Control Tools for Distributed Optimization

Refresher on passivity theory

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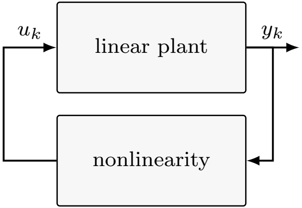
The Lur'e problem (1944)

The Lur'e problem studies the (absolute) stability of the origin $x = 0_n$ for a dynamical system obtained as the *interconnection* of a LTI system

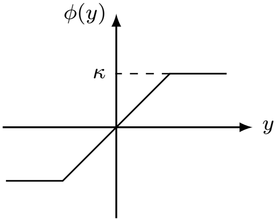
$$x_{k+1} = Ax_k + Bu_k, \qquad x_0 = x^0$$
$$y_k = Cx_k + Du_k$$



with $x_k \in \mathbb{R}^n$, $u_k \in \mathbb{R}^m$ and $y_k \in \mathbb{R}^p$, in feedback with a static nonlinearity

$$u_k = \phi(y_k)$$

with $\phi : \mathbb{R}^p \to \mathbb{R}^m$ well behaved (namely, sector bounded)



Example. A saturation can be modeled as a sector bounded nonlinearity

$$u_k = \begin{cases} y_k, & \text{if } |y_k| < \kappa \\ \kappa \cdot \mathrm{sign}(y_k), & \text{if } |y_k| \geq \kappa \end{cases}$$

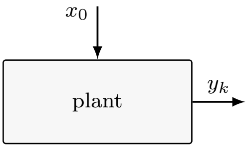
Goal. Study the interconnection stability based on the properties of the *individual* components

Lyapunov theory (recall)

Lyapunov theory focuses on the equilibrium stability of unforced systems as

$$x_{k+1} = f(x_k), \qquad x_0 = x^0$$

with state $x_k \in \mathbb{R}^n$ and a well-behaved vector field $f : \mathbb{R}^n \to \mathbb{R}^n$



Study whether a generalized energy function $V : \mathbb{R}^n \to \mathbb{R}$ decreases along trajectories to certify stability of the equilibrium $x = x_{\mathrm{eq}}$

Equivalently, check if

- the value of V at $T > 0$ is less than the initial value

$$V(x_T) - V(x_0) \leq 0$$

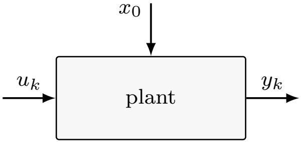
- the sign of the *increment* of V along trajectories of the system for all $k \in \mathbb{N}$

$$V(x_{k+1}) - V(x_k) \leq 0$$

Energy-based approach for input-output systems

Consider *input-output* ($u_k \in \mathbb{R}^m$ and $y_k \in \mathbb{R}^p$) systems in the form

$$
\begin{aligned}
x_{k+1} &= f(x_k, u_k), \qquad x_0 = x^0 \\
y_k &= g(x_k, u_k)
\end{aligned}
$$



with well-behaved $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ and $g : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^p$

Some questions arise when considering the energy balance of the system

- Does the system (internally) *produce* energy?
- Does the system *dissipate* energy?
- Does the system *store* the externally supplied energy?

Focus on systems in which the increase in *internally stored* energy is less than the *externally supplied* energy provided through the input

Definition of dissipative system

Definition. A system is said to be *dissipative* from input u to output y with respect to the *supply rate* $\varphi : \mathbb{R}^p \times \mathbb{R}^m \to \mathbb{R}$ if there exists a *storage function* $V : \mathbb{R}^n \to \mathbb{R}$ satisfying

$$V(x_T) - V(x_0) \leq \sum_{k=0}^{T} \varphi(y_k, u_k)$$

for all admissible trajectories and all $T > 0$ (aka *dissipation inequality*)

