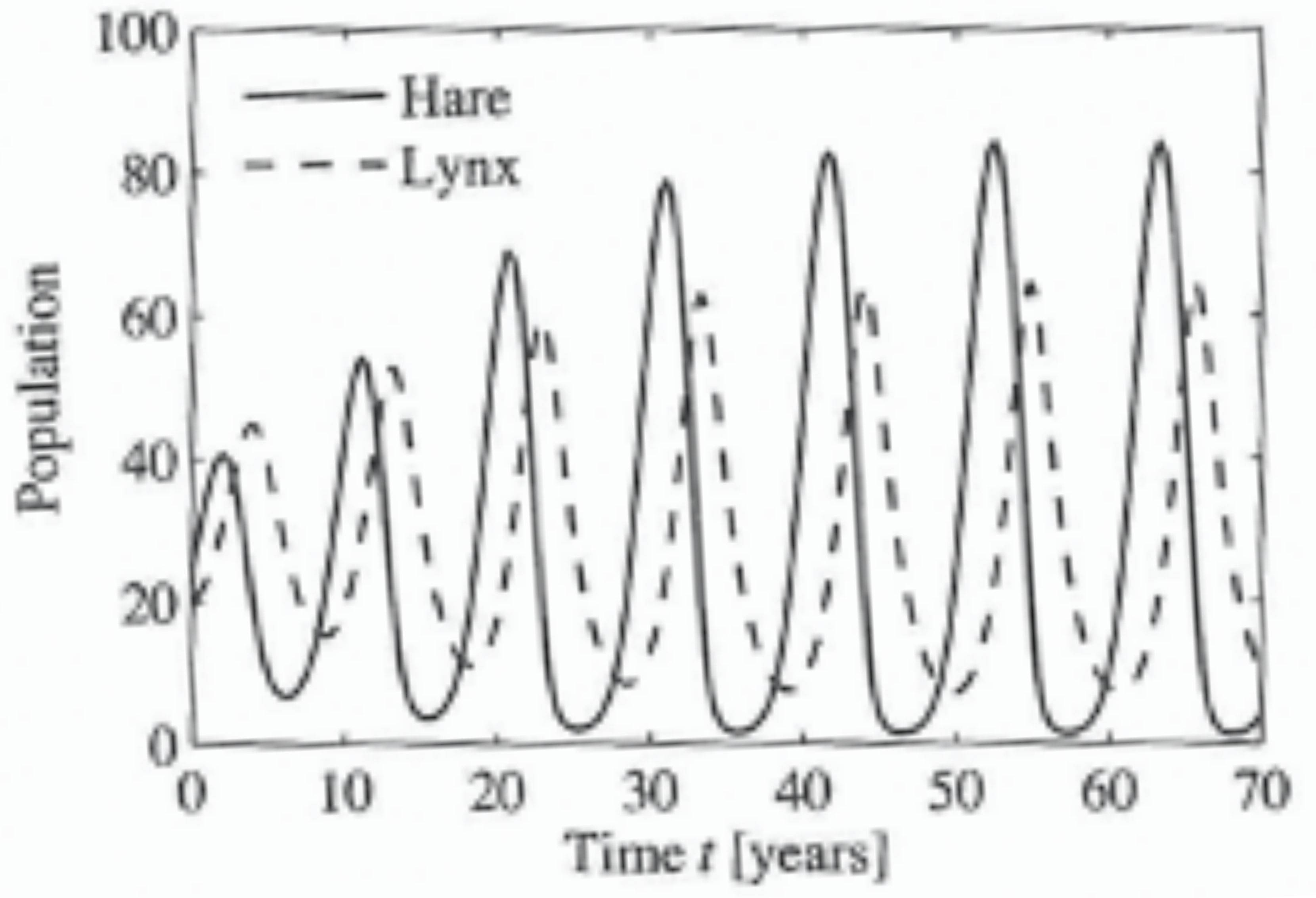


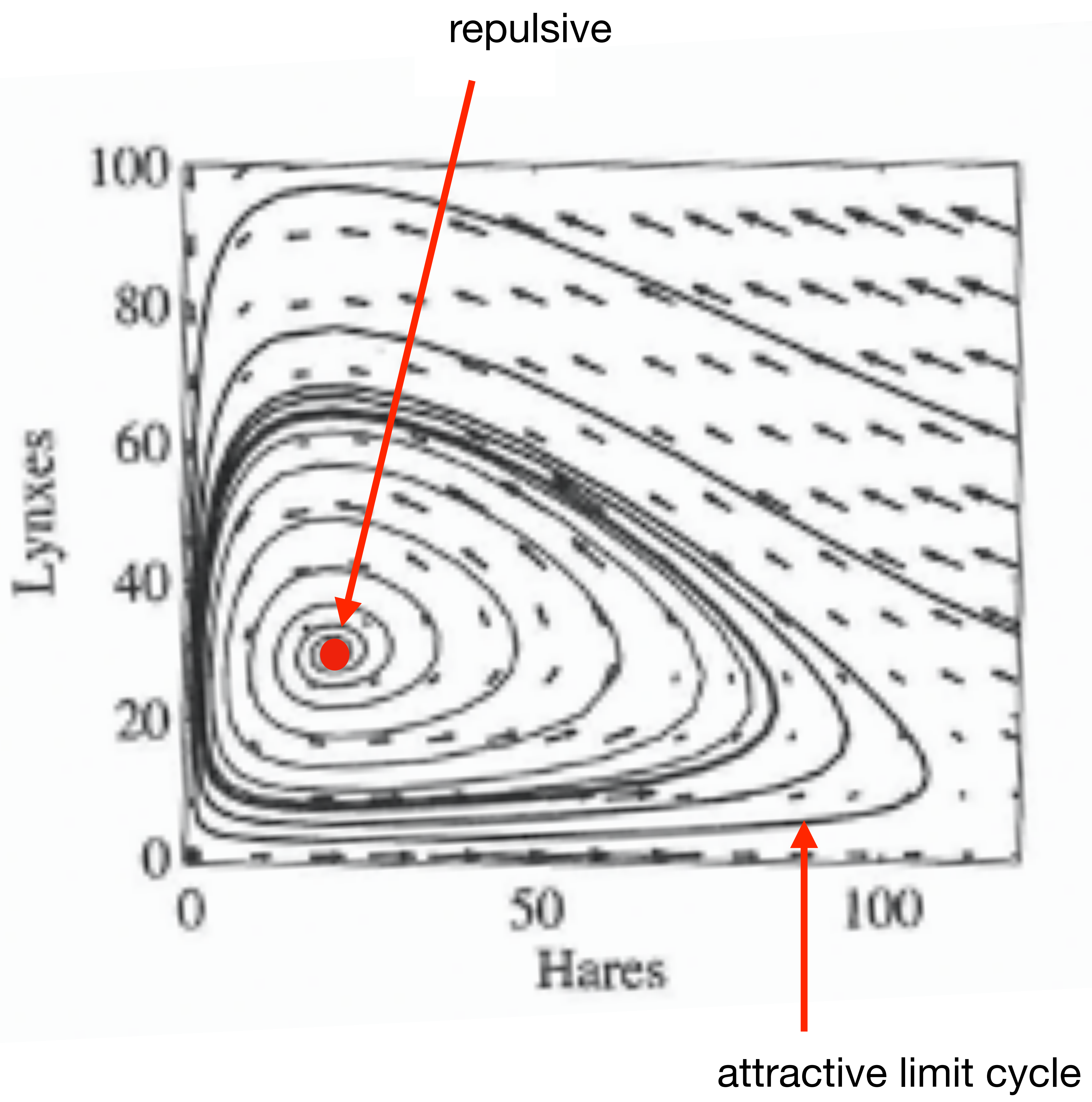
$a = 3.2, b = 0.6, c = 50, d = 0.56, k = 125, r = 1.6$

Predator-Prey system

$$\dot{x} = r\,x\,(1 - \frac{x}{k}) - \frac{a x\, y}{c + x}$$

$$\dot{y} = b\frac{a x\, y}{c + x} - d\,y$$



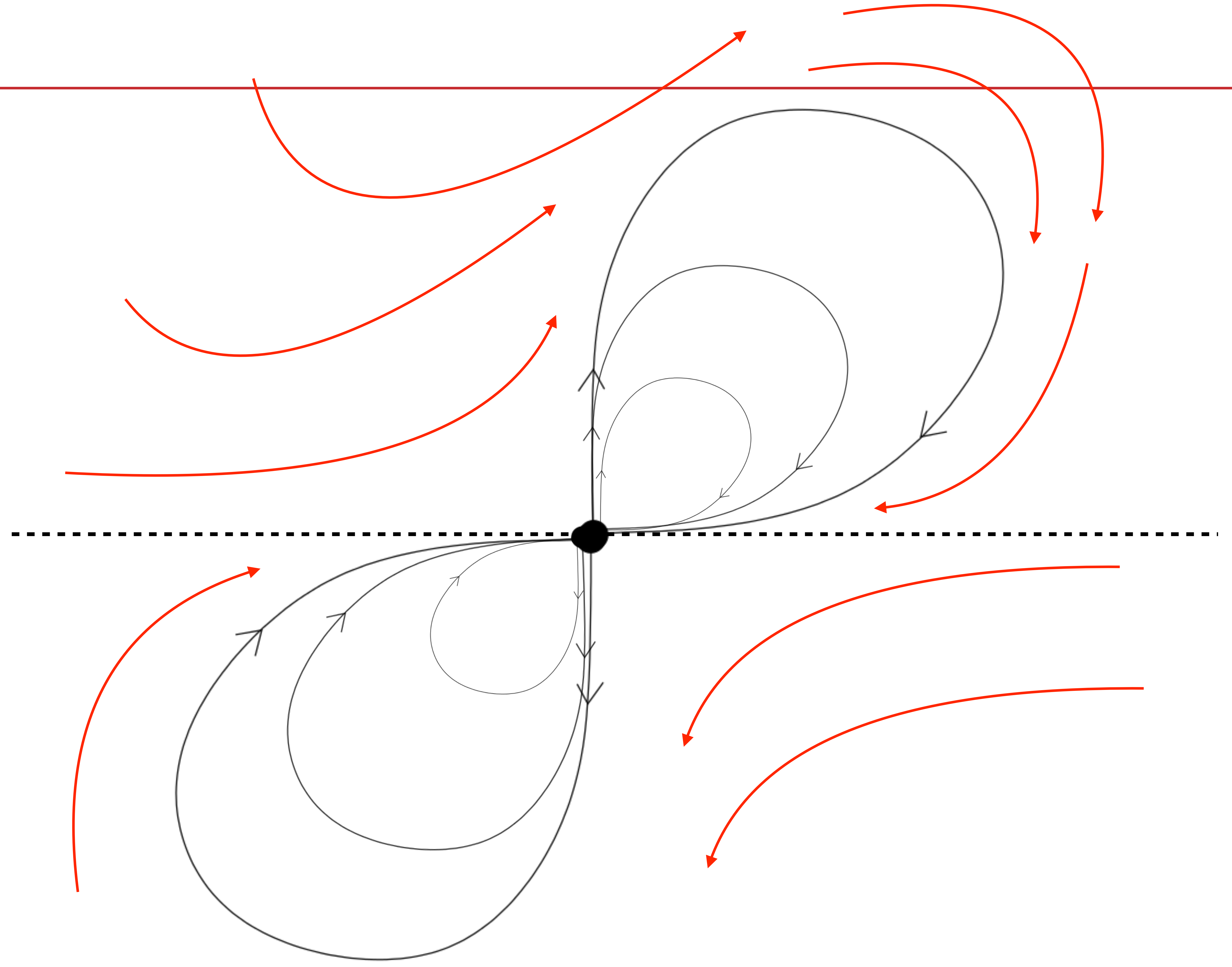


Example of a system with homoclinic orbit

The Vinograd's example (1957)

