

Control Tools for Distributed Optimization

The gradient method analysis via control tools

Prof. Ivano Notarnicola

Dept. of Electrical, Electronic, Information Eng.
Università di Bologna
ivano.notarnicola@unibo.it

Prof. Ruggero Carli

Dept. of Information Engineering
Università di Padova
ruggero.carli@unipd.it

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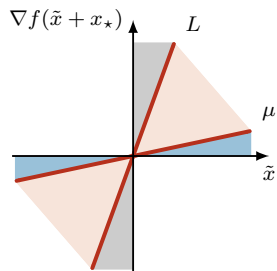
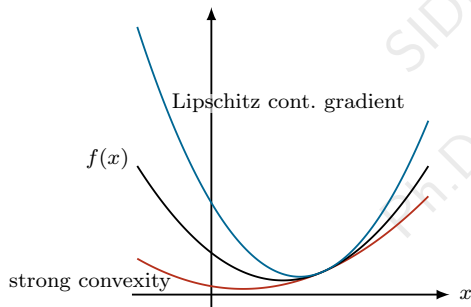
Convex optimization problem (recall)

Consider the unconstrained optimization problem

$$\min_{x \in \mathbb{R}^n} f(x)$$

where $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is μ -strongly convex and has L -Lipschitz continuous gradient ($L > \mu > 0$)

For all x_A, x_B , it holds $(\nabla f(x_A) - \nabla f(x_B))^\top (x_A - x_B) \geq \frac{1}{\mu+L} \|\nabla f(x_A) - \nabla f(x_B)\|^2 + \frac{\mu L}{\mu+L} \|x_A - x_B\|^2 \geq 0$



The gradient method in error coordinates

Let $\tilde{x} = x - x_*$, then the gradient method reads

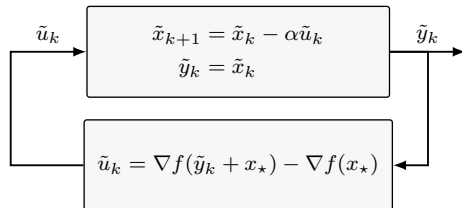
$$\tilde{x}_{k+1} = \tilde{x}_k - \alpha \tilde{u}_k$$

$$\tilde{y}_k = \tilde{x}_k$$

$$\tilde{u}_k = \nabla f(\tilde{y}_k + x_*) - \nabla f(x_*)$$

with $\alpha > 0$ being the so-called *stepsize* and an arbitrary initial condition $x_0 \in \mathbb{R}^n$

The *convergence* analysis amounts to studying the *stability* properties of the origin $\tilde{x} = 0_n$ (unique equilibrium)



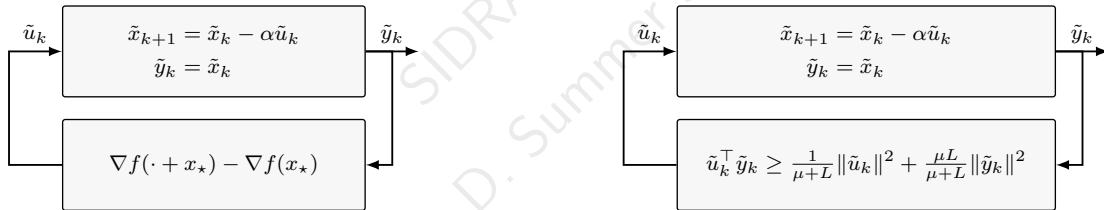
Equivalent representation of the gradient method analysis

The gradient method is a feedback interconnection of a discrete-time integrator and a static nonlinear map

$$\tilde{y} \mapsto \tilde{u} = \nabla f(\tilde{y} + x_\star) - \nabla f(x_\star)$$

which can be “replaced” with its sector-bounded characterization ($x_A = \tilde{y}$ and $x_B = 0$)

$$(\nabla f(\tilde{y}_k + x_\star) - \nabla f(x_\star))^\top \tilde{y}_k \geq \frac{1}{\mu+L} \|\nabla f(\tilde{y}_k + x_\star) - \nabla f(x_\star)\|^2 + \frac{\mu L}{\mu+L} \|\tilde{y}_k\|^2$$



Are the two individual components passive?

Gradient method analysis: first loop transformation

First, actuate a (positive) *feedback action* on the plant

$$\tilde{u} = \hat{u} + \mu \tilde{y}$$

“stealing” strong convexity from the cost function f

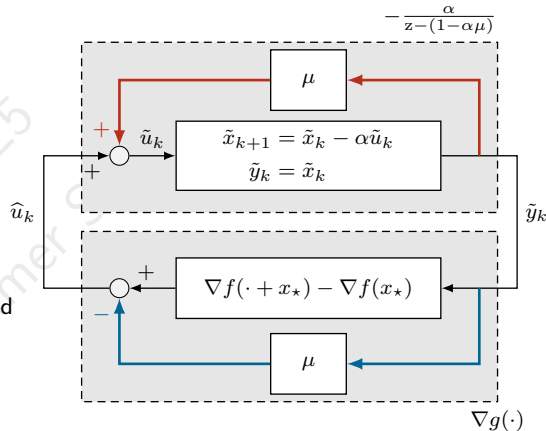
The resulting (linear) dynamics is

$$\tilde{x}_{k+1} = (1 - \alpha\mu)\tilde{x}_k - \alpha\hat{u}_k$$

$$\tilde{y}_k = \tilde{x}_k$$

with $\hat{u}_k = \tilde{u}_k - \mu \tilde{y}_k$ and $\tilde{y}_k$ satisfying the co-coercivity bound

$$\hat{u}_k^\top \tilde{y}_k \geq \frac{1}{L-\mu} \|\hat{u}_k\|^2$$



Remark. The static nonlinearity (lower subsystem) can be thought of as being associated with a convex function $g(\cdot)$ having a $(L - \mu)$ -Lipschitz continuous gradient