Definition. A system is *strictly dissipative* from u to y wrt φ if it also exists $\epsilon > 0$ such that V satisfies

$$V(x_T) - V(x_0) \leq \sum_{k=0}^{T} \varphi(y_k, u_k) - \underbrace{\epsilon \sum_{k=0}^{T} (\|x_k\|^2 + \|u_k\|^2)}$$

for all admissible trajectories and all $T > 0$

Remark. The negative term is called *dissipation rate* and measures the energy lost in the system, typically

$$\varphi(y_k, u_k) = y_k^\top u_k - \gamma_1 \|y_k\|^2 - \gamma_2 \|u_k\|^2$$

where γ_1 and γ_2 are referred to as *passivity indices*

Remark. Passivity seamlessly applies to linear/nonlinear static/dynamical systems

More definitions about passivity

A system is said to be

- *passive* if the storage function satisfies

$$V(x_{k+1}) - V(x_k) \leq y_k^\top u_k$$

- *lossless* if the storage function satisfies

$$V(x_{k+1}) - V(x_k) = y_k^\top u_k$$

- *input strictly passive* if there exists $\gamma_1 > 0$ such that

$$V(x_{k+1}) - V(x_k) \leq u_k^\top y_k - \gamma_1 \|u_k\|^2$$

  (input-feedforward passive (IFP) for general $u^\top \varphi_1(u)$)

- *output strictly passive* if there exists $\gamma_2 > 0$ such that

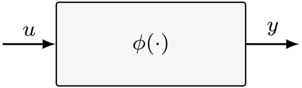
$$V(x_{k+1}) - V(x_k) \leq u_k^\top y_k - \gamma_2 \|y_k\|^2$$

  (output-feedback passive (OFP) for general $y^\top \varphi_2(y)$)

Passivity for algebraic maps

Consider an algebraic (aka static or memoryless) map

$$u \longmapsto y = \phi(u)$$



- It is *passive* (or monotone) if, for all u, it satisfies

$$\phi(u)^\top u \geq 0$$

- It is *input striclty passive* if there exists $\gamma_2 > 0$ such that

$$\phi(u)^\top u - \gamma_1 \|u\|^2 \geq 0$$

- It is *output striclty passive* if there exists $\gamma_1 > 0$ such that

$$\phi(u)^\top u - \gamma_2 \|\phi(u)\|^2 \geq 0$$

- It is *very strictly passive* if there exists $\gamma_1, \gamma_2 > 0$ such that

$$\phi(u)^\top u - \gamma_1 \|u\|^2 - \gamma_2 \|\phi(u)\|^2 \geq 0$$

Example: discrete-time integrator/accumulator

Consider the system

$$\begin{aligned} x_{k+1} &= x_k + u_k \\ y_k &= x_k \end{aligned}$$

with state, input and output $x_k, u_k, y_k \in \mathbb{R}^n$ $(m = n)$

Consider the storage function

$$V(x_k) = \tfrac{1}{2}\|x_k\|^2$$

The increment of V along system trajectories satisfies

$$\begin{aligned} V(x_{k+1}) - V(x_k) &= \tfrac{1}{2}\|x_{k+1}\|^2 - \tfrac{1}{2}\|x_k\|^2 \\ &= \tfrac{1}{2}\|x_k + u_k\|^2 - \tfrac{1}{2}\|x_k\|^2 \\ &= x_k^\top u_k + \tfrac{1}{2}\|u_k\|^2 \\ &= y_k^\top u_k + \tfrac{1}{2}\|u_k\|^2 \end{aligned}$$