$$\dot{x} = \frac{x^2(y-x)+y^5}{(x^2+y^2)(1+(x^2+y^2)^2)}$$

$$\dot{y} = \frac{y^2(y-2x)}{(x^2+y^2)(1+(x^2+y^2)^2)}$$

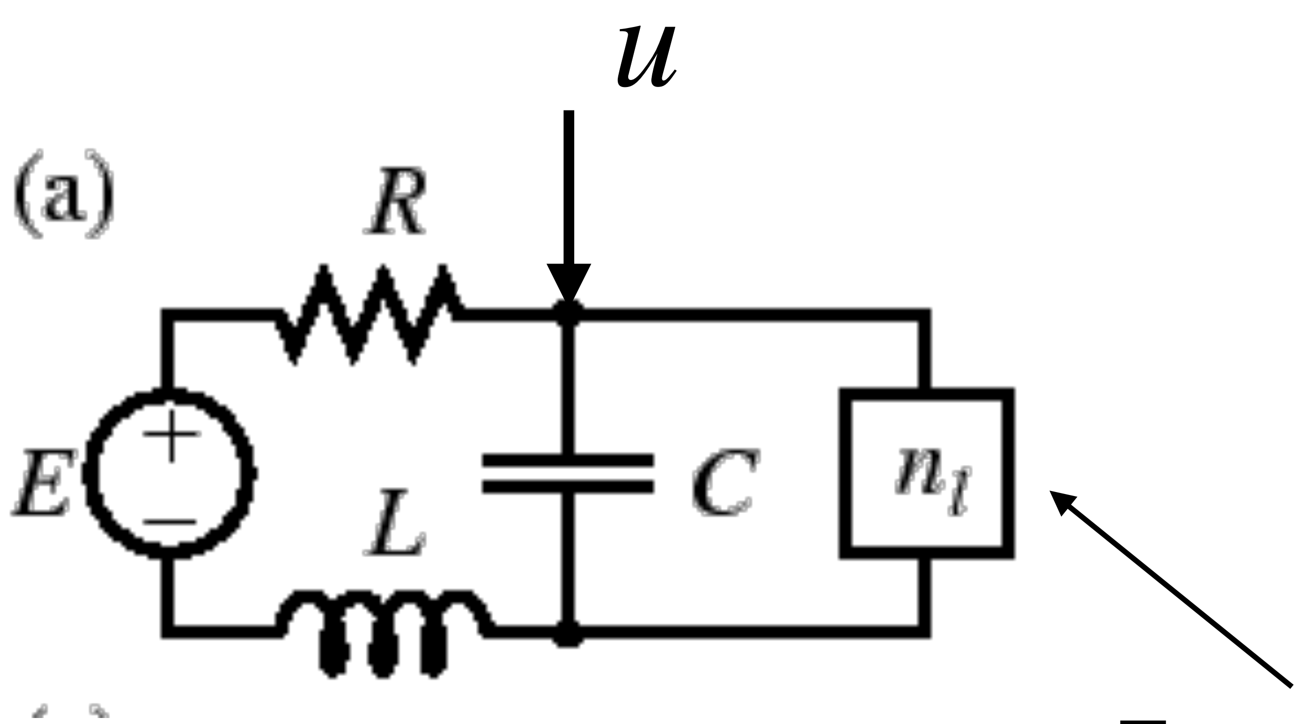


The origin is the only equilibrium point. It is "globally attractive". From an engineering perspective such an equilibrium point is not "nice" (small perturbations of the initial state from the equilibrium lead to transients whose amplitude is not related to the amplitude of the perturbation)

Example: FitzHugh-Nagumo circuit

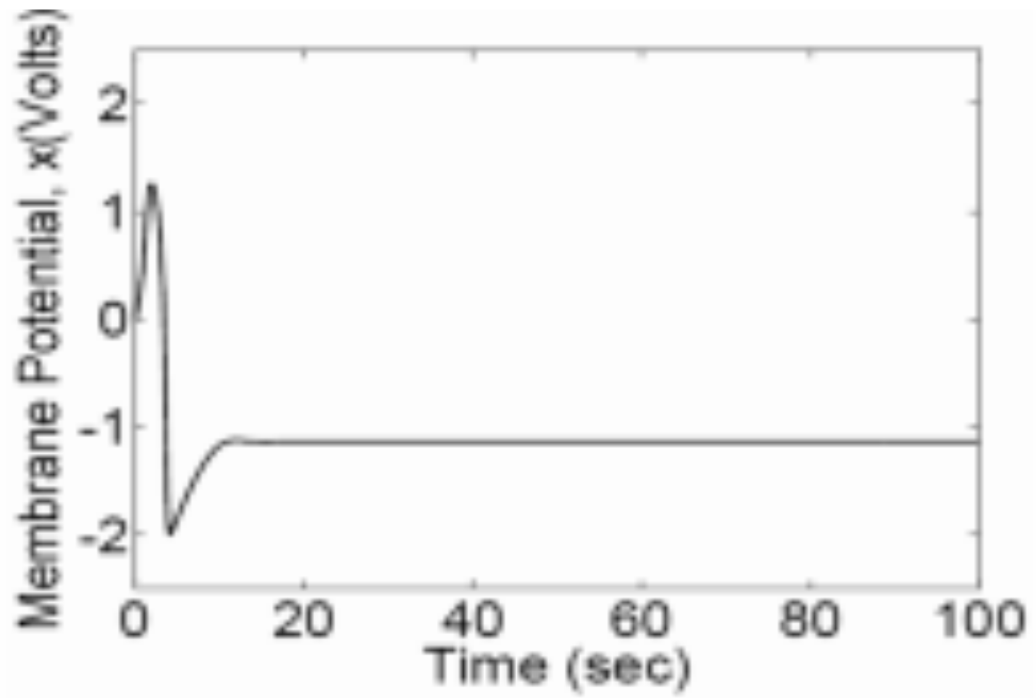
Prototype of relaxation oscillator (when the activation input u exceeds a certain threshold the state variables exhibit a characteristic oscillatory behaviours). The circuit is a relevant benchmark able to capture typical **spike behaviours** observed in neurons after stimulation by an external input current

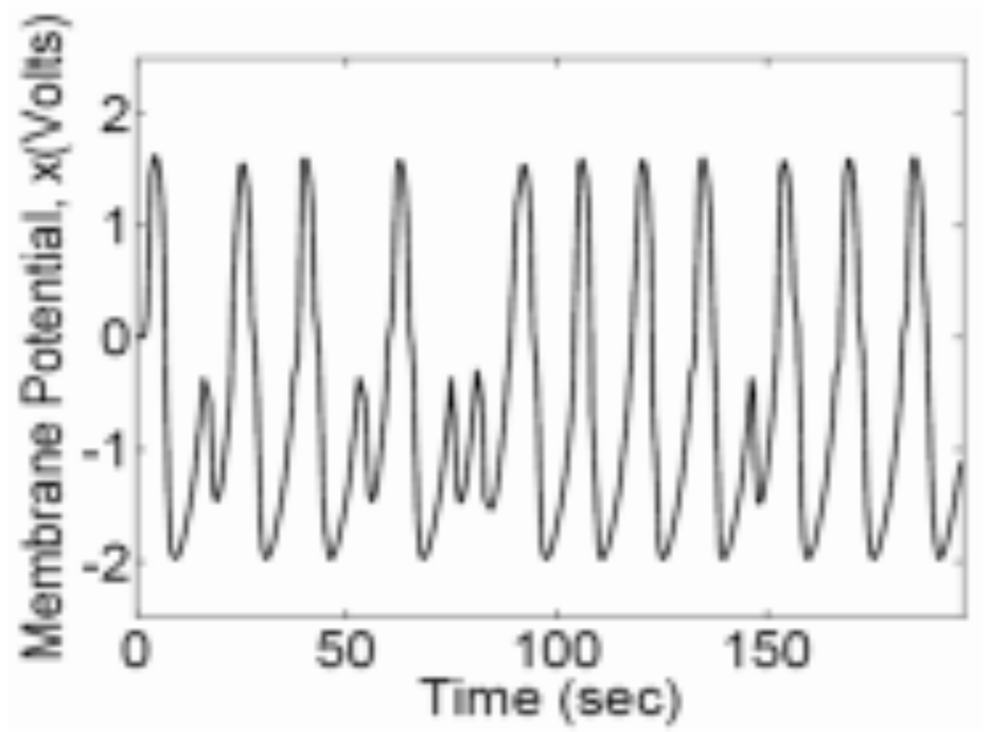
$$
\begin{aligned}
C\dot{V} &= \kappa V - \frac{V^3}{3} - I + u \\
L\dot{I} &= -I + RV
\end{aligned}
$$