Gradient method analysis: second loop transformation

Second, consider a *feedforward action* on the plant

$$\tilde{y} \mapsto \hat{y} = \tilde{y} - \frac{1}{L-\mu} \hat{u}$$

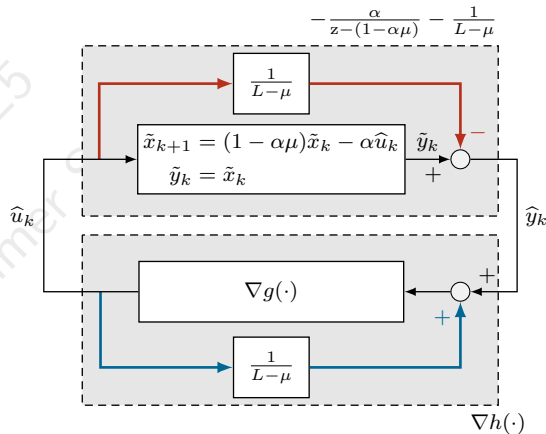
The resulting (linear) dynamics is

$$\tilde{x}_{k+1} = (1 - \alpha\mu)\tilde{x}_k - \alpha\hat{u}_k$$

$$\hat{y}_k = \tilde{x}_k - \frac{1}{L-\mu} \hat{u}_k$$

with $\hat{u}_k$ and $\hat{y}_k$ satisfying the monotonicity bound

$$\hat{u}_k^\top \hat{y}_k \geq 0$$

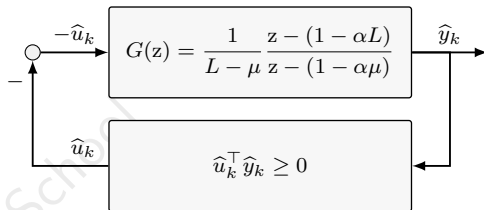


Remark. The static nonlinearity can be thought as associated to a merely convex function $\nabla h(\cdot)$

Gradient method analysis: zero-pole map

The transfer function from $-\hat{u}_k$ to $\hat{y}_k$ is

$$\begin{aligned} G(z) &= \frac{\hat{Y}(z)}{-\hat{U}(z)} = \frac{\alpha}{z - (1 - \alpha\mu)} + \frac{1}{L - \mu} \\ &= \frac{1}{L - \mu} \frac{z - (1 - \alpha L)}{z - (1 - \alpha\mu)} \end{aligned}$$



The open-loop plant has a zero at $z = 1 - \alpha L$ and a pole at $z = 1 - \alpha\mu$

Recall that $G(z) + G(\bar{z})$ is positive outside the disk

- of radius $\frac{|z_0 - p_0|}{2} = \frac{|1 - \alpha L - 1 + \alpha\mu|}{2} = \alpha \frac{L - \mu}{2}$ and
- centered at $z = \frac{z_0 + p_0}{2} = 1 - \alpha \frac{L + \mu}{2}$

For a sufficiently small stepsize $\alpha > 0$, the mentioned disk is entirely contained in the unit disk

Hence $G(z)$ is *strictly* discrete positive real

Gradient method analysis: convergence proof

Consider a *minimal state-space realization* of the transfer function $G(z)$ given by

$$\begin{aligned}\tilde{x}_{k+1} &= (1 - \alpha\mu)\tilde{x}_k + \alpha(-\hat{u}_k) \\ \hat{y}_k &= \tilde{x}_k + \frac{1}{L-\mu}(-\hat{u}_k)\end{aligned}$$

Then, by the KYP lemma, there exists a storage function $V(\tilde{x}) = \frac{1}{2}\tilde{x}^\top P\tilde{x}$, with $P = P^\top > 0$, such that

$$V(\tilde{x}_{k+1}) - V(\tilde{x}_k) = \hat{y}_k^\top (-\hat{u}_k) - \frac{1}{2}\|M_y\hat{y}_k + M_u(-\hat{u}_k)\|^2 - \left(1 - \frac{1}{\rho}\right)V(\tilde{x}_k)$$

with $\rho > 1$

When feedback interconnected with a passive nonlinear map (i.e., satisfying $\hat{y}_k^\top \hat{u}_k \geq 0$) it follows

$$V(\tilde{x}_{k+1}) - V(\tilde{x}_k) \leq -\left(1 - \frac{1}{\rho}\right)V(\tilde{x}_k)$$

certifying *exponential stability* of the equilibrium $\tilde{x}=0$, and hence convergence of y_k to the optimal solution $x_\star$