Example: modified discrete-time integrator

Consider the system

$$\begin{aligned} x_{k+1} &= x_k + u_k \\ y_k &= x_k + u_k \end{aligned}$$

with state, input and output $x_k, u_k, y_k \in \mathbb{R}^n$

Consider the storage function

$$V(x_k) = \frac{1}{2}\|x_k\|^2$$

The increment of V along system trajectories satisfies

$$\begin{aligned} V(x_{k+1}) - V(x_k) &= \frac{1}{2}\|x_{k+1}\|^2 - \frac{1}{2}\|x_k\|^2 \\ &= \frac{1}{2}\|x_k + u_k\|^2 - \frac{1}{2}\|x_k\|^2 \\ &= x_k^\top u_k + \frac{1}{2}\|u_k\|^2 \\ &= y_k^\top u_k - \frac{1}{2}\|u_k\|^2 \end{aligned}$$

Storage functions for linear systems

Consider an input-output linear system

$$\begin{aligned} x_{k+1} &= Ax_k + Bu_k, \qquad x_0 = x^0 \\ y_k &= Cx_k + Du_k \end{aligned}$$

and a *quadratic storage function* $V(x) := \frac{1}{2}x^\top Px$, with $P \in \mathbb{R}^{n\times n}$ and $P = P^\top > 0$

Then, the increment of V along system trajectories satisfies

$$\begin{aligned} V(x_{k+1}) - V(x_k) &= \tfrac{1}{2}x_{k+1}^\top Px_{k+1} - \tfrac{1}{2}x_k^\top Px_k \\ &= \tfrac{1}{2}x_k^\top(A^\top PA - P)x_k + x_k^\top(A^\top PB)u_k + \tfrac{1}{2}u_k^\top B^\top PBu_k \end{aligned}$$

where

- the first Lyapunov-like term (quadratic in x_k) is possibly negative (if A is Schur)
- the last term (quadratic in u_k) is always positive, depending also on P
- the cross term must be exploited, e.g., reconstructing the output y_k

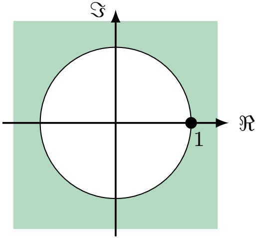
Frequency domain: discrete positive realness

An alternative input-output description of linear systems is related to the *frequency domain*
(Focus on square systems only, i.e., with $p = m$)

A linear time-invariant system can be represented with its transfer matrix $G(\mathrm{z})$

The $m \times m$ transfer matrix $G(\mathrm{z})$ of a LTI system can be computed from its state-space realization (A, B, C, D) as

$$G(\mathrm{z}) = \frac{Y(\mathrm{z})}{U(\mathrm{z})} = C(\mathrm{z}I_n - A)^{-1}B + D, \qquad \mathrm{z} \in \mathbb{C}$$



Definition. A transfer matrix $G(\mathrm{z})$ is *discrete positive real* if

- $G(\mathrm{z})$ has no poles outside the unit disk and simple poles (if any) on the unit disk
- $G(\mathrm{z}) + G(\bar{\mathrm{z}})^\top \geq 0$ for all $\mathrm{z} \in \mathbb{C}$ such that $|z| > 1$

Definition. A transfer matrix $G(\mathrm{z})$ is *strictly discrete positive real* if there exists $\rho > 1$ such that $G(\mathrm{z}/\rho)$ is DPR

Example: discrete-time integrator (revisited)

Consider the system

$$x_{k+1} = x_k + u_k$$
$$y_k = x_k$$

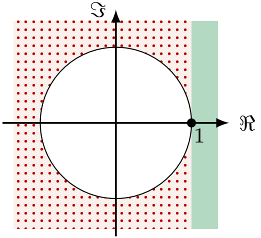
with state, input and output $x_k, u_k, y_k \in \mathbb{R}^n$ $(m = n)$

The transfer matrix is

$$G(\mathrm{z}) = \frac{Y(\mathrm{z})}{U(\mathrm{z})} = \frac{1}{\mathrm{z}-1} I_n$$

To check DPR, we study the sign of

$$G(\mathrm{z}) + G(\bar{\mathrm{z}})^\top = \frac{1}{\mathrm{z}-1} I_n + \frac{1}{\bar{\mathrm{z}}-1} I_n = \frac{\mathrm{z}-1+\bar{\mathrm{z}}-1}{|\mathrm{z}-1|^2} I_n = 2\frac{\Re(\mathrm{z})-1}{|\mathrm{z}-1|^2} I_n$$