Tunnel diode

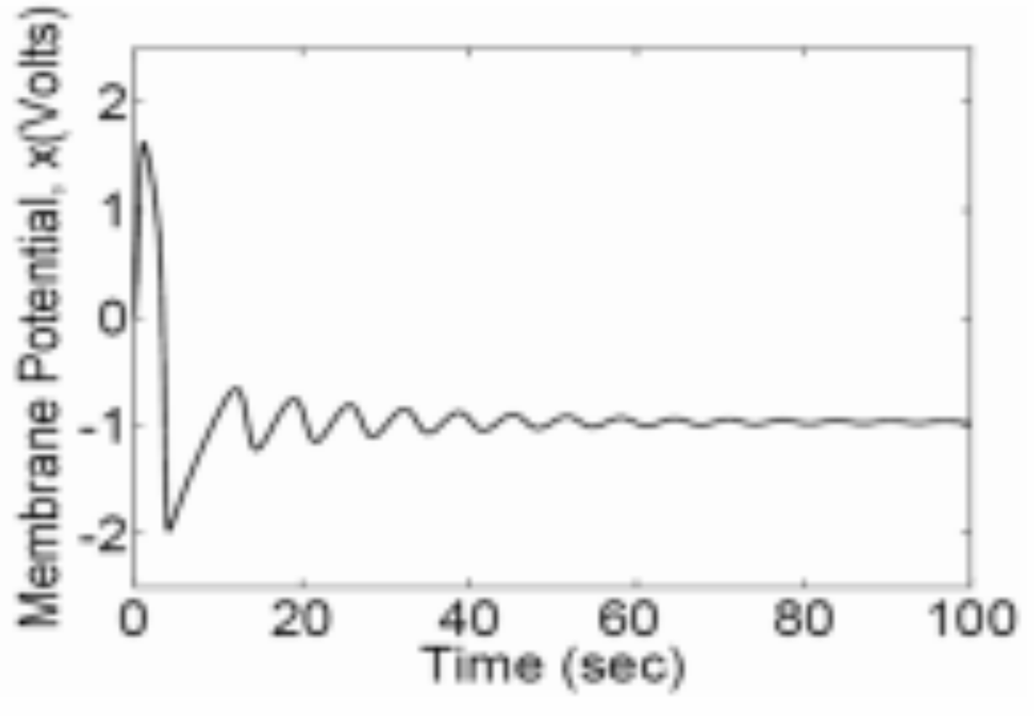
$u = 0.1$

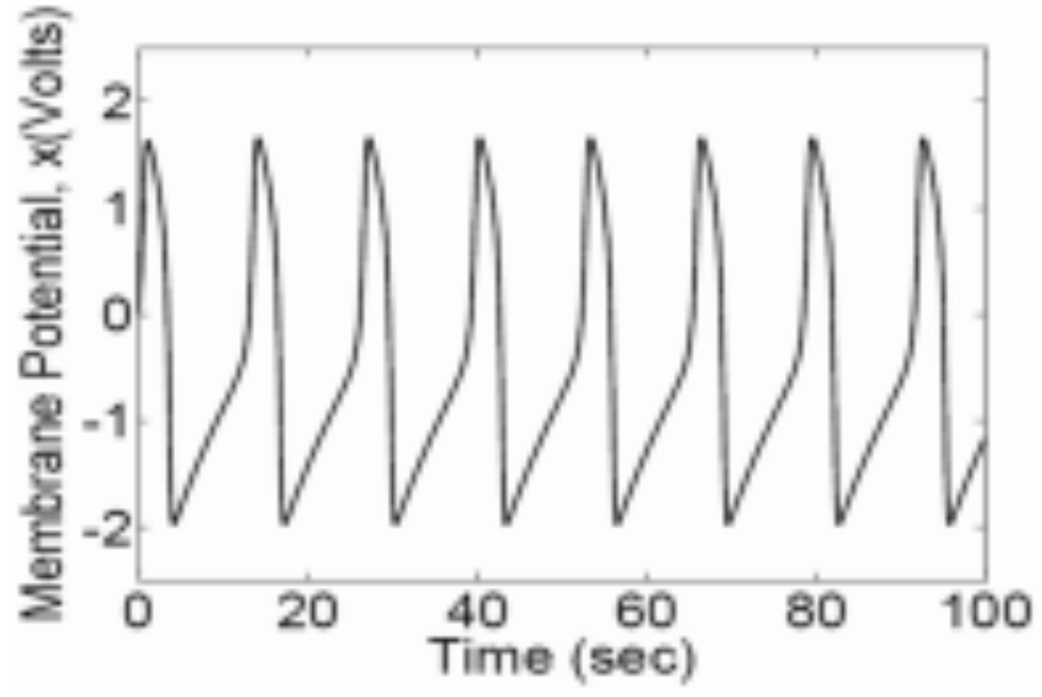




$u = 0.337$

$u = 0.33$





$u = 0.34$

Homework: plot the phase portrait for the different values of u

Example: Susceptible-Infected-Resolved Model (COVID'19)

A simple model for epidemics spread (**SIR**):

$$\dot{S} = -\beta S I$$
$$\dot{I} = \beta S I - \gamma I$$
$$\dot{R} = \gamma I$$

$$S(0) + I(0) + R(0) = 1$$

where:

- S = percentage of **susceptible** individuals (not infected but that can be infected)
- I= percentage of **infected** individuals
- R = percentage of **resolved** individuals (dead or recovered)

and:

- β = average number of contacts per person per time $\times$ infection probability per contact
- γ = recovery rate (e.g., γ= (average duration of infection)$^{-1}$)

- from $dS = -(\beta SI)dt$ we conclude that βSI is the average percentage of susceptible individuals that become infected at each unit of time (taken from S and added to I)
- from $dI = (\beta SI)dt - \gamma I dt$ we conclude that γI is the average percentage of infected that heal or die per unit of time (taken from I and added to R)

Epidemic indicators

Basic reproductive number: $R_0 \doteq \dfrac{\beta}{\gamma}$

Effective reproductive number: $R_{eff}(t) \doteq R_0 S(t) = \dfrac{\beta}{\gamma} S(t)$

Example: Susceptible-Infected Model (COVID'19)

Let us neglect the R dynamics (it is only an accumulator, does not affect S or I)

$$\dot{S} = -\beta S I \qquad\qquad S(0) + I(0) = 1$$
$$\dot{I} = \beta S I - \gamma I$$

$$\begin{bmatrix} \dot{S}(t) \\ \dot{I}(t) \end{bmatrix} = \underbrace{\begin{bmatrix} -\beta I(t) & 0 \\ 0 & \beta S(t) - \gamma \end{bmatrix}}_{A(t)} \begin{bmatrix} S(t) \\ I(t) \end{bmatrix}$$

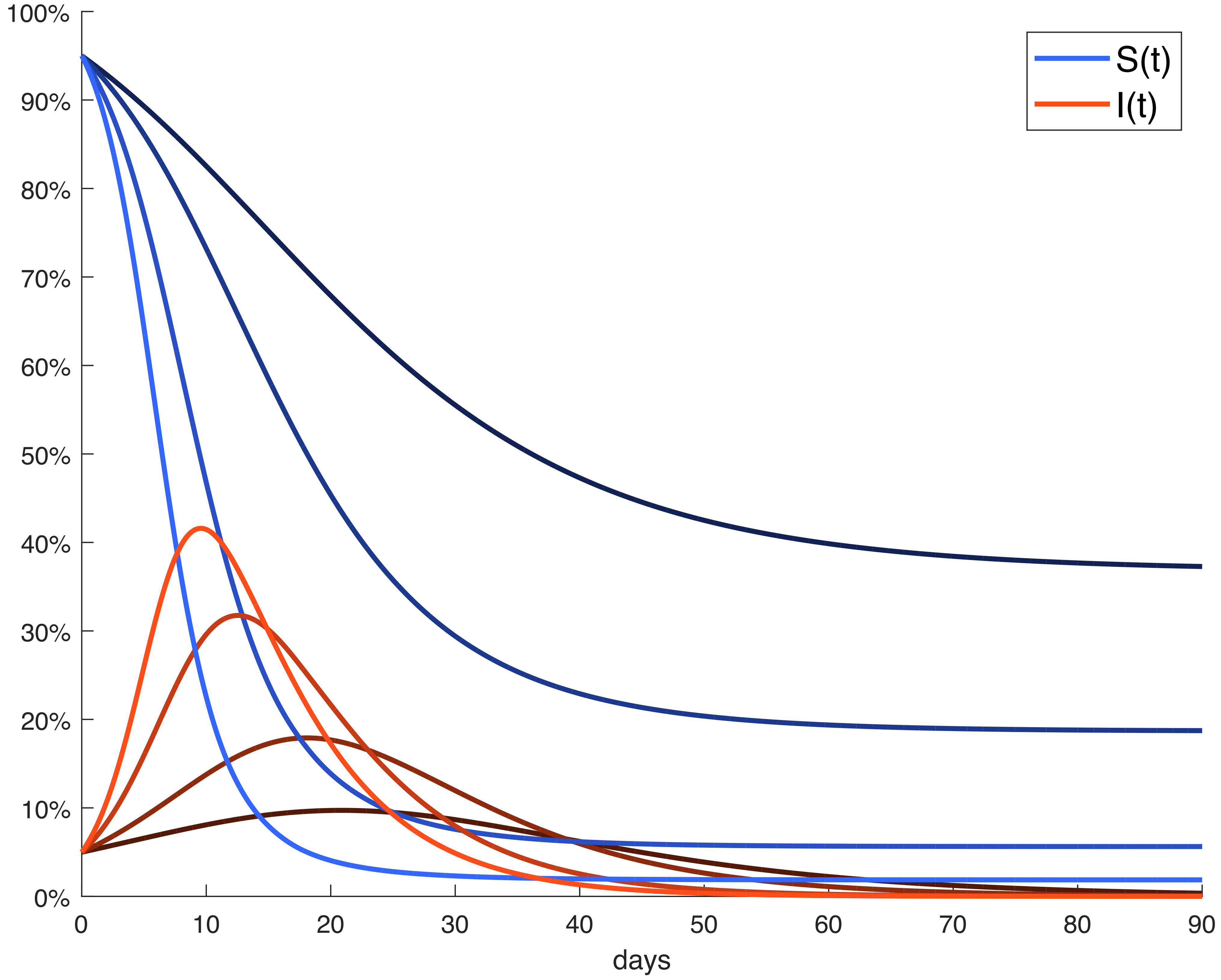
- The term in the first dynamics ,$-\beta S(t)I(t)$, is always negative (the susceptible population always decreases)

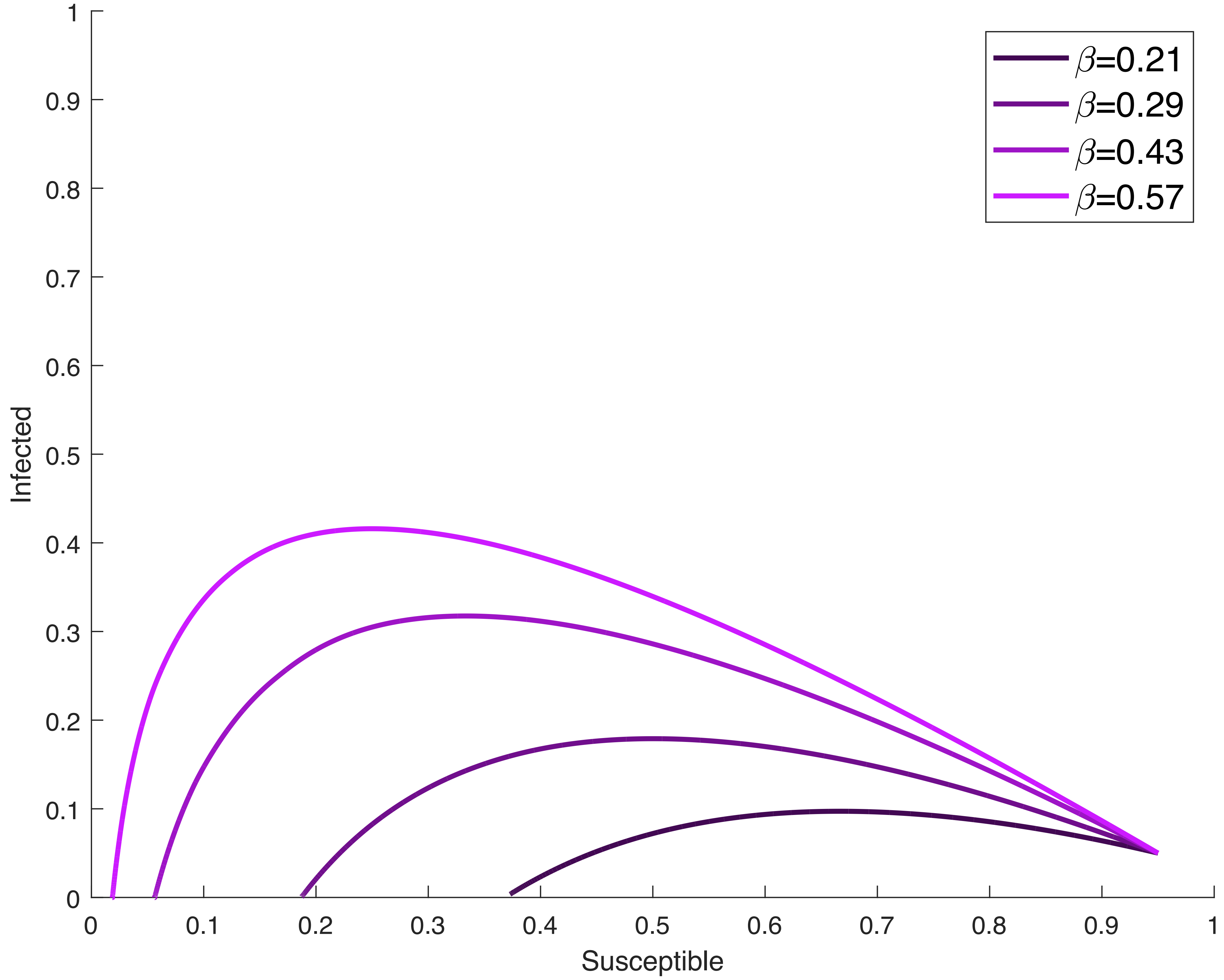
- The second is positive if $R_{eff}(t) > 1$; At the beginning of the infection ($S(0) \approx 1,\ I(0) \approx 0,\ R(0) = 0$),
$R_{eff}(0) \approx R_0 \ \rightarrow$ the disease spreads if $R_0 > 1$, otherwise it dies out! (in the early COVID case $R_0 \in [1.5,\ 4]$)

Example: Susceptible-Infected Model (COVID'19)

$$I(0) = 0.05, \;\; S(0) = 0.95 \qquad\qquad R_0 = \{1.5, 2, 3, 4\}$$

The equilibrium point reached depends on R_0 but $I_{\rm eq} = 0$ always





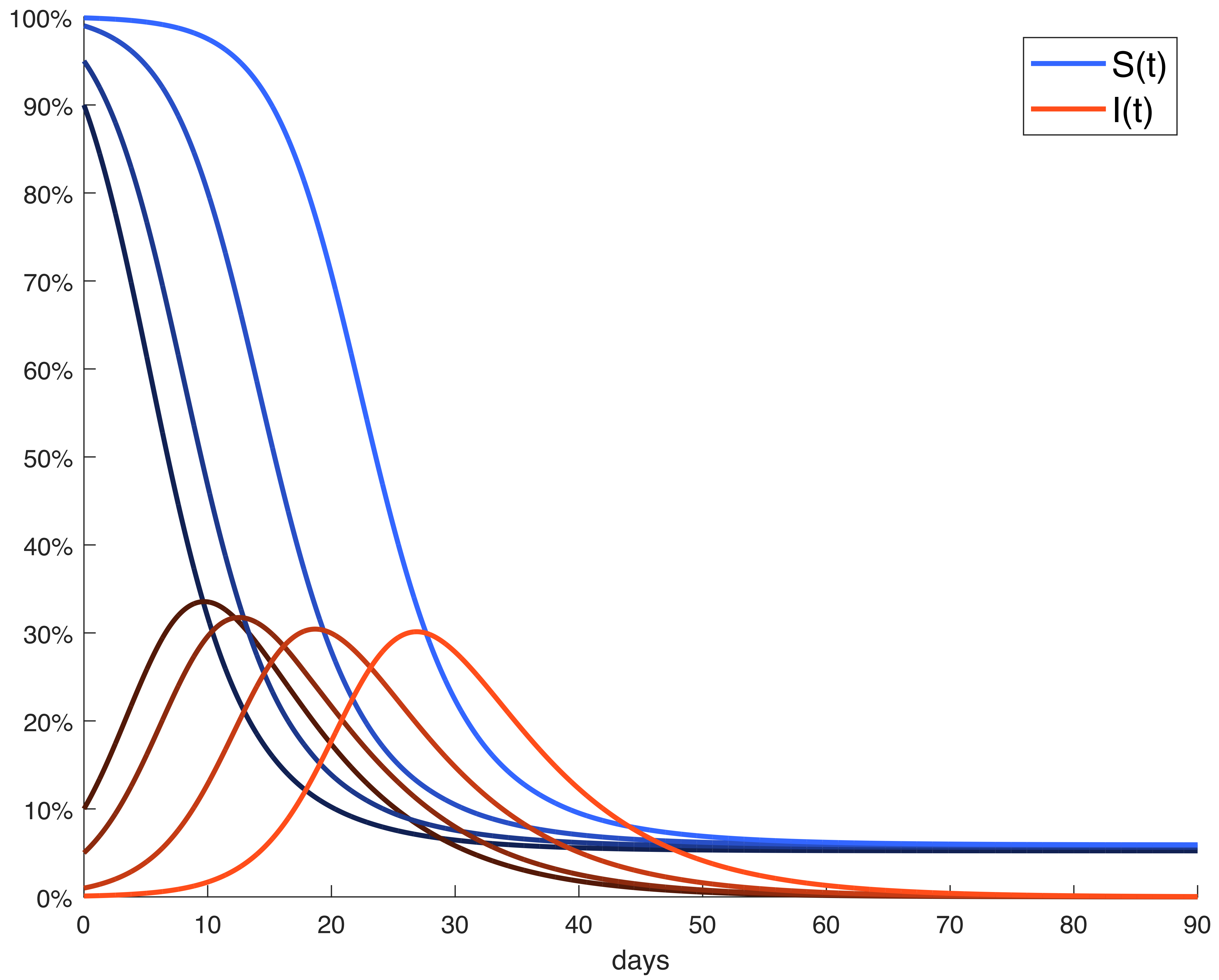
Example: Susceptible-Infected Model (COVID'19)

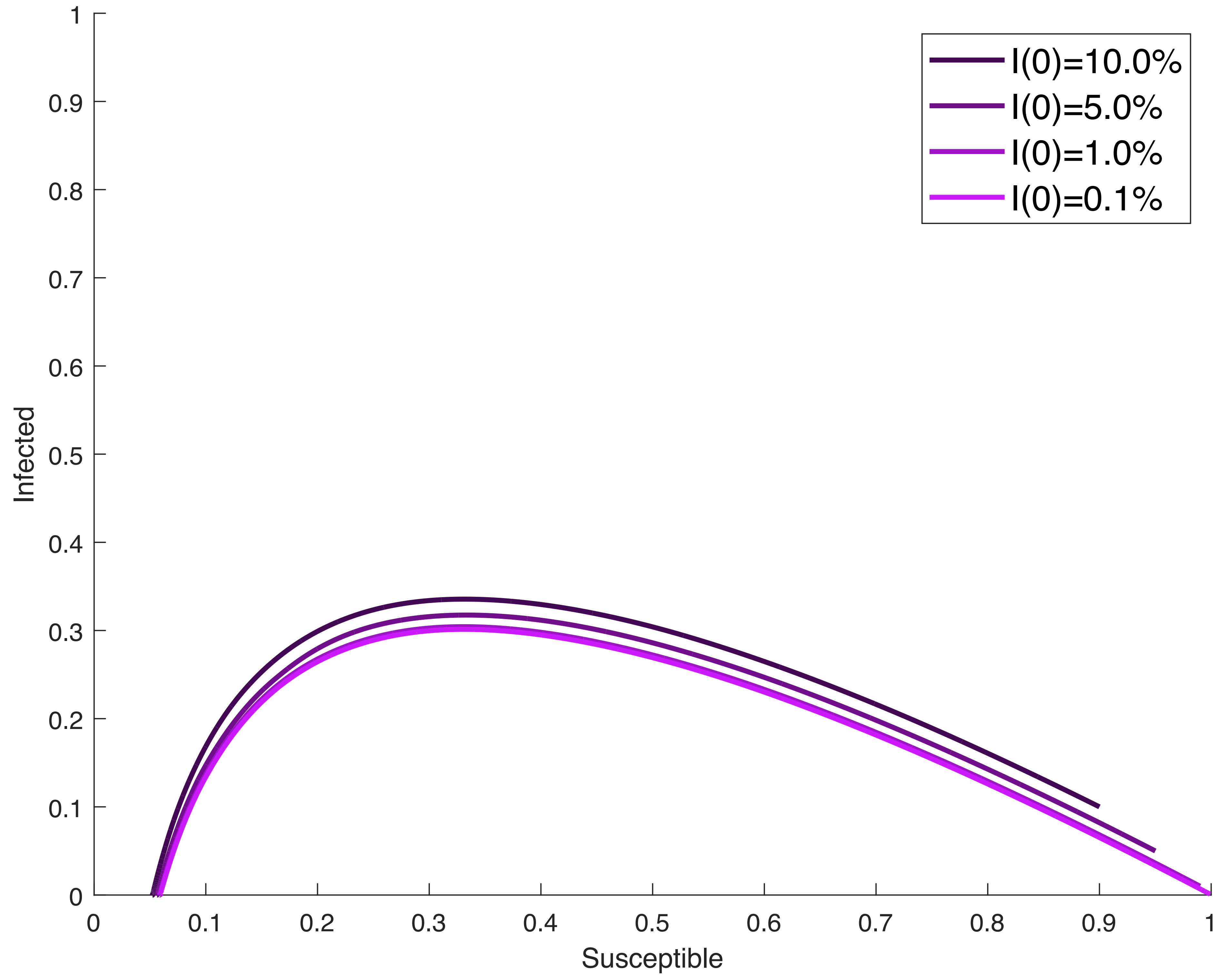
$$I(0) = \{0.1, 0.05, 0.01, 0.001\}$$
$$S(0) = 1 - I(0)$$

$$R_0 = 3$$

The equilibrium point reached does not depend on $I(0)$

Smaller $I(0)$ only delay the peak but its amplitude remains the same





Example: Susceptible-Infected Model (COVID'19)

A more realistic model (Reinfections and Variable Rate)

$$\dot{S} = \beta(t)\,S\,I + \delta R$$
$$\dot{I} = \beta(t)\,S\,I - \gamma\,I$$
$$\dot{R} = \gamma\,I - \delta R$$

$$S(0) + I(0) + R(0) = 1$$

- We assume all infected heal ($I \to R$) and, with rate δ, become susceptible again ($R \to S$)
- $\beta(t)$ is periodic to model different infection rates in different periods of the year

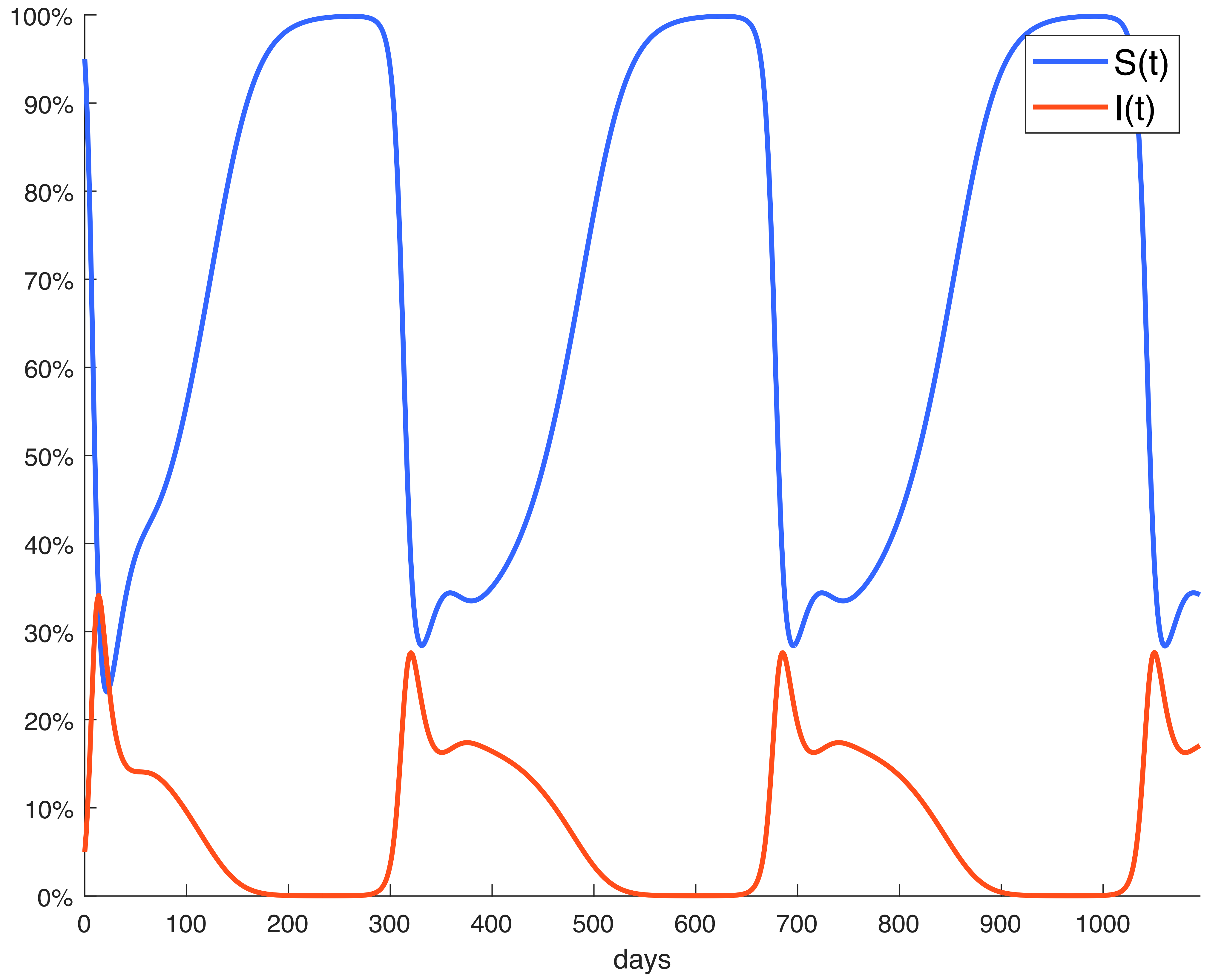
$$\dot{z}_1 = z_2$$
$$\dot{z}_2 = -\,\omega^2 z_1$$
$$\beta(t) = \beta^- + \frac{1 + z_1(t)}{2}\beta^+$$

with $z_1(0) = 1,\ z_2(0) = 0,\ \beta^+, \beta^- > 0,\ \ \omega \doteq \dfrac{2\pi}{365}$