It is positive exclusively for $\Re(z) > 1$, and not generally for all $|\mathrm{z}| > 1$

Remark. The state-space description is $A = 0$, $B = I$, $C = I$ and $D = 0$, thus it is not strictly proper

Remark. Considering $n = 1$ is practically wlog

Example: modified discrete-time integrator (revisited)

Consider the system

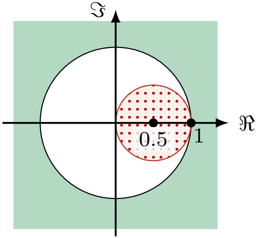
$$x_{k+1} = x_k + u_k$$
$$y_k = x_k + u_k$$

The transfer function is

$$G(\mathrm{z}) = \frac{Y(\mathrm{z})}{U(\mathrm{z})} = \frac{1}{\mathrm{z}-1} + 1 = \frac{\mathrm{z}}{\mathrm{z}-1}$$

To check DPR, we study the sign of

$$G(\mathrm{z}) + G(\bar{\mathrm{z}}) = \frac{\mathrm{z}}{\mathrm{z}-1} + \frac{\bar{\mathrm{z}}}{\bar{\mathrm{z}}-1} = \frac{2\mathrm{z}\bar{\mathrm{z}} - \mathrm{z} - \bar{\mathrm{z}}}{|\mathrm{z}-1|^2} = 2\frac{|\mathrm{z}|^2 - \Re(\mathrm{z})}{|\mathrm{z}-1|^2}$$
$$= 2\frac{(\Re(\mathrm{z}) - \frac{1}{2})^2 + \Im(\mathrm{z})^2 - \frac{1}{4}}{|\mathrm{z}-1|^2}$$



It is positive outside the disk of radius $\frac{1}{2}$ centered at $\mathrm{z} = \frac{1}{2}$, hence also for all $|\mathrm{z}| > 1$

Positive real lemma

Is there a connection between passivity and discrete positive realness?

Positive-real lemma. Let (A, B, C, D) be a minimal realization of a square transfer matrix $G(\mathrm{z})$, with no poles outside the unit disk and simple poles (if any) on the unit disk

If there exist a (describing) matrix $P \in \mathbb{R}^{n\times n}$, $P = P^\top > 0$, and matrices $M_y \in \mathbb{R}^{\ell\times n}$ and $M_u \in \mathbb{R}^{\ell\times m}$ such that

$$
\begin{aligned}
A^\top P A - P &= -M_y^\top M_y \\
A^\top P B &= C^\top - M_y^\top M_u \\
B^\top P B &= D + D^\top - M_u^\top M_u
\end{aligned}
$$

then, the transfer matrix $G(\mathrm{z})$ is discrete positive real

Conversely, if $G(\mathrm{z})$ is discrete positive real, then for any minimal realization of $G(\mathrm{z})$ there exist $P = P^\top > 0$, M_y and M_u satisfying the previously stated matrix equations

Remark. For general rectangular systems, the so-called *bounded-real lemma* must be considered

Implications of discrete positive realness

The increment along trajectories of the storage function $V(x) := \frac{1}{2}x^\top P x$ satisfies

$$\begin{aligned}
V(x_{k+1}) - V(x_k) &= \tfrac{1}{2}x_k^\top (A^\top PA - P)x_k + x_k^\top (A^\top PB)u_k + \tfrac{1}{2}u_k^\top B^\top PBu_k \\
&= \tfrac{1}{2}x_k^\top (-M_y^\top M_y)x_k + x_k^\top (C^\top - M_y^\top M_u)u_k + \tfrac{1}{2}u_k^\top (D + D^\top - M_u^\top M_u)u_k \\
&= y_k^\top u_k - \tfrac{1}{2}\|M_y x_k + M_u u_k\|^2 + \tfrac{1}{2}u_k^\top \underbrace{(D - D^\top)}_{} u_k
\end{aligned}$$