$\boxed{\text{System of order 5}}$

Example: Susceptible-Infected Model (COVID'19)

Simulations with $\gamma = \dfrac{1}{7}, \delta = \dfrac{1}{20}, \beta^{-} = \dfrac{1}{2}\gamma, \beta^{+} = 3\gamma$



Example: Susceptible-Infected Model (COVID'19)

Homework: plot and interpret the phase portrait of the SI model

$$\dot{S} = -\beta S I$$
$$\dot{I} = \beta S I - \gamma I$$

Homework: plot and interpret the **3D** phase portrait of the SI model with reinfections (and constant rate β)

$$\dot{S} = \beta S I + \delta R$$
$$\dot{I} = \beta S I - \gamma I$$
$$\dot{R} = \gamma I - \delta R$$

Qualitative Behaviour of Trajectories

Finding limit cycles, in general, is a very difficult problem. There exist a many theoretical results predicting the existence or the absence of limit cycles of **two-dimensional** nonlinear dynamical systems. Examples:

$$\dot{x}_1 = f_1(x_1, x_2)$$
$$\dot{x}_2 = f_2(x_1, x_2)$$

Result

Every closed trajectory contains within its interior at least an equilibrium.

Bendixson's criterion

If on a simply connected region $\mathscr{D} \subset \mathbb{R}^2$ (i.e. $\mathscr{D}$ has no holes in it) the function

$$\nabla f(x) := \frac{\partial f_1(x_1, x_2)}{\partial x_1} + \frac{\partial f_2(x_1, x_2)}{\partial x_2}$$

is not identically zero and does not change sign, then the system has no closed orbits lying entirely in $\mathscr{D}$.

Divergence of the vector field

Example

$$\dot{x}_1 = x_2 + x_1 x_2^2$$
$$\dot{x}_2 = -x_1 + x_1^2 x_2$$

$$\nabla f(x) = x_1^2 + x_2^2$$

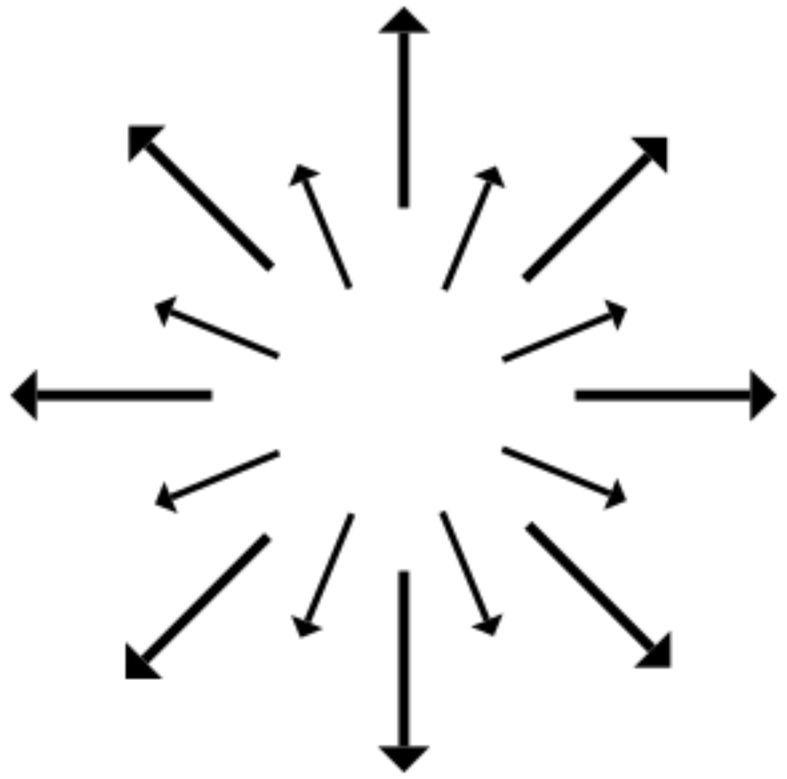
No closed orbits

Homework. For the linear system $\dot{x}_1 = a_{11}x_1 + a_{12}x_2,\ \ \dot{x}_2 = a_{21}x_1 + a_{22}x_2$, find conditions on a_{ij} so that the system has not periodic solutions by using the Bendixson's criterion

Qualitative Behaviour of Trajectories

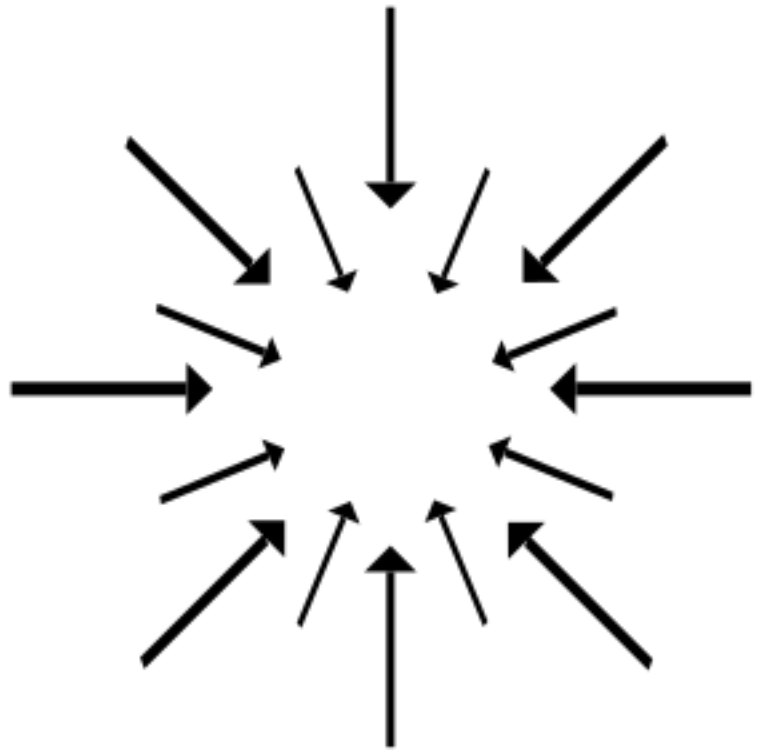
Interpretation of divergence of a vector field at a point: the extent to which the vector field flux behaves like a source locally around that point (local measure of its “outgoingness”).

Source

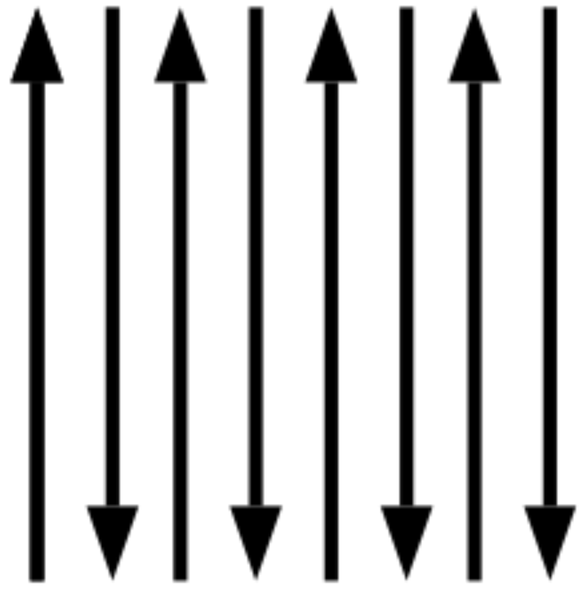


$$\frac{\partial f_1}{\partial x_1} > 0$$
$$\frac{\partial f_2}{\partial x_2} > 0$$
$$\nabla f > 0$$

Sink



$$\frac{\partial f_1}{\partial x_1} < 0$$
$$\frac{\partial f_2}{\partial x_2} < 0$$
$$\nabla f < 0$$



$$\frac{\partial f_1}{\partial x_1} = 0$$
$$\frac{\partial f_2}{\partial x_2} = 0$$
$$\nabla f = 0$$

Qualitative Behaviour of Trajectories: Bifurcations

Qualitative behaviour determined by the pattern of its equilibirum points and periodic orbits and their stability properties

The system maintains its qualitative behavior under infinitesimally small perturbations?

If yes —> **Structural stability .** If not —> **Bifurcations**

Bifurcation is change in the equilibrium points or periodic orbits, or in their stability properties, as a parameter is varied

Qualitative Behaviour of Trajectories: Bifurcations

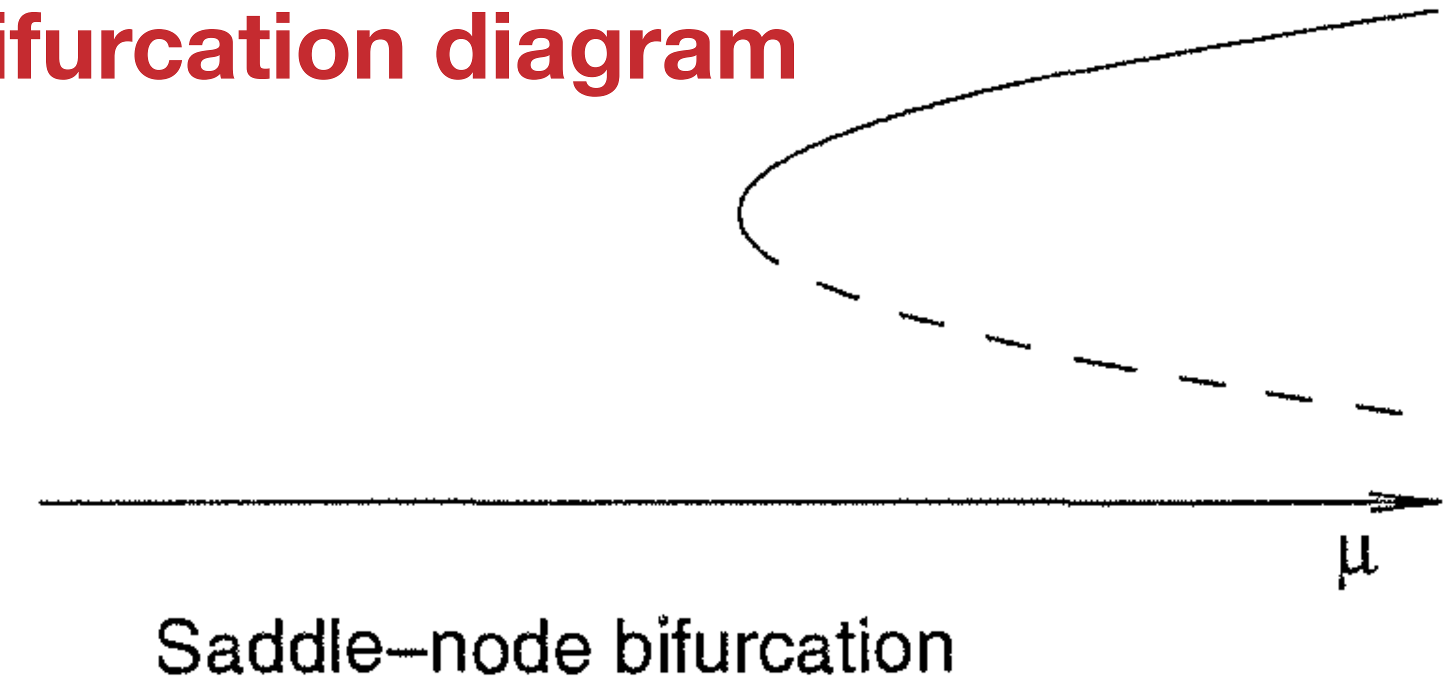
Example

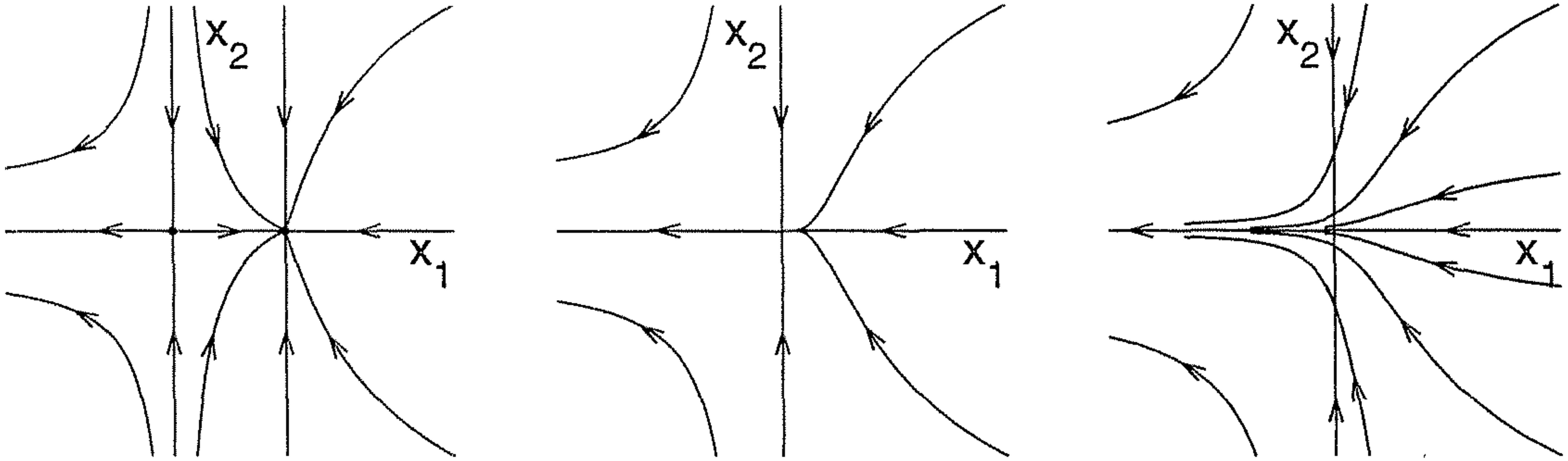
$$\dot{x}_1 = \mu - x_1^2$$
$$\dot{x}_2 = -x_2$$

μ : Bifurcation parameter

"Dangerous" bifurcation

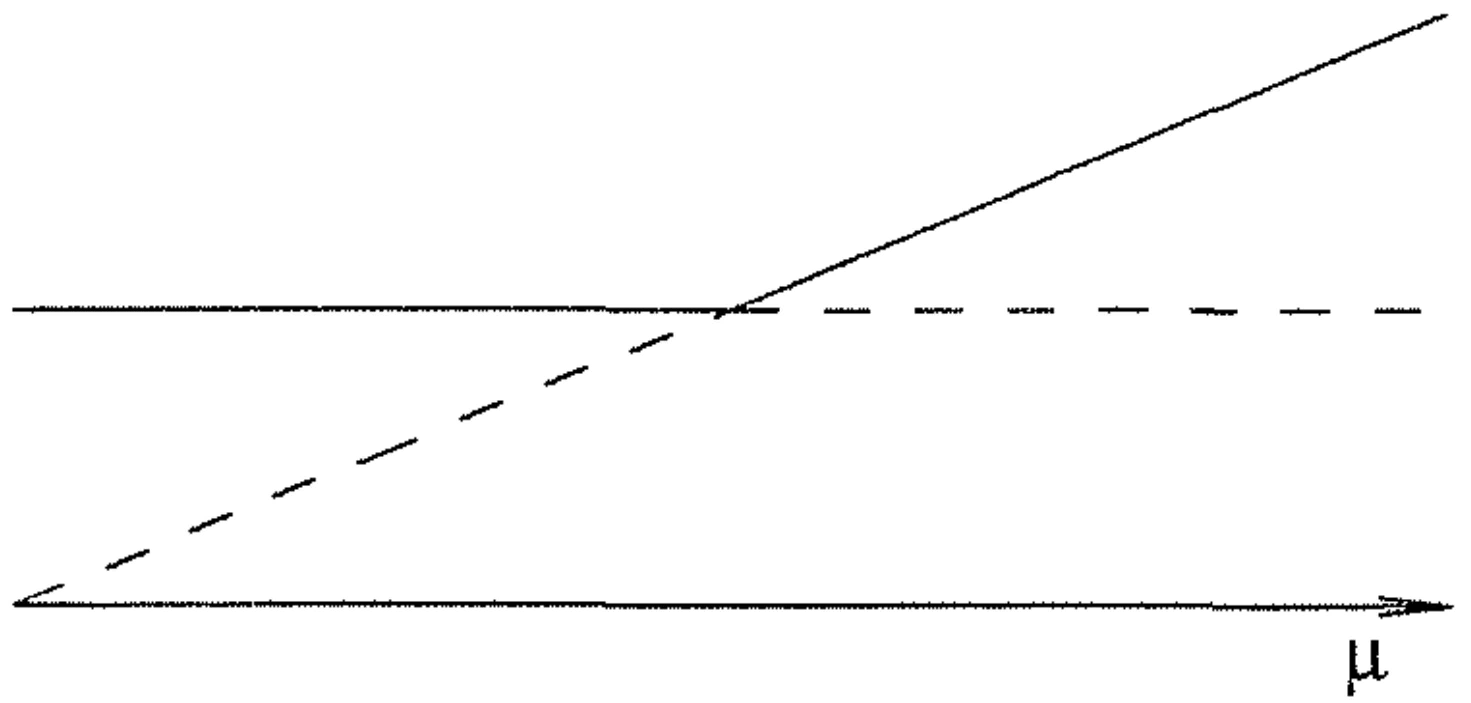
Bifurcation diagram





$\mu > 0$ $\qquad$ $\mu = 0$ $\qquad$ $\mu < 0$

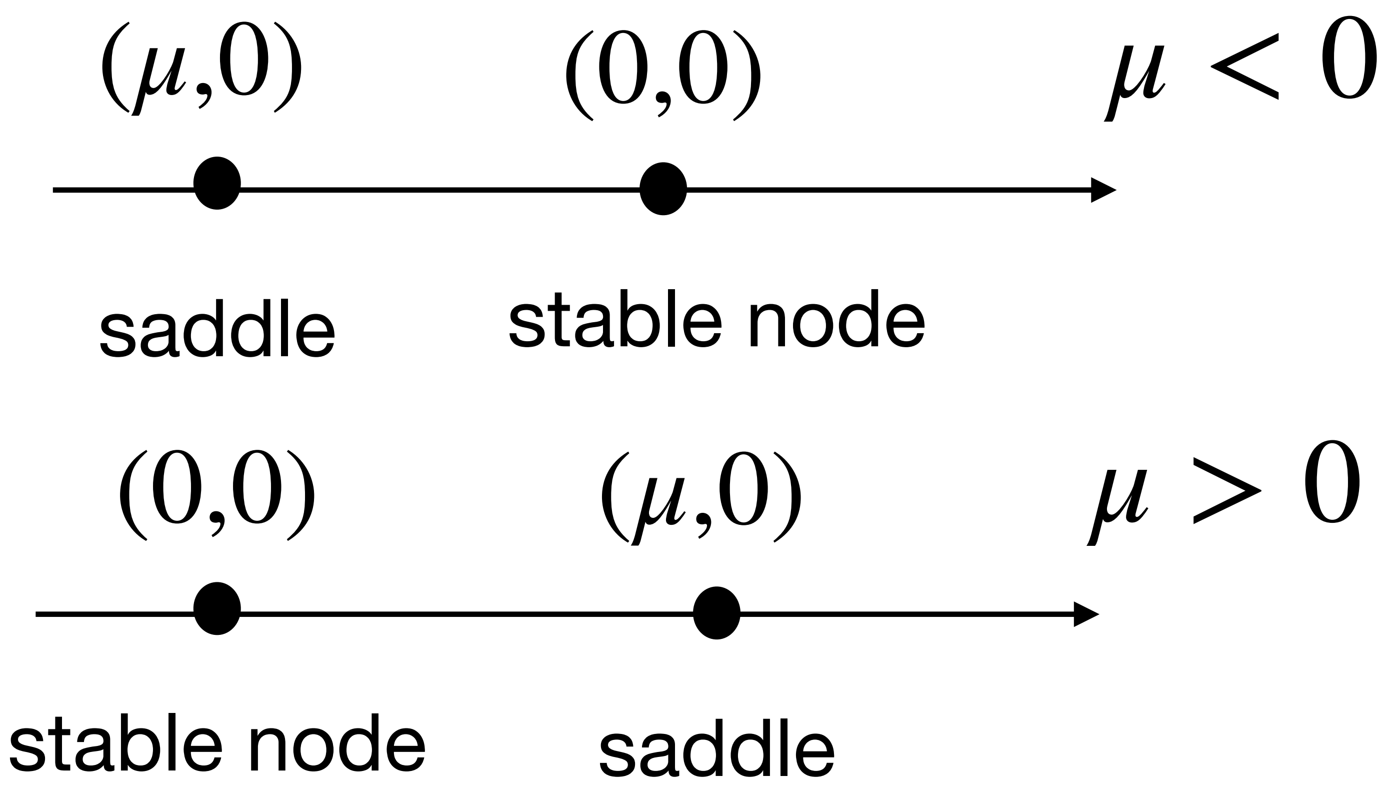
Qualitative Behaviour of Trajectories: Bifurcations