Remark. The matrix $D - D^\top$ is skew-symmetric

Remark. The vector $M_y x_k + M_u u_k$ represents a particular "output" of the system

Kalman-Yakubovich-Popov lemma

Kalman-Yakubovich-Popov lemma. Let (A, B, C, D) be a minimal realization of a square transfer matrix $G(\mathrm{z})$, with no poles outside the unit disk and simple poles (if any) on the unit disk

If there exist $P = P^\top > 0$, M_y, M_u, and $\rho > 1$ such that

$$
\begin{aligned}
A^\top P A - \tfrac{1}{\rho} P &= -M_y^\top M_y \\
A^\top P B &= C^\top - M_y^\top M_u \\
B^\top P B &= D + D^\top - M_u^\top M_u
\end{aligned}
$$

then, the transfer matrix $G(\mathrm{z})$ is *strictly* discrete positive real

Conversely, if $G(\mathrm{z})$ is strictly discrete positive real, then for any minimal realization of $G(\mathrm{z})$ there exist $P = P^\top > 0$, M_y, M_u and $\rho > 1$ satisfying the previously stated matrix equations

Implications of the KYP lemma

The increment along trajectories of the storage function $V(x) := \frac{1}{2}x^\top Px$ satisfies

$$V(x_{k+1}) - V(x_k) = y_k^\top u_k - \tfrac{1}{2}\|M_y x_k + M_u u_k\|^2 - \left(1 - \tfrac{1}{\rho}\right)V(x_k)$$

Remark. If the closed-loop input u_k satisfies $y_k^\top u_k \leq 0$, then *exponential stability* of the origin is certified

Remark. The negative term $-\frac{1}{2}\|M_y x_k + M_u u_k\|^2$ can be neglected

Example: pure delay system

Consider the system

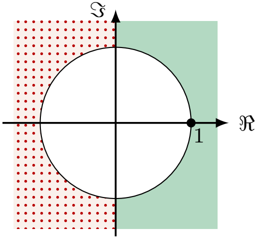
$$x_{k+1} = u_k$$
$$y_k = x_k$$

The transfer function is

$$G(\mathrm{z}) = \frac{1}{\mathrm{z}}$$

To check DPR, we study the sign of

$$G(\mathrm{z}) + G(\bar{\mathrm{z}}) = \frac{1}{\mathrm{z}} + \frac{1}{\bar{\mathrm{z}}} = 2\frac{\Re(\mathrm{z})}{|\mathrm{z}|^2}$$



It is positive exclusively when $\Re(\mathrm{z}) > 0$, and not in general for all $|\mathrm{z}| > 1$

Remark. The state-space description is $A = 0$, $B = I$, $C = I$ and $D = 0$, thus it is not strictly proper

Example: modified delay system

Consider the system

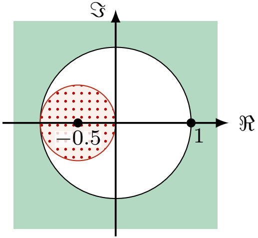
$$x_{k+1} = u_k$$
$$y_k = x_k + u_k$$

The transfer function is

$$G(\mathrm{z}) = \frac{1}{\mathrm{z}} + 1 = \frac{\mathrm{z}+1}{\mathrm{z}}$$

To check DPR, we study the sign of

$$G(\mathrm{z}) + G(\bar{\mathrm{z}}) = \frac{\mathrm{z}+1}{\mathrm{z}} + \frac{\bar{\mathrm{z}}+1}{\bar{\mathrm{z}}} = \frac{(\mathrm{z}+1)\bar{\mathrm{z}} + (\bar{\mathrm{z}}+1)\mathrm{z}}{|\mathrm{z}|^2} = 2\frac{|\mathrm{z}|^2 + \Re(\mathrm{z})}{|\mathrm{z}|^2}$$
$$= 2\frac{(\Re(\mathrm{z}) + \frac{1}{2})^2 + \Im(\mathrm{z})^2 - \frac{1}{4}}{|\mathrm{z}|^2}$$