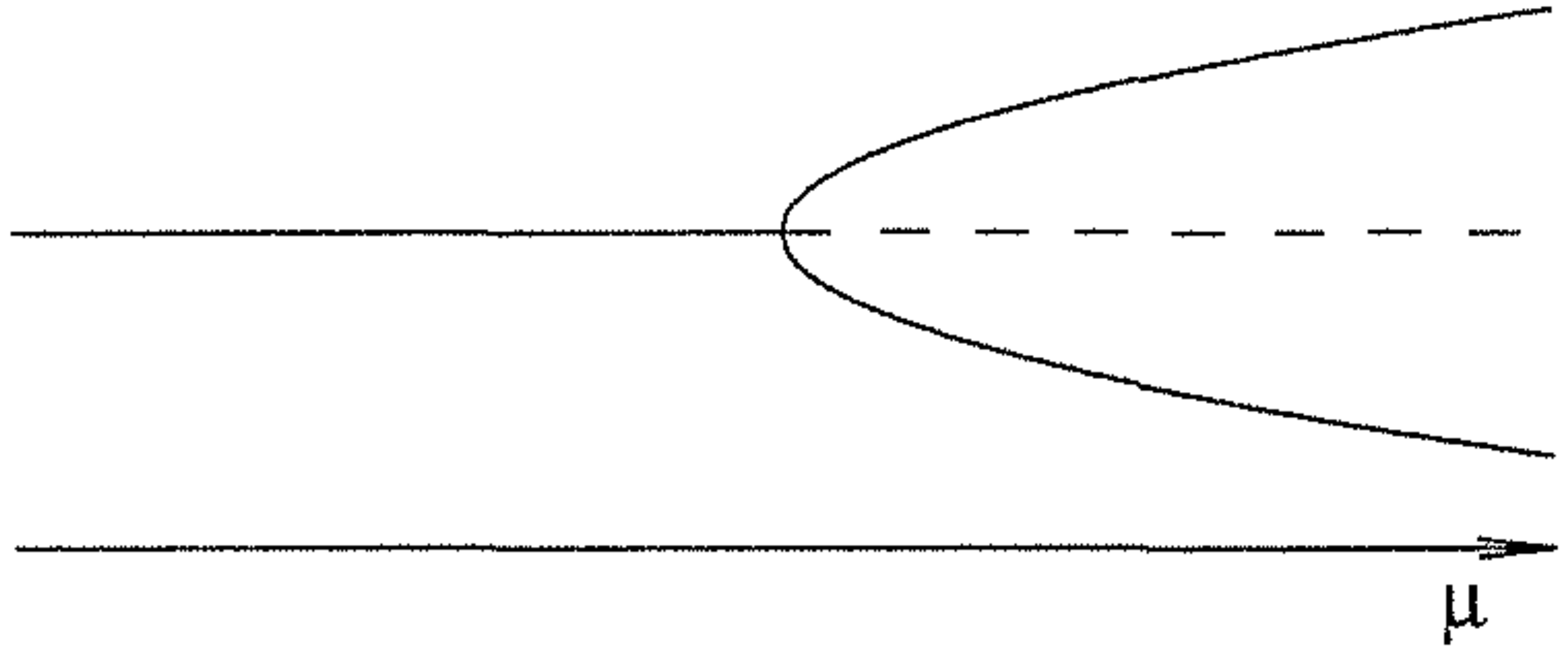


Transcritical bifurcation

$$\dot{x}_1 = \mu x_1 - x_1^2$$
$$\dot{x}_2 = -x_2$$

“Safe” bifurcation

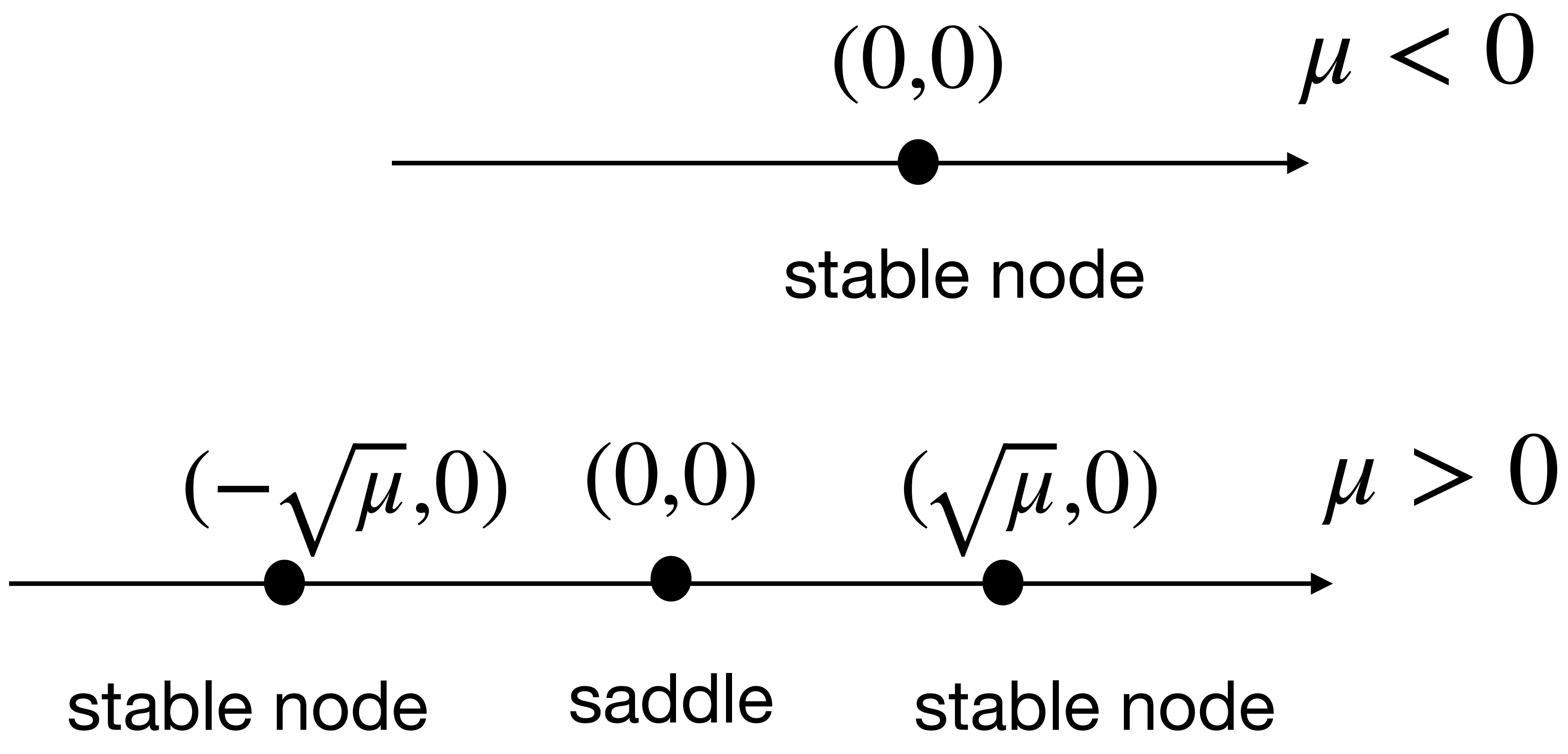


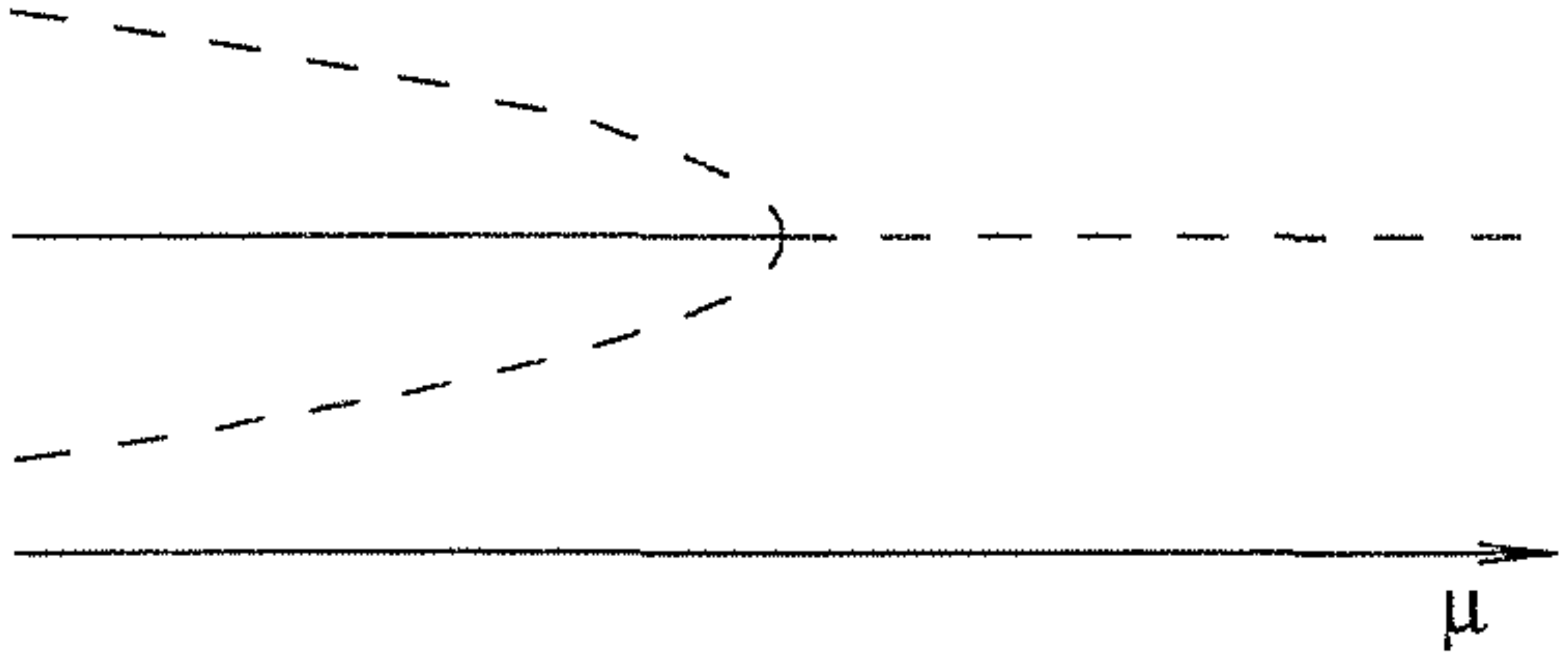


Supercritical pitchfork bifurcation

$$\dot{x}_1 = \mu x_1 - x_1^3$$
$$\dot{x}_2 = -x_2$$

“Safe” bifurcation

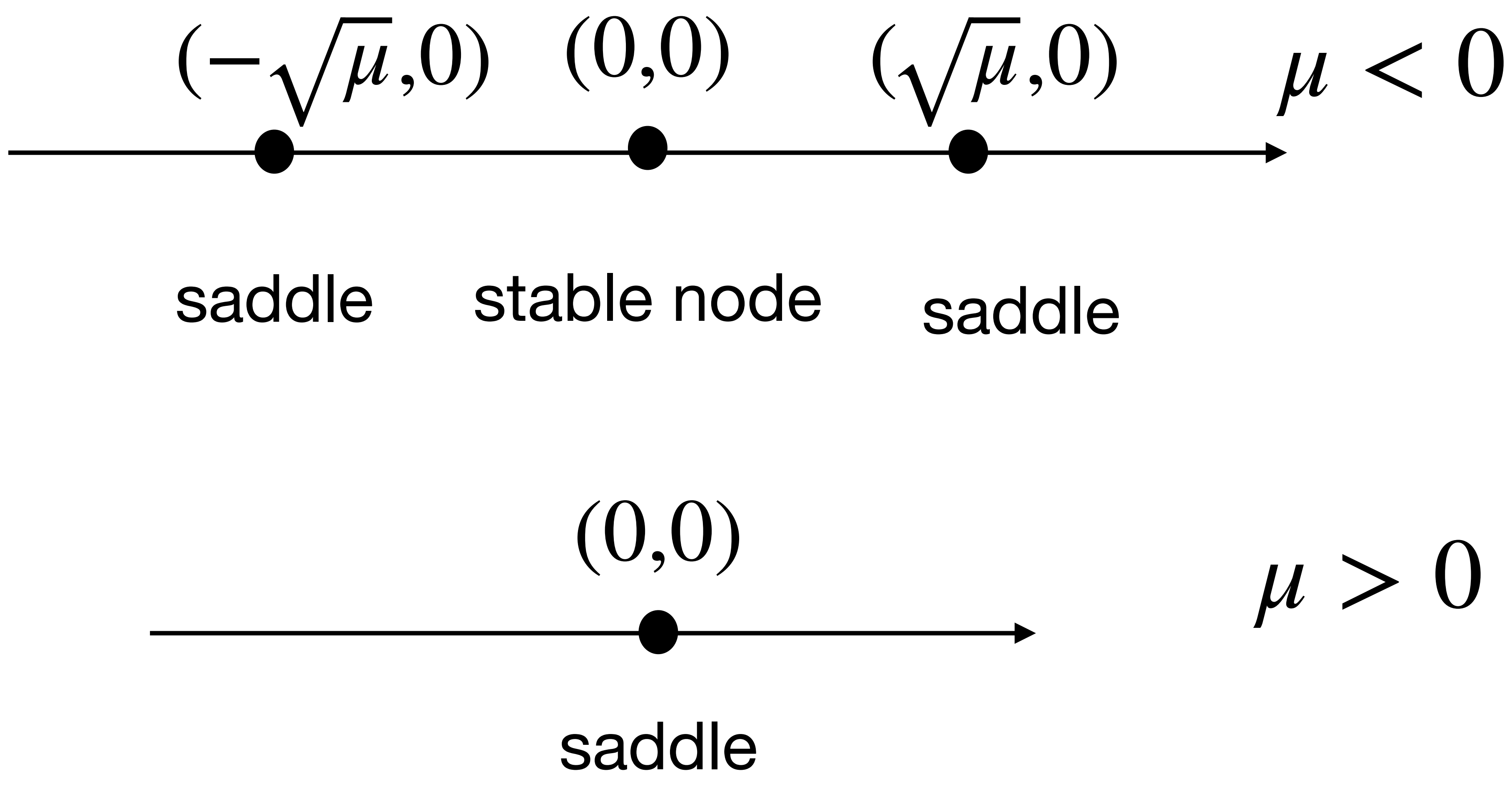


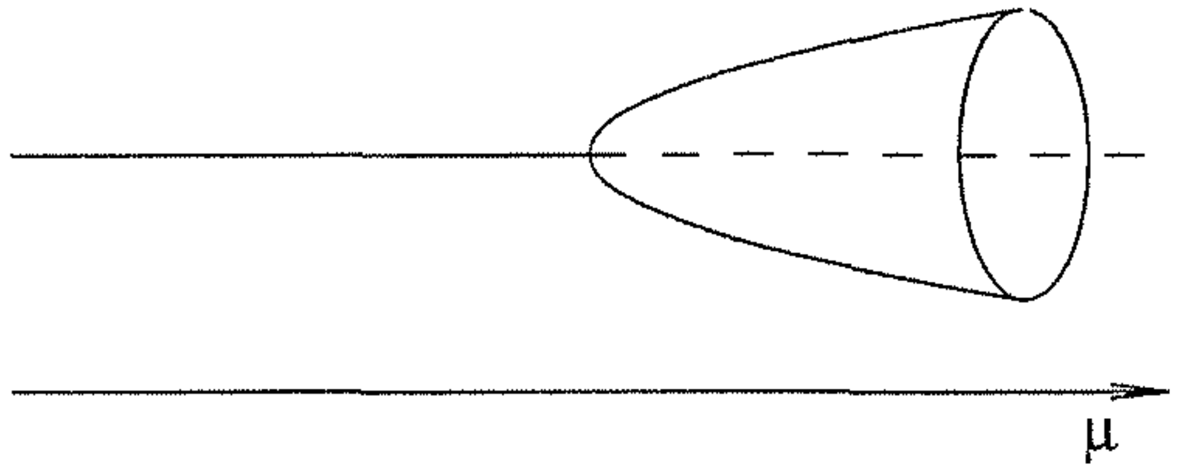


(d) Subcritical pitchfork bifurcation

$$\dot{x}_1 = \mu x_1 + x_1^3$$
$$\dot{x}_2 = -x_2$$