It is positive outside the disk of radius $\frac{1}{2}$ centered at $\mathrm{z} = -\frac{1}{2}$, hence for all $|\mathrm{z}| > 1$

Remark. For strict DPR, scale the transfer function with $\rho > 1$, i.e., $G(\mathrm{z}/\rho) = \frac{\mathrm{z}/\rho+1}{\mathrm{z}/\rho} = \frac{\mathrm{z}+\rho}{\mathrm{z}}$: nm-phase

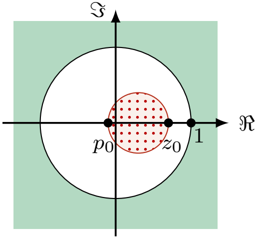
Example: zero-pole system

Consider the system with a (real) zero at $\mathrm{z} = z_0$ and a (real) pole at $\mathrm{z} = p_0$

$$G(\mathrm{z}) = \frac{\mathrm{z} - z_0}{\mathrm{z} - p_0}$$

To check DPR, we study the sign of

$$\begin{aligned} G(\mathrm{z}) + G(\bar{\mathrm{z}}) &= \frac{\mathrm{z} - z_0}{\mathrm{z} - p_0} + \frac{\bar{\mathrm{z}} - z_0}{\bar{\mathrm{z}} - p_0} \\ &= \frac{(\mathrm{z} - z_0)(\bar{\mathrm{z}} - p_0) + (\bar{\mathrm{z}} - z_0)(\mathrm{z} - p_0)}{|\mathrm{z} - p_0|^2} \\ &= 2\frac{|\mathrm{z}|^2 - (z_0 + p_0)\Re(\mathrm{z}) + z_0 p_0}{|\mathrm{z} - p_0|^2} \\ &= 2\frac{\left(\Re(\mathrm{z}) - \frac{z_0 + p_0}{2}\right)^2 + \Im(\mathrm{z}) - \left(\frac{z_0 - p_0}{2}\right)^2}{|\mathrm{z} - p_0|^2} \end{aligned}$$



It is positive outside the disk of radius $\frac{|z_0 - p_0|}{2}$ centered at $\mathrm{z} = \frac{z_0 + p_0}{2}$

Remark. For the DPR condition to hold, z_0 and p_0 must lie in the interval $[-1, 1]$

Stability and control of a passive (linear) system

Consider a passive LTI system (A, B, C, D)

$$\begin{aligned} x_{k+1} &= Ax_k + Bu_k \\ y_k &= Cx_k + Du_k \end{aligned}$$

with a storage function V satisfying the dissipation inequality

$$V(x_{k+1}) - V(x_k) \leq y_k^\top u_k$$

- If $u_k \equiv 0$, then $V(x_{k+1}) - V(x_k) \leq 0$, i.e., marginal stability of the equilibrium is guaranteed
- If $y_k \equiv 0$, then $V(x_{k+1}) - V(x_k) \leq 0$, i.e., the zero-dynamics of the system is stable
- A passive system can be easily stabilized using a static output feedback

$$u_k = -\alpha y_k$$

with an *arbitrary* gain $\alpha > 0$

The passivity theorem

Theorem. Given two passive (nonlinear) systems Σ_1 and Σ_2, their (negative) feedback interconnection (whenever well-posed) is also passive from (u_k^1, u_k^2) to (y_k^1, y_k^2)

Remark. The equilibrium of the interconnected system is stable (possibly also asymptotically stable)

Remark. An *excess* of passivity in one subsystem can compensate for a *shortage* in the other