“Dangerous” bifurcation

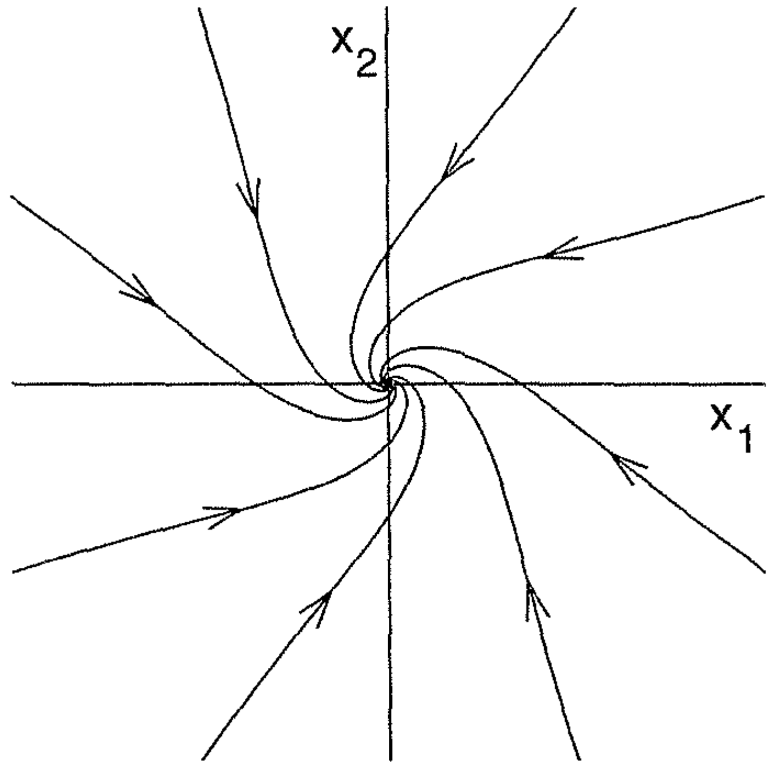




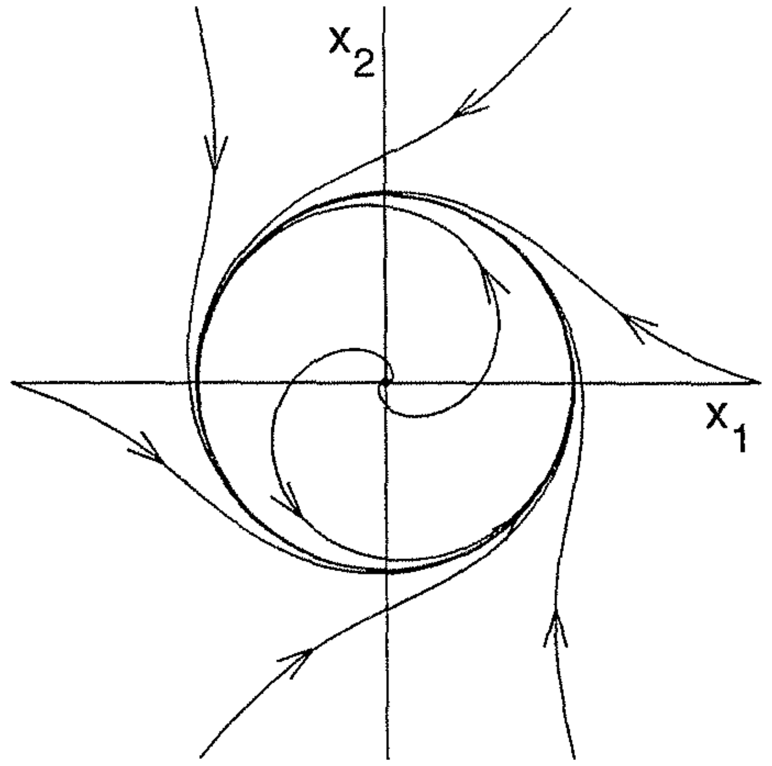
(e) Supercritical Hopf bifurcation

$$\dot{x}_1 = x_1(\mu - x_1^2 - x_2^2) - x_2$$
$$\dot{x}_2 = x_2(\mu - x_1^2 - x_2^2) + x_1$$

“Safe” bifurcation

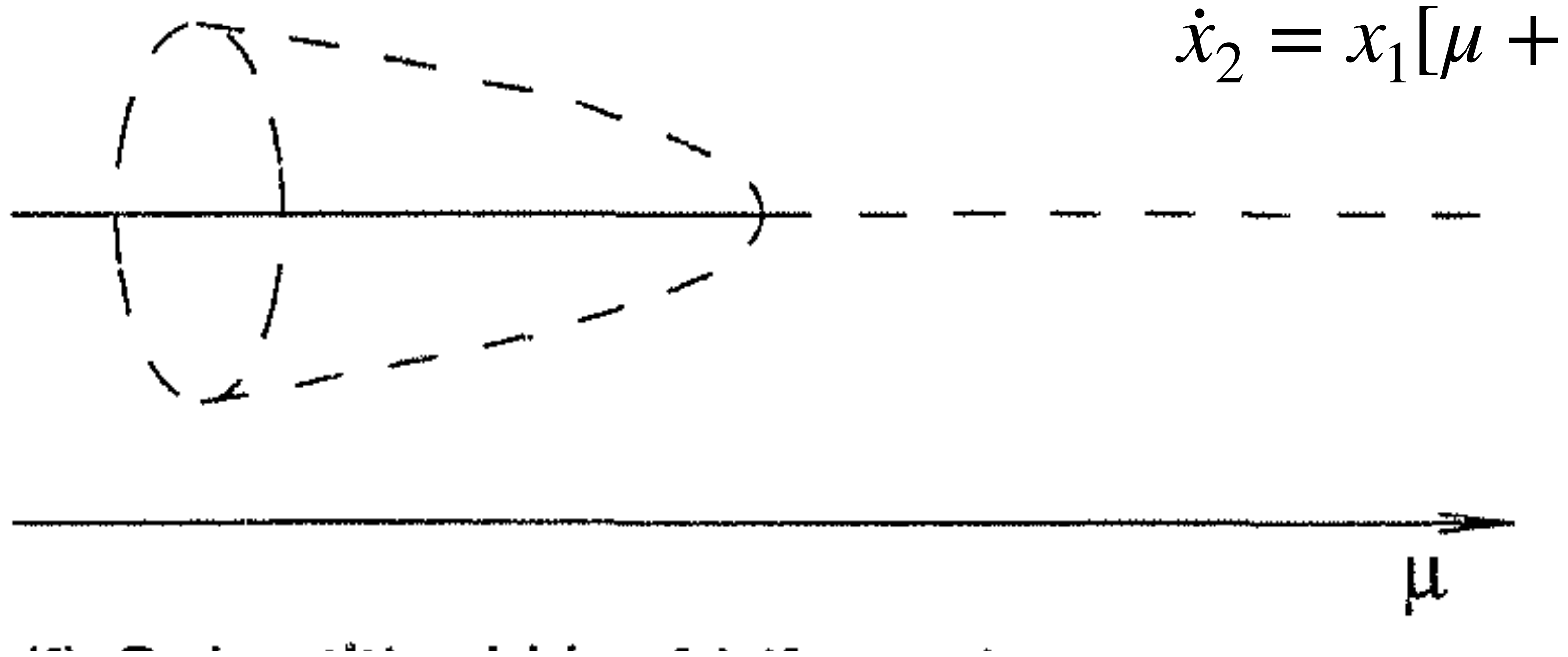


$\mu < 0$



$\mu > 0$

$$\dot{x}_1 = x_1[\mu + (x_1^2 + x_2^2) - (x_1^2 + x_2^2)^2] - x_2$$
$$\dot{x}_2 = x_1[\mu + (x_1^2 + x_2^2) - (x_1^2 + x_2^2)^2] + x_1$$



? Homework…

“Dangerous” bifurcation

Extremely rich